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# 量子代数 $SU_q(4)$ 的不可约表示*

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**摘 要**

本文利用类张量的方法计算了量子代数 $SU_q(4)$ 在 $SU_q(4)\supset SU_q(2)\oplus SU_q(2)$ 基上的不可约表示,得到了求矩阵元的递推公式.

## 一、引 言

量子代数是一种特殊的 Hopf 代数,Drinfeld 称之为准三角 Hopf 代数. Drinfeld[1] 和 Jimbo[2] 证明了,对于量子代数的每一个不可约表示,都存在一个 Yang-Baxter 方程(YBE)的解,因此许多物理学家对量子代数的表示论发生了兴趣. 最近我们给出了一个计算 $SU_q(3)$ 和 $SU_q(2/1)$ 的不可约表示的新方法[3-5],这个方法的要点是:先引入满足我们称之为类 Serre 关系的某些算符,它们可视作 $SU_q(2)$ 的 1/2 阶张量算符,这些算符连同原本满足 Serre 关系的算符一起,可以在类 Elliott 基上导出它们的约化矩阵元的递推公式,用此公式就可确定 $SU_q(3)$ 和 $SU_q(2/1)$ 的表示以及计算 $SU_q(3)$ 的 Clebsch-Gordan(C、G)系数. 在这篇文章中我们将用此方法计算 $SU_q(4)\supset SU_q(2)\oplus SU_q(2)$ 的不可约表示.

这篇文章的组织如下:第 2 节先写出量子代数 $SU_q(4)$ 以及它的 q 振子实现. 第 3 节讨论 $SU_q(4)\supset SU_q(2)\oplus SU_q(2)$ 的不可约表示,并给出了一些约化矩阵元的数值表.

## 二、$SU_q(4)$ 量子代数

$SU_q(4)$ 量子代数的一般关系由 Jimbo[2] 给出如下:

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$$\left.\begin{aligned}
&[h_i,h_j]=0,\\
&[h_i,e_{\pm j}]=\pm A_{ij}e_{\pm j},\\
&[e_i,e_{-j}]=\delta_{ij}[h_i], \qquad (i,j)=1,2,3.\\
&[e_{\pm 1},e_{\pm 3}]=0,\\
&e_{\pm1}^2e_{\pm2}+e_{\pm2}e_{\pm1}^2=[2]e_{\pm1}e_{\pm2}e_{\pm1},\\
&e_{\pm2}^2e_{\pm1}+e_{\pm1}e_{\pm2}^2=[2]e_{\pm2}e_{\pm1}e_{\pm2},\\
&e_{\pm2}^2e_{\pm3}+e_{\pm3}e_{\pm2}^2=[2]e_{\pm2}e_{\pm3}e_{\pm2},\\
&e_{\pm3}^2e_{\pm2}+e_{\pm2}e_{\pm3}^2=[2]e_{\pm3}e_{\pm2}e_{\pm3}.
\end{aligned}\right\} \tag{1}$$

式中 $(A_{ij})$ 为 Cartan 矩阵，$A_{ii}=2, A_{ij}=-1(j=i\pm1), A_{ij}=0(|i-j|\geqslant 2)$；$[n]=(q^n-q^{-n})/(q-q^{-1})$

q 玻色子满足下列关系式：

$$\left.\begin{aligned}
&a_ia_i^+-qa_i^+a_i=q^{-N_i}; \quad a_ia_i^+-q^{-1}a_i^+a_i=q^{N_i},\\
&[N_i,a_j^+]=\delta_{ij}a_i^+; \quad [N_i,a_j]=-\delta_{ij}a_i,\\
&[N_i,N_j]=[b_i,b_j]=[b_i^+,b_j^+]=0,\\
&[b_i,b_j^+]=0 \qquad (i\neq j).
\end{aligned}\right\} \tag{2}$$

由上 q 玻色子可得 $SU_q(4)$ 的玻色实现[6]：

$$e_i=a_i^+a_{i+1}, \quad e_{-i}=a_{i+1}^+a_i, \qquad h_i=N_i-N_{i+1} \qquad (i=1,2,3).$$

直接验证可知 $e_{\pm i}, h_i \quad (i=1,2,3)$ 满足关系式(1)

定义：

$$e_4=a_1^+a_3, \quad e_5=a_1^+a_4, \quad e_6=a_2^+a_4, \quad e_{-i}=(e_i)^+ \quad (i=4,5,6),$$

则由 $a_i^+, a_i, h_i$ 的关系式(2)可得类 Serre 关系式[3—5]

$$\left.\begin{aligned}
&e_{\pm4}e_{\mp1}^2+e_{\mp1}^2e_{\pm4}=[2]e_{\mp1}e_{\pm4}e_{\mp1}; \quad e_{\pm4}e_{\pm3}^2+e_{\pm3}^2e_{\pm4}=[2]e_{\pm3}e_{\pm4}e_{\pm3},\\
&e_{\pm5}e_{\mp1}^2+e_{\mp1}^2e_{\pm5}=[2]e_{\mp1}e_{\pm5}e_{\mp1}; \quad e_{\pm5}e_{\mp3}^2+e_{\mp3}^2e_{\pm5}=[2]e_{\mp3}e_{\pm5}e_{\mp3},\\
&e_{\pm6}e_{\pm1}^2+e_{\pm1}^2e_{\pm6}=[2]e_{\pm1}e_{\pm6}e_{\pm1}; \quad e_{\pm6}e_{\mp3}^2+e_{\mp3}^2e_{\pm6}=[2]e_{\mp3}e_{\pm6}e_{\mp3}.
\end{aligned}\right\} \tag{3}$$

以及关系式：

$$\left.\begin{aligned}
&[h_1,e_{\pm4}]=\pm e_{\pm4}; \quad [h_2,e_{\pm4}]=\pm e_{\pm4}; \quad [h_3,e_{\pm4}]=\mp e_{\pm4},\\
&[h_1,e_{\pm5}]=\pm e_{\pm5}; \quad [h_2,e_{\pm5}]=0; \quad [h_3,e_{\pm5}]=\pm e_{\pm5},\\
&[h_1,e_{\pm6}]=\mp e_{\pm6}; \quad [h_2,e_{\pm6}]=\pm e_{\pm6}; \quad [h_3,e_{\pm6}]=\pm e_{\pm6},\\
&[e_4,e_{-4}]=[h_1+h_2]; \quad [e_5,e_{-5}]=[h_1+h_2+h_3]; \quad [e_6,e_{-6}]=[h_2+h_3].
\end{aligned}\right\} \tag{4}$$

类如文献[3—5]，我们令：

$$\tau_0=h_1/2, \tau_{\pm1}=e_{\pm1}; \quad \sigma_0=h_3/2, \sigma_{\pm1}=e_{\pm3}; \quad Q=h_1+2h_2+h_3,$$

$$T_{\frac12\frac12}=e_5, T_{\frac12\frac{-1}{2}}=e_4, T_{\frac{-1}{2}\frac12}=e_6, T_{\frac{-1}{2}\frac{-1}{2}}=e_2,$$

$$V_{\frac{-1}{2}\frac{-1}{2}}=e_{-5}, V_{\frac{-1}{2}\frac12}=-e_{-4}, V_{\frac12\frac{-1}{2}}=-e_{-6}, V_{\frac12\frac12}=e_{-2},$$

则有：

$$T_{p,s}=(-1)^{1+p+s}(V_{-p,-s})^+,$$

$$
\begin{aligned}
&[Q,\tau_i]=[Q,\sigma_i]=0,[\tau_i,\sigma_j]=0,[\tau_{+1},\tau_{-1}]=[2\tau_0],[\sigma_{+1},\sigma_{-1}]=[2\sigma_0],\\
&\left.\begin{aligned}
&[\tau_0,T_{ps}]=PT_{ps},[\sigma_0,T_{ps}]=ST_{ps},[\tau_0,V_{ps}]=PV_{ps},\\
&[\sigma_0,V_{ps}]=SV_{ps},[Q,T_{ps}]=2T_{ps},[Q,V_{ps}]=-2V_{ps},\\
&\tau_1^2T_{-1/2,s}+T_{-1/2}\tau_1^2=[2]\tau_1T_{-1/2,s}\tau_1;\ \tau_1^2V_{-1/2,s}+V_{-1/2,s}\tau_1^2=[2]\tau_1V_{-1/2,s}\tau_1,\\
&\sigma_1^2T_{p-1/2}+T_{p-1/2}\sigma_1^2=[2]\sigma_1T_{p,-1/2}\sigma_1;\ \sigma_1^2V_{p,-1/2}+V_{p-1/2}\sigma_1^2=[2]\sigma_1V_{p-1/2}\sigma_1\\
&\tau_{-1}^2T_{1/2,s}+T_{1/2,s}\tau_{-1}^2=[2]\tau_{-1}T_{1/2,s}\tau_{-1};\ \tau_{-1}^2V_{1/2,s}+V_{1/2,s}\tau_{2-1}=[2]\tau_{-1}V_{1/2,s}\tau_{-1},\\
&\sigma_{-1}^2T_{p,1/2}+T_{p,1/2}\sigma_{-1}^2=[2]\sigma_{-1}T_{p,1/2}\sigma_{-1};\ \sigma_{-1}^2V_{p,1/2}+V_{p,1/2}\sigma_{-1}^2=[2]\sigma_{-1}V_{p,1/2}\sigma_{-1}
\end{aligned}\right\}
\end{aligned}
\tag{5}
$$

由关系式(5)可知：$\tau_0,\tau_{\pm1},\sigma_0,\sigma_{\pm1}$ 分别构成了 $SU_q(2)$ 量子代数.

## 三、$SU_q(4)$ 的不可约表示

$SU_q(4)$ 的不可约表示空间用 $(\lambda_1\lambda_2\lambda_3)$ 来标记[7]，令表示空间的基矢取为 $\left|\begin{smallmatrix}(\lambda)\\ \varepsilon JMjmi\end{smallmatrix}\right\rangle$，满足：

$$
\begin{aligned}
&\left\langle\begin{smallmatrix}(\lambda)\\ \varepsilon'J'M'j'm'i'\end{smallmatrix}\right|Q\left|\begin{smallmatrix}(\lambda)\\ \varepsilon JMjmi\end{smallmatrix}\right\rangle=\delta_{\varepsilon\varepsilon'}\delta_{JJ'}\delta_{MM'}\delta_{jj'}\delta_{mm'}\delta_{ii'}\varepsilon,\\
&\left\langle\begin{smallmatrix}(\lambda)\\ \varepsilon'J'M'j'm'i'\end{smallmatrix}\right|\tau_0\left|\begin{smallmatrix}(\lambda)\\ \varepsilon JMjmi\end{smallmatrix}\right\rangle=\delta_{\varepsilon\varepsilon'}\delta_{JJ'}\delta_{MM'}\delta_{jj'}\delta_{mm'}\delta_{ii'}M,\\
&\left\langle\begin{smallmatrix}(\lambda)\\ \varepsilon'J'M'j'm'i'\end{smallmatrix}\right|\sigma_0\left|\begin{smallmatrix}(\lambda)\\ \varepsilon JMjmi\end{smallmatrix}\right\rangle=\delta_{\varepsilon\varepsilon'}\delta_{JJ'}\delta_{MM'}\delta_{jj'}\delta_{mm'}\delta_{ii'}m,\\
&\left\langle\begin{smallmatrix}(\lambda)\\ \varepsilon'J'M'j'm'i'\end{smallmatrix}\right|\tau_{\pm1}\left|\begin{smallmatrix}(\lambda)\\ \varepsilon JMjmi\end{smallmatrix}\right\rangle=\delta_{\varepsilon\varepsilon'}\delta_{JJ'}\delta_{M\pm1M'}\delta_{jj'}\delta_{mm'}\delta_{ii'}\sqrt{[J\mp M][J\pm M+1]},\\
&\left\langle\begin{smallmatrix}(\lambda)\\ \varepsilon'J'M'j'm'i'\end{smallmatrix}\right|\sigma_{\pm1}\left|\begin{smallmatrix}(\lambda)\\ \varepsilon JMjmi\end{smallmatrix}\right\rangle=\delta_{\varepsilon\varepsilon'}\delta_{JJ'}\delta_{MM'}\delta_{jj'}\delta_{m\pm1m'}\delta_{ii'}\sqrt{[j\mp m][j\pm m+1]}
\end{aligned}
\tag{6}
$$

利用(5)中的 $T_{ps}$ 与 $\tau_{\pm1},\sigma_{\pm1}$ 的关系式可以得到[3—5]：

$$
\left\langle\begin{smallmatrix}(\lambda)\\ \varepsilon'J'M'j'm'i'\end{smallmatrix}\right|Tps\left|\begin{smallmatrix}(\lambda)\\ \varepsilon JMjmi\end{smallmatrix}\right\rangle=q^{A/2}C_q(JM1/2P|J'M')C_q(jm1/2S|j'm')\times
\frac{\left\langle\begin{smallmatrix}(\lambda)\\ \varepsilon+2J'j'i'\end{smallmatrix}\right\|T\left\|\begin{smallmatrix}(\lambda)\\ \varepsilon Jji\end{smallmatrix}\right\rangle\delta\varepsilon',\varepsilon_{+2}}{\sqrt{[2J'+1][2j'+1]}},
\tag{7}
$$

式中 $A=(-1)^{p+J-J'}(J+1/2)+(-1)^{s+j-j'}(j+1/2)-(M'+m)$；　$C_q(JM\,1/2\,P|J'M')$.　$C_q(jm1/2s|j'm')$ 均是 $SU_q(2)$ 的 $C$、$G$ 系数[8]，$(\lambda)\equiv(\lambda_1,\lambda_2,\lambda_3)$，$i$ 为流动指标.

由 $T_{ps}=(-1)^{1+p+s}(V_{-p-s})^+$ 可得[3—5]

$$
\left\langle\begin{smallmatrix}(\lambda)\\ \varepsilon+2J'j'i'\end{smallmatrix}\right\|T\left\|\begin{smallmatrix}(\lambda)\\ \varepsilon Jji\end{smallmatrix}\right\rangle=(-1)^{1+J'+j'-J-j}\left\langle\begin{smallmatrix}(\lambda)\\ \varepsilon Jji\end{smallmatrix}\right\|V\left\|\begin{smallmatrix}(\lambda)\\ \varepsilon+J'j'i\end{smallmatrix}\right\rangle.
\tag{8}
$$

如文献[3—5]，下面计算约化矩阵元 $\left\langle\begin{smallmatrix}(\lambda)\\ \varepsilon+2J'j'i'\end{smallmatrix}\right\|T\left\|\begin{smallmatrix}(\lambda)\\ \varepsilon Jji\end{smallmatrix}\right\rangle$ 或 $\left\langle\begin{smallmatrix}(\lambda)\\ \varepsilon Jji\end{smallmatrix}\right\|V\left\|\begin{smallmatrix}(\lambda)\\ \varepsilon+2J'j'i'\end{smallmatrix}\right\rangle$ 由于

$$\left.\begin{aligned}
&[T_{\frac{-1}{2}\frac{-1}{2}},V_{\frac{1}{2}\frac{1}{2}}]=[\frac{Q}{2}-\tau_0-\sigma_0],\\
&[T_{\frac{-1}{2}\frac{1}{2}},V_{\frac{1}{2}\frac{-1}{2}}]=-[\frac{Q}{2}-\tau_0+\sigma_0],\\
&[T_{\frac{1}{2}\frac{-1}{2}},V_{\frac{-1}{2}\frac{1}{2}}]=-[\frac{Q}{2}+\tau_0-\sigma_0],\\
&[T_{\frac{1}{2}\frac{1}{2}},V_{\frac{-1}{2}\frac{-1}{2}}]=[\frac{Q}{2}+\tau_0+\sigma_0],
\end{aligned}\right\} \tag{9}$$

因此得到

$$\left.\begin{aligned}
&\langle {}^{(\lambda)}_{\varepsilon\tilde J M\tilde j m\tilde i}|[T_{\frac{-1}{2}\frac{-1}{2}}V_{\frac{1}{2}\frac{1}{2}}]|{}^{(\lambda)}_{\varepsilon JMjmi}\rangle=\delta_{J\tilde J}\delta_{j\tilde j}\delta_{i\tilde i}[\frac{\varepsilon}{2}-M-m],\\
&\langle {}^{(\lambda)}_{\varepsilon\tilde J M\tilde j m\tilde i}|[T_{\frac{-1}{2}\frac{1}{2}}V_{\frac{1}{2}\frac{-1}{2}}]|{}^{(\lambda)}_{\varepsilon JMjmi}\rangle=-\delta_{J\tilde J}\delta_{j\tilde j}\delta_{i\tilde i}[\frac{\varepsilon}{2}-M+m],\\
&\langle {}^{(\lambda)}_{\varepsilon\tilde J M\tilde j m\tilde i}|[T_{\frac{1}{2}\frac{-1}{2}}V_{\frac{-1}{2}\frac{1}{2}}]|{}^{(\lambda)}_{\varepsilon JMjmi}\rangle=-\delta_{J\tilde J}\delta_{j\tilde j}\delta_{i\tilde i}[\frac{\varepsilon}{2}+M-m],\\
&\langle {}^{(\lambda)}_{\varepsilon\tilde J M\tilde j m\tilde i}|[T_{\frac{1}{2}\frac{1}{2}}V_{\frac{-1}{2}\frac{-1}{2}}]|{}^{(\lambda)}_{\varepsilon JMjmi}\rangle=\delta_{J\tilde J}\delta_{j\tilde j}\delta_{i\tilde i}[\frac{\varepsilon}{2}+M+m].
\end{aligned}\right\} \tag{10}$$

利用关系式(7-8),求解上式可得

$$\begin{aligned}
D_\varepsilon(J'j';Jji\tilde J\tilde j\tilde i)=&F_0\,\delta_{\tilde J J}\delta_{\tilde i i}+F_1D_{\varepsilon+2}(Jji,\tilde J\tilde j\tilde i;J+1/2j+1/2)\\
&+F_2D_{\varepsilon+2}(Jji,\tilde J\tilde j\tilde i;J+1/2j-1/2)\\
&+F_3D_{\varepsilon+2}(Jji,\tilde J\tilde j\tilde i;J-1/2j+1/2)+F_4D_{\varepsilon+2}(Jji,\tilde J\tilde j\tilde i;J-1/2j\text{-}1/2),
\end{aligned} \tag{11}$$

式中

$$D_\varepsilon(J'j';Jji,\tilde J\tilde j\tilde i)=\sum_{i'}\langle{}^{(\lambda)}_{\varepsilon-2J'j'i'}\|V\|{}^{(\lambda)}_{\varepsilon\tilde J\tilde j\tilde i}\rangle\langle{}^{(\lambda)}_{\varepsilon-2J'j'i'}\|V\|{}^{(\lambda)}_{\varepsilon Jji}\rangle,$$

$$D_{\varepsilon+2}(Jji,\tilde J\tilde j\tilde i,J'j')=\sum_{i'}\langle{}^{(\lambda)}_{\varepsilon Jji}\|V\|{}^{(\lambda)}_{\varepsilon+2J'j'i'}\rangle\langle{}^{(\lambda)}_{\varepsilon\tilde J\tilde j\tilde i}\|V\|{}^{(\lambda)}_{\varepsilon+2J'j'i'}\rangle,$$

$F_0$、$F_1$、$F_2$、$F_3$、$F_4$ 是常数,列于表 1.

**表 1**

| $J'$ | $j'$ | $J$ | $j$ | $\tilde J$ | $\tilde j$ | $F_0$ | $F_1$ | $F_2$ | $F_3$ | $F4$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $J+\frac{1}{2}$ | $j+\frac{1}{2}$ | $J$ | $j$ | $J$ | $j$ | $[2J+2][2j+2]$ $[\frac{\varepsilon}{2}-J-j]$ | $\frac{1}{[2J+1][2j+1]}$ | $\frac{[2j+2]}{[2J+1][2j+1]}$ | $\frac{[2J+2]}{[2J+1][2j+1]}$ | $\frac{[2J+2][2j+2]}{[2J+1][2j+1]}$ |
| $J+\frac{1}{2}$ | $j-\frac{1}{2}$ | $J$ | $j$ | $J$ | $j$ | $[2J+2][2j]$ $[\frac{\varepsilon}{2}-J+j+1]$ | $\frac{[2j]}{[2J+1][2j+1]}$ | $-\frac{1}{[2J+1][2j+1]}$ | $\frac{[2J+2][2j]}{[2J+1][2j+1]}$ | $-\frac{[2J+2]}{[2J+1][2j+1]}$ |
| $J-\frac{1}{2}$ | $j+\frac{1}{2}$ | $J$ | $j$ | $J$ | $j$ | $[2J][2j+2]$ $[\frac{\varepsilon}{2}+J-j+1]$ | $\frac{[2J]}{[2J+1][2j+1]}$ | $-\frac{[2J][2j+2]}{[2J+1][2j+1]}$ | $-\frac{1}{[2J+1][2j+1]}$ | $-\frac{[2j+2]}{[2J+1][2j+1]}$ |
| $J-\frac{1}{2}$ | $j-\frac{1}{2}$ | $J$ | $j$ | $J$ | $j$ | $[2J][2j]$ $[\frac{\varepsilon}{2}+J+j+2]$ | $\frac{[2J][2j]}{[2J+1][2j+1]}$ | $-\frac{[2J]}{[2J+1][2j+1]}$ | $-\frac{[2j]}{[2J+1][2j+1]}$ | $\frac{1}{[2J+1][2j+1]}$ |

续表

| $J'$ | $j'$ | $J$ | $j$ | $\tilde{J}$ | $\tilde{j}$ | $F_0$ | $F_1$ | $F_2$ | $F_3$ | $F4$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $J+\frac{1}{2}$ | $j+\frac{1}{2}$ | $J$ | $j$ | $J+1$ | $j$ | | $\frac{1}{[2j+1]}$ | $\frac{[2j+2]}{[2j+1]}$ | | |
| $J+\frac{1}{2}$ | $j-\frac{1}{2}$ | $J$ | $j$ | $J+1$ | $j$ | | $\frac{[2j]}{[2j+1]}$ | $-\frac{1}{[2J+1]}$ | | |
| $J+\frac{1}{2}$ | $j+\frac{1}{2}$ | $J$ | $j$ | $J$ | $j+1$ | | $\frac{1}{[2J+1]}$ | | $\frac{[2J+2]}{[2J+1]}$ | |
| $J-\frac{1}{2}$ | $j+\frac{1}{2}$ | $J$ | $j$ | $J$ | $j+1$ | | $\frac{[2J]}{[2J+1]}$ | | $-\frac{1}{[2J+1]}$ | |
| $J+\frac{1}{2}$ | $j-\frac{1}{2}$ | $J$ | $j$ | $J$ | $j-1$ | | | $\frac{1}{[2J+1]}$ | | $\frac{[2J+2]}{[2J+1]}$ |
| $J-\frac{1}{2}$ | $j-\frac{1}{2}$ | $J$ | $j$ | $J$ | $j-1$ | | | $\frac{[2J]}{[2J+1]}$ | | $-\frac{1}{[2J+1]}$ |
| $J-\frac{1}{2}$ | $j+\frac{1}{2}$ | $J$ | $j$ | $j-1$ | $j$ | | | | $\frac{1}{[2j+1]}$ | $\frac{[2j+2]}{[2j+1]}$ |
| $J-\frac{1}{2}$ | $j-\frac{1}{2}$ | $J$ | $j$ | $J-1$ | $j$ | | | | $\frac{[2j]}{[2j+1]}$ | $-\frac{1}{[2j+1]}$ |
| $J+\frac{1}{2}$ | $j+\frac{1}{2}$ | $J$ | $j$ | $J+1$ | $j+1$ | | 1 | | | |
| $J+\frac{1}{2}$ | $j-\frac{1}{2}$ | $J$ | $j$ | $J+1$ | $j-1$ | | | 1 | | |
| $J-\frac{1}{2}$ | $j+\frac{1}{2}$ | $J$ | $j$ | $J-1$ | $j+1$ | | | | 1 | |
| $J-\frac{1}{2}$ | $j-\frac{1}{2}$ | $J$ | $j$ | $J-1$ | $j-1$ | | | | | 1 |

对于最高权态，$\varepsilon_{\max} = \lambda_1 + \lambda_2 - \lambda_3, J_0 = (\lambda_1 - \lambda_2)/2, j_0 = \lambda_{3/2}$，此时仅有唯一的 $i$，记 $i=1$.

将 $\varepsilon_{\max}, J_0, j_0$，代入递推式(11)，并按下列方法规定 $i$，就可以求出 $SU_q(4)$ 的不可约表示$(\lambda_1, \lambda_2, \lambda_3)$.

设 $\varepsilon$ 确定时，所有基矢 $|{}^{(\lambda)}_{\varepsilon JMjmi}\rangle$ 均已确定，可以计算出所有量 $D\varepsilon(J'j';Jji,\tilde{J}\,\tilde{j}\,\tilde{i})$，为方便起见，规定 $Jji$ 的大小，假设 $J_1j_1i_1, J_2j_2i_2$ 满足 $J_1 < J_2$ 或 $J_1 = J_2, j_1 < j_2$ 或 $J_1 = J_2$, $j_1 = j_2, i_1 < i_2$，则 $J_1j_1i_1 < J_2j_2i_2$

在表示空间，基矢有如下关系：

$$\sum_{\tilde{i}} \langle {}^{(\lambda)}_{\varepsilon-2\tilde{J}\tilde{j}\tilde{i}} \| V \| {}^{(\lambda)}_{\varepsilon Jji} \rangle \, |{}^{(\lambda)}_{\varepsilon-2\tilde{J}M'\tilde{j}m'\tilde{i}}\rangle = \sum_{ps} G(J,p,\tilde{J}\,M')G(j,s,\tilde{j}\,m')V_{ps}|{}^{(\lambda)}_{\varepsilon JMjmi}\rangle, \quad (12)$$

式中 $G(JP\tilde{J}\,M'), G(js\,\tilde{j}\,m')$ 是系数，由下式给出：

$$\left.\begin{aligned}
G(JP\tilde{J}\,M') &= \begin{cases} \dfrac{F(P,J+1/2) - K(J+1/2)F(P,J-1/2)}{1 + K(J+1/2)\cdot K(J-1/2)} & (\tilde{J} = J + 1/2), \\[2ex] \dfrac{K(J-1/2)F(P,J+1/2) + F(P,J-1/2)}{1 + K(J+1/2)\cdot K(J-1/2)} & (\tilde{J} = J - 1/2), \end{cases} \\
G(js\,\tilde{j}\,m') &= \begin{cases} \dfrac{F(s,j+1/2) - K(j+1/2)F(s,j-1/2)}{1 + K(j+1/2)K(j-1/2)} & (\tilde{j} = j + 1/2), \\[2ex] \dfrac{K(j-1/2)F(sj+1/2) + F(sj-1/2)}{1 + K(j+1/2)K(j-1/2)} & (\tilde{j} = j - 1/2). \end{cases}
\end{aligned}\right\}$$

(13)

$$K(j+1/2)=q^{m'}\frac{[j+1/2]}{[2j+1]}\sqrt{\frac{[j+m'+1/2][j-m'+1/2][2j+2]}{[2j]}}(q-q^{-1}),$$

$$K(j-1/2)=q^{m'}\frac{[j+1/2]}{[2j+1]}\sqrt{\frac{[j+m'+1/2][j-m'+1/2][2j]}{[2j+2]}}(q-q^{-1}),$$

$$F(s,j')=\sqrt{[2j'+1]}C_q(jm1/2S|j'm')q^{-\frac{(-1)^{s+j-j'}(j+1/2)-m'}{2}},$$

$$K(J+1/2)=q^{M'}\frac{[J+1/2]}{[2J+1]}\sqrt{\frac{[J+M'+1/2][J-M'+1/2][2J+2]}{[2J]}}(q-q^{-1}),$$

$$K(J-1/2)=q^{M'}\frac{[J+1/2]}{[2J+1]}\sqrt{\frac{[J+M'+1/2][J-M'+1/2][2J]}{[2J+2]}}(q-q^{-1}),$$

$$F(P,J')=\sqrt{[2J'+1]}C_q(JM1/2P|J'M')q^{-\frac{(-1)^{P+J-J'}(J+1/2)-M'}{2}}.$$

设 $J_1j_1i_1$ 是满足 $D_\varepsilon(J'j',Jji,Jji)>0$ 的最小 $Jji$ 值,定义:

$$\sum_{ps}G(J_1P\tilde{J}M')G(j_1s\tilde{j}m')V_{ps}|{}^{(\lambda)}_{\varepsilon J_1M_1j_1mi_1}\rangle=\langle{}^{(\lambda)}_{\varepsilon-2J'j'1}\|V\|{}^{(\lambda)}_{\varepsilon J_1j_1i_1}\rangle|{}^{(\lambda)}_{\varepsilon-2J'M'j'm'1}\rangle,\quad(14)$$

则有:$\langle{}^{(\lambda)}_{\varepsilon-2J'j'i'}\|V\|{}^{(\lambda)}_{\varepsilon J_1j_1i_1}\rangle=0\qquad(i'\geqslant 2)$,

$$\langle{}^{(\lambda)}_{\varepsilon-2J'j'1}\|V\|{}^{(\lambda)}_{\varepsilon Jji}\rangle=\frac{D_\varepsilon(J'j';J_1j_1i_1,Jji)}{\{D_\varepsilon(J'j';J_1j_1i_1,J_1j_1i_1)\}^{1/2}}.\quad(15)$$

再设 $J_2j_2i_2$ 是满足 $D_\varepsilon(J'j';JjiJji)-\langle{}^{(\lambda)}_{\varepsilon-2J'j'1}\|V\|{}^{(\lambda)}_{\varepsilon Jji}\rangle^2>0$ 的最小 $Jji$ . 则有:

$$\langle{}^{(\lambda)}_{\varepsilon-2J'j'2}\|V\|{}^{(\lambda)}_{\varepsilon Jji}\rangle=0\qquad(Jji<J_2j_2i_2).$$

定义:

$$\sum_{ps}G(J_2PJ'M')G(j_2sj'm')V_{ps}|{}^{(\lambda)}_{\varepsilon J_2Mj_2mi_2}\rangle=\sum_{k=1}^{2}\langle{}^{(\lambda)}_{\varepsilon-2J'j'k}\|V\|{}^{(\lambda)}_{\varepsilon J_2j_2i_2}\rangle|{}^{(\lambda)}_{\varepsilon-2J'M'j'm'k}\rangle,\quad(16)$$

$$\langle{}^{(\lambda)}_{\varepsilon-2J'j'2}\|V\|{}^{(\lambda)}_{\varepsilon Jji}\rangle$$

$$=\begin{cases}0 & (Jji<J_2j_2i_2).\\ \dfrac{D_\varepsilon(J'j;J_2j_2i_2,Jji)-\langle{}^{(\lambda)}_{\varepsilon-2J'j'1}\|V\|{}^{(\lambda)}_{\varepsilon J_2j_2i_2}\rangle\langle{}^{(\lambda)}_{\varepsilon-2J'j'1}\|V\|{}^{(\lambda)}_{\varepsilon Jji}\rangle}{\{D_\varepsilon(J'j';J_2j_2i_2,J_2j_2i_2)-\langle{}^{(\lambda)}_{\varepsilon-2J'j'1}\|V\|{}^{(\lambda)}_{\varepsilon J_2j_2i_2}\rangle^2\}^{1/2}} & (Jji\geqslant J_2j_2i_2).\end{cases}\quad(17)$$

依此类推:设 $J_lj_li_l$ 是满足下式的最小 $Jji$ .

$$D_\varepsilon(J'j';Jji,Jji)-\sum_{k=1}^{l-1}\langle{}^{(\lambda)}_{\varepsilon-2J'j'k}\|V\|{}^{(\lambda)}_{\varepsilon Jji}\rangle^2>0.$$

定义:

$$\sum_{ps}G(J_lP'J'M')G(j_lsj'm')V_{ps}|{}^{(\lambda)}_{\varepsilon J_lMj_lmi_l}\rangle$$

$$=\sum_{k=1}^{l}\langle{}^{(\lambda)}_{\varepsilon-2J'j'k}\|V\|{}^{(\lambda)}_{\varepsilon J_lj_li_l}\rangle|{}^{(\lambda)}_{\varepsilon-2J'M'j'm'R}\rangle\quad(Jji<J_lj_li_l),\quad(18)$$

$$\langle{}^{(\lambda)}_{\varepsilon-2J'j'_l}\|V\|{}^{(\lambda)}_{\varepsilon Jji}\rangle=$$

$$
\begin{cases} 0 \\ \dfrac{D_\epsilon(J'j';J_lj_li_lJji) - \sum\limits_{k=1}^{l-1} \langle \begin{smallmatrix}(\lambda)\\ \epsilon-2J'j'k\end{smallmatrix} \| V \| \begin{smallmatrix}(\lambda)\\ \epsilon J_lj_li_l\end{smallmatrix}\rangle \langle \begin{smallmatrix}(\lambda)\\ \epsilon-2J'j'k\end{smallmatrix} \| V \| \begin{smallmatrix}(\lambda)\\ \epsilon Jji\end{smallmatrix}\rangle}{\{D_\epsilon(J'j';J_lj_li_lJ_lj_li_l) - \sum\limits_{k=1}^{l-1} \langle \begin{smallmatrix}(\lambda)\\ \epsilon-2J'j'k\end{smallmatrix} \| V \| \begin{smallmatrix}(\lambda)\\ \epsilon J_lj_li_l\end{smallmatrix}\rangle^2\}^{1/2}} \qquad (Jji \geqslant J_lj_li_l). \end{cases} \tag{19}
$$

因此可以求出所有 $SU_q(4)$ 的不可约表示. 在下面的表 2 列出了一些不可约表示的约化矩阵元.

以上我们讨论了 $SU_q(4)$ 量子代数在 $SU_q(4) \supset SU_q(2) \oplus SU_q(2)$ 基上的不可约表示，得到了求约化矩阵元的递推公式. 以后的工作将计算 $SU_q(4) \supset SU_q(2) \oplus SU_q(2)$ 基上的约化系数.

**表 2**

| $(\lambda)\equiv(\lambda_1\lambda_2\lambda_3)$ | $\epsilon_{max}$ | $J_0$ | $j_0$ | $\epsilon$ | $J$ | $j$ | $i$ | $\epsilon'$ | $J'$ | $j'$ | $i'$ | $\langle \begin{smallmatrix}(\lambda)\\ \epsilon Jji\end{smallmatrix} \| v \| \begin{smallmatrix}(\lambda)\\ \epsilon'J'j'i'\end{smallmatrix}\rangle$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (100) | 1 | 1/2 | 0 | −1 | 0 | 1/2 | 1 | 1 | 1/2 | 0 | 1 | $[2]$ |
| (200) | 2 | 1 | 0 | 0 | 1/2 | 1/2 | 1 | 2 | 1 | 0 | 1 | $[2]\sqrt{[3]}$ |
| (200) | 2 | 1 | 0 | −2 | 0 | 1 | 1 | 0 | 1/2 | 1/2 | 1 | $[2]\sqrt{[3]}$ |
| (110) | 2 | 0 | 0 | 0 | 1/2 | 1/2 | 1 | 2 | 0 | 0 | 1 | $[2]$ |
| (110) | 2 | 0 | 0 | −2 | 0 | 0 | 1 | 0 | 1/2 | 1/2 | 1 | $[2]$ |
| (300) | 3 | 3/2 | 0 | 1 | 1 | 1/2 | 1 | 3 | 3/2 | 0 | 1 | $\sqrt{[4]!}$ |
| (300) | 3 | 3/2 | 0 | −1 | 1/2 | 1 | 1 | 1 | 1 | 1/2 | 1 | $[3]!$ |
| (300) | 3 | 3/2 | 0 | −3 | 0 | 3/2 | 1 | −1 | 1/2 | 1 | 1 | $\sqrt{[4]!}$ |
| (210) | 3 | 1/2 | 0 | 1 | 1 | 1/2 | 1 | 3 | 1/2 | 0 | 1 | $\sqrt{[3]!}$ |
| (210) | 3 | 1/2 | 0 | 1 | 0 | 1/2 | 1 | 3 | 1/2 | 0 | 1 | $\sqrt{[3]!}$ |
| (210) | 3 | 1/2 | 0 | −1 | 1/2 | 1 | 1 | 1 | 0 | 1/2 | 1 | $[3]$ |
| (210) | 3 | 1/2 | 0 | −1 | 1/2 | 0 | 1 | 1 | 0 | 1/2 | 1 | 1 |
| (210) | 3 | 1/2 | 0 | −1 | 1/2 | 1 | 1 | 1 | 1 | 1/2 | 1 | $[3]$ |
| (210) | 3 | 1/2 | 0 | −1 | 1/2 | 0 | 1 | 1 | 1 | 1/2 | 1 | $-[3]$ |
| (210) | 3 | 1/2 | 0 | −3 | 0 | 1/2 | 1 | −1 | 1/2 | 0 | 1 | $\sqrt{[3]!}$ |
| (210) | 3 | 1/2 | 0 | −3 | 0 | 1/2 | 1 | −1 | 1/2 | 1 | 1 | $-\sqrt{[3]!}$ |
| (111) | 1 | 0 | 1/2 | −1 | 1/2 | 0 | 1 | 1 | 0 | 1/2 | 1 | $[2]$ |
| (400) | 4 | 2 | 0 | 2 | 3/2 | 1/2 | 1 | 4 | 2 | 0 | 1 | $\sqrt{[5][4][2]}$ |
| (400) | 4 | 2 | 0 | 0 | 1 | 1 | 1 | 2 | 3/2 | 1/2 | 1 | $\sqrt{[4]!\,[3]}$ |
| (400) | 4 | 2 | 0 | −2 | 1/2 | 3/2 | 1 | 0 | 1 | 1 | 1 | $\sqrt{[4]!\,[3]}$ |
| (400) | 4 | 2 | 0 | −4 | 0 | 2 | 1 | −2 | 1/2 | 3/2 | 1 | $\sqrt{[5][4][2]}$ |
| (310) | 4 | 1 | 0 | 2 | 3/2 | 1/2 | 1 | 4 | 1 | 0 | 1 | $\sqrt{[4][2]}$ |
| (310) | 4 | 1 | 0 | 2 | 1/2 | 1/2 | 1 | 4 | 1 | 0 | 1 | $\sqrt{[4][2]}$ |
| (310) | 4 | 1 | 0 | 0 | 0 | 1 | 1 | 2 | 1/2 | 1/2 | 1 | $\sqrt{[3][2]}$ |
| (310) | 4 | 1 | 0 | 0 | 1 | 0 | 1 | 2 | 1/2 | 1/2 | 1 | $\sqrt{[2]}$ |
| (310) | 4 | 1 | 0 | 0 | 1 | 1 | 1 | 2 | 1/2 | 1/2 | 1 | $\sqrt{[4][3]}$ |
| (310) | 4 | 1 | 0 | 0 | 1 | 0 | 1 | 2 | 3/2 | 1/2 | 1 | $-[4]$ |
| (310) | 4 | 1 | 0 | 0 | 1 | 1 | 1 | 2 | 3/2 | 1/2 | 1 | $\sqrt{[4]!}$ |

续表

| $(\lambda)\equiv(\lambda_1\lambda_2\lambda_3)$ | $\varepsilon_{max}$ | $J_0$ | $j_0$ | $\varepsilon$ | $J$ | $j$ | $i$ | $\varepsilon'$ | $J'$ | $j'$ | $i'$ | $\langle {(\lambda) \atop \varepsilon Jji} \Vert v \Vert {(\lambda) \atop \varepsilon' J' j' i'} \rangle$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (310) | 4 | 1 | 0 | −2 | 1/2 | 3/2 | 1 | 0 | 0 | 1 | 1 | $[4]$ |
| (310) | 4 | 1 | 0 | −2 | 1/2 | 1/2 | 1 | 0 | 0 | 1 | 1 | $\sqrt{[2]}$ |
| (310) | 4 | 1 | 0 | −2 | 1/2 | 3/2 | 1 | 0 | 1 | 1 | 1 | $\sqrt{[4]!}$ |
| (310) | 4 | 1 | 0 | −2 | 1/2 | 1/2 | 1 | 0 | 1 | 0 | 1 | $[3]\sqrt{[2]}$ |
| (310) | 4 | 1 | 0 | −2 | 1/2 | 1/2 | 1 | 0 | 1 | 1 | 1 | $-\sqrt{[4][3]}$ |
| (310) | 4 | 1 | 0 | −4 | 0 | 1 | 1 | −2 | 1/2 | 1/2 | 1 | $[2]\sqrt{[4]}$ |
| (310) | 4 | 1 | 0 | −4 | 0 | 1 | 1 | −2 | 1/2 | 3/2 | 1 | $-\sqrt{[4][2]}$ |
| (220) | 4 | 0 | 0 | 2 | 1/2 | 1/2 | 1 | 4 | 0 | 0 | 1 | $[2]\sqrt{[2]}$ |
| (220) | 4 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 1/2 | 1/2 | 1 | $[3]\sqrt{[2]}$ |
| (220) | 4 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 1/2 | 1/2 | 1 | $\sqrt{[3]!}$ |
| (220) | 4 | 0 | 0 | −2 | 1/2 | 1/2 | 1 | 0 | 0 | 0 | 1 | $\sqrt{[3]!}$ |
| (220) | 4 | 0 | 0 | −2 | 1/2 | 1/2 | 1 | 0 | 1 | 1 | 1 | $[3]\sqrt{[2]}$ |
| (220) | 4 | 0 | 0 | −4 | 0 | 0 | 1 | −2 | 1/2 | 1/2 | 1 | $[2]\sqrt{[2]}$ |
| (211) | 2 | 1/2 | 1/2 | 0 | 1 | 0 | 1 | 2 | 1/2 | 1/2 | 1 | $\sqrt{[3]!}$ |
| (211) | 2 | 1/2 | 1/2 | 0 | 0 | 1 | 1 | 2 | 1/2 | 1/2 | 1 | $\sqrt{[3]!}$ |
| (211) | 2 | 1/2 | 1/2 | 0 | 0 | 0 | 1 | 2 | 1/2 | 1/2 | 1 | $\sqrt{[4]}$ |
| (211) | 2 | 1/2 | 1/2 | −2 | 1/2 | 1/2 | 1 | 0 | 0 | 0 | 1 | $\sqrt{[4]}$ |
| (211) | 2 | 1/2 | 1/2 | −2 | 1/2 | 1/2 | 1 | 0 | 0 | 1 | 1 | $\sqrt{[3]!}$ |
| (211) | 2 | 1/2 | 1/2 | −2 | 1/2 | 1/2 | 1 | 0 | 1 | 0 | 1 | $\sqrt{[3]!}$ |

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# Irreducible Representation of Quantum Algebra $SU_q(4)$


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ABSTRACT

Irreducible representations and a recurrent formula to calculate reduced matrix elements has been obtained through a tensor-like method. Some of the reduced matrix elements are tabulated.