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六极磁铁的三级 Lie 映射

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摘要　用 Lie 代数方法分析了相对论带电粒子在六极磁铁中的运动，得到在六维相空间中粒子的三级近似轨迹.

关键词　六极磁铁　Lie 映射　三级像差

1　引言

六极透镜常用于高分辨率的分析系统、微米束系统和高能加速器的储存环等装置. 众所周知，在一级近似下六极场相当于一个漂浮空间，因此这种磁铁常常用来修正离子光学系统的二极像差. 当需要计算系统的三级像差或用八极场修正系统的三级像差时，就需要得到六极场三级像差的解析表达式. 本文用 Lie 代数方法对六极磁铁的三级像差作了分析，如果需要，还可以扩展到更高级像差.

2　六极磁场的 Hamilton 函数及其 Taylor 展开

在直角坐标系 (x, y, z) 中，纯六极场的磁矢势 A 可表示为

$$A = \frac{B_0}{a_0^2}(xy^2 - x^3/3)\boldsymbol{e}_z, \tag{1}$$

其中 B_0 为极顶处的磁感应强度，a_0 为六极磁铁的内孔半径. 取时间 t 为独立变量，$\eta = (x, y, z, P_x, P_y, P_z)$ 为六维相空间坐标，其中 P_x, P_y, P_z 为正则动量在 3 个坐标方向的分量，定义为

$$P_x = p_x - qA_x,\ P_y = p_y - qA_y,\ P_z = p_z - qA_z, \tag{2}$$

式中 p_x, p_y, p_z 分别为机械动量在 3 个坐标方向的分量，则以 t 为独立变量的 Hamilton 函数为

$$H_t = c[p_x^2 + p_y^2 + (p_z - qA_z)^2 + m_0^2c^2]^{\frac{1}{2}} = \\ c\left\{p_x^2 + p_y^2 + \left[p_z - q\frac{B_0}{a_0^2}(xy^2 - x^3/3)\right]^2 + m_0^2c^2\right\}^{\frac{1}{2}} \tag{3}$$

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其中 c 为光速,m_0 为粒子的静止质量,q 为电荷. 若取 z 为独立变量,则 $\zeta=(x,x',y,y',\tau,\tau')$为一组新的六维相空间中坐标变量,其中,

$$x' = \mathrm{d}x/\mathrm{d}z = p_x/p_z, \quad y' = \mathrm{d}y/\mathrm{d}z = p_y/p_z,$$
$$\tau = T - z/\beta_0 = ct - z/\beta_0 = z(1/\beta - 1/\beta_0); \ \tau' = P_T - P_T^0, \tag{4}$$

其中 $\beta=v/c$(v 为任意粒子的速度),$\beta_0=v_0/c$(v_0 为参考粒子的速度),$P_T=p_t/(p_0c)$,$p_t=-H_t$;P_T^0 为 P_T 在参考轨道上的值,$p_0=m_0\gamma_0 v_0$(参考粒子的动量). 式(4)所表示的正则变换下,与 $\zeta=(x,x',y,y',\tau,\tau')$相应的 Hamilton 函数为

$$H = -[(\tau' + P_T^0)^2 - x'^2 - y'^2 - m_0^2c^2/p_0^2]^{\frac{1}{2}} - k^2(xy^2 - x^3/3) - (\tau' + P_T^0)/\beta_0, \tag{5}$$

其中 $k=\left(\dfrac{qB_0}{a_0^2 p_0}\right)^{\frac{1}{2}}$ 如果将 Hamilton 函数在平衡轨道 $\zeta=(0,0,0,0,0,0)$附近作幂级数展开,并取到四级项,则有

$$H = \sum_{i=0}^{\infty} H_i, \tag{6}$$

其中

$$\begin{aligned}
&H_0 = \frac{1}{\beta_0^2\gamma_0^2},\ H_1 = 0,\quad H_2 = \frac{1}{2}(x'^2 + y'^2) + \frac{1}{2\beta_0^2\gamma_0^2}\tau'^2,\\
&H_3 = \frac{1}{3}k^2x^3 - k^2xy^2 + \frac{1}{2\beta_0}(x'^2 + y'^2)\tau' + \frac{1}{2\beta_0^3\gamma_0^2}\tau'^3,\\
&H_4 = \frac{1}{8}(x'^4 + y'^4) + \frac{1}{4}x'^2y'^2 + \frac{1}{4}\left(\frac{3}{\beta_0^2} - 1\right)(x'^2 + y'^2)\tau'^2 +\\
&\qquad \frac{1}{8}\left(\frac{5}{\beta_0^4\gamma_0^2} - \frac{1}{\beta_0^2\gamma_0^2}\right)\tau'^4\cdots\cdots.
\end{aligned} \tag{7}$$

根据文献[1],当粒子在相空间(x,x',y,y',τ,τ')中从初点 ζ 运动到终点 ζ^{f} 时满足以下关系

$$\zeta^{\mathrm{f}} = \boldsymbol{M}\zeta, \tag{8}$$

其中上标 f 表示终值,$\boldsymbol{M}$ 为一映射,表示为

$$\boldsymbol{M} = \exp\left[-:\int_0^l H\mathrm{d}z:\right] = \cdots\boldsymbol{M}_4\boldsymbol{M}_3\boldsymbol{M}_2, \tag{9}$$

积分限 l 为参考粒子沿中心轨道的运动距离,$\boldsymbol{M}_2$,$\boldsymbol{M}_3$ 和 $\boldsymbol{M}_4$ 分别为轨迹的一级映射、二级映射和三级映射,表示为

$$\boldsymbol{M}_2 = \exp(:f_2:),\ \boldsymbol{M}_3 = \exp(:f_3:),\ \boldsymbol{M}_4 = \exp(:f_4:), \tag{10}$$

这里

$$\begin{aligned}
&f_2 = -lH_2,\quad f_3 = -\int_0^l h_3^{\mathrm{int}}(\zeta,z_1)\mathrm{d}z_1,\\
&f_4 = -\int_0^l h_4^{\mathrm{int}}(\zeta,z_1)\mathrm{d}z_1 + \frac{1}{2}\int_0^l \mathrm{d}z_1\int_0^{z_1}\mathrm{d}z_2[-h_3^{\mathrm{int}}(\zeta,z_2),\ -h_3^{\mathrm{int}}(\zeta,z_1)],
\end{aligned} \tag{11}$$

而且

$$h_n^{\mathrm{int}}(\zeta,z) = H_n(\boldsymbol{M}_2(z\leftarrow 0)\zeta,z), \tag{12}$$

其中下标 n 表示 Hamilton 函数展开的幂次。

3　一级轨迹计算

将 $\boldsymbol{M}_2$ 作用于初始坐标 ζ，可得轨迹的一级近似解 ζ_1：

$$x_1 = x + lx',\quad x_1' = x',$$
$$y_1 = y + ly',\quad y_1' = y',$$
$$\tau_1 = \tau + \frac{l}{\beta_0^2\gamma_0^2}\tau',\quad \tau_1' = \tau'.$$

将上式写成矩阵形式即为

$$\begin{bmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \\ \tau_1 \\ \tau_1' \end{bmatrix} = \begin{bmatrix} 1 & l & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & l & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dfrac{l}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \\ y \\ y' \\ \tau \\ \tau' \end{bmatrix},$$

显然，在一级近似下六极磁铁相当于一个长为 l 的漂浮空间.

4　二级轨迹计算

在计算二级像差系数时需要根据式(10)计算 f_3，其中 h_3^{int} 需要从式(12)算出，即

$$h_3^{\text{int}}(\zeta, z) = H_3(\boldsymbol{M}_2(z \leftarrow 0)\zeta, z)|_{z=l},$$

得到 f_3 后，将 f_3 作用于上节中算出的 ζ_1，可得到轨迹的二级近似解，用下标 2 表示，即

$$\begin{aligned}
x_2 = :f_3:x_1 &= -\frac{1}{2}k^2l^2x^2 - \frac{1}{3}k^2l^3xx' - \frac{1}{12}k^2l^4x'^2 + \frac{1}{2}k^2l^2y^2 + \\
&\quad \frac{1}{3}k^2l^3yy' + \frac{1}{12}k^2l^4y'^2 + \frac{l}{\beta_0}x'\tau' \\
x_2' = :f_3:x_1' &= -k^2lx^2 - k^2l^2xx' - \frac{1}{3}k^2l^3x'^2 + k^2ly^2 + k^2l^2yy' + \frac{1}{3}k^2l^3y'^2, \\
y_2 = :f_3:y_1 &= k^2l^2xy + \frac{1}{3}k^2l^3xy' + \frac{1}{3}k^2l^3x'y + \frac{1}{6}k^2l^4x'y' + \frac{l}{\beta_0}y'\tau', \qquad (16) \\
y_2' = :f_3:y_1' &= 2k^2lxy + k^2l^2xy' + k^2l^3x'y + \frac{2}{3}k^2l^3x'y' \\
\tau_2 = :f_3:\tau_1 &= \frac{l}{2\beta_0}x'^2 + \frac{l}{2\beta_0}y'^2 + \frac{3l}{2\beta_0^3\gamma_0^2}\tau'^2, \\
\tau_2' = :f_3:\tau_1' &= :f_3:\tau' = 0.
\end{aligned}$$

5 三级轨迹计算

根据文献[1]，三级轨迹由两部分组成：一部分为 f_3 的贡献，另一部分为 f_4 的贡献，即

$$\zeta_3 = :f_4:\zeta_1 + \frac{1}{2}:f_3:\zeta_2 = :f_4:\zeta_1 + \frac{1}{2}:f_3:^2\zeta_1, \tag{17}$$

其中 f_3 为式(10)中的第 2 式，f_4 为式(10)中的第 3 式. 由此可得映射的三次项，用下标 3 表示，即

$$\begin{aligned}
x_3 =& \frac{1}{12}k^4l^4x^3 + \frac{1}{12}k^4l^5x^2x' + \frac{1}{36}k^4l^6xx'^2 + \frac{1}{12}k^4l^4xy^2 + \frac{5}{12}k^4l^6xy'^2 + \\
& \frac{1}{9}k^4l^6xyy' + \left(\frac{1}{252}k^4l^7 + \frac{1}{2}l\right)x'^3 - \frac{1}{4}k^2l^4x'^2\tau' - \frac{1}{60}k^4l^5x'y^2 + \\
& \left(\frac{1}{252}k^4l^7 + \frac{1}{2}l\right)x'y'^2 - \frac{1}{2\beta_0}k^2l^2x^2\tau' - \frac{2}{3\beta_0}k^2l^3xx'\tau' + \\
& \frac{1}{2\beta_0}k^2l^2y^2\tau' + \frac{2}{3\beta_0}k^2l^3yy'\tau' + \frac{1}{4\beta_0}k^2l^4y'^2\tau' + \frac{1}{2}l\left(\frac{3}{\beta_0^2} - 1\right)x'\tau'^2, \\
x'_3 =& \frac{1}{3}k^4l^3x^3 + \frac{5}{12}k^4l^4x^2x' + \frac{1}{6}k^4l^5xx'^2 + \frac{1}{3}k^4l^3xy^2 + \frac{1}{2}k^4l^4xyy' + \\
& \frac{1}{2}k^4l^5xy'^2 + \frac{1}{36}k^4l^6x'^3 - \frac{1}{12}k^4l^4x'y^2 + \frac{1}{6}k^4l^5x'yy' + \frac{1}{36}k^4l^6x'y'^2 - \\
& \frac{k^2l^2}{\beta_0}xx'\tau' - \frac{2k^2l^3}{3\beta_0}x'^2\tau' + \frac{k^2l^2}{\beta_0}yy'\tau' + \frac{2k^2l^3}{3\beta_0}y'^2\tau', \\
y_3 =& \frac{1}{12}k^4l^4x^2y - \frac{5}{12}k^4l^5x^2y' + \frac{1}{9}k^4l^6xx'y' + \frac{1}{60}k^4l^6x'^2y + \\
& \left(\frac{1}{252}k^4l^7 + \frac{l}{2}\right)x'^2y' + \frac{1}{12}k^4l^4y^3 + \frac{3}{16}k^4l^5y^2y' + \frac{1}{36}k^4l^6yy'^2 + \\
& \left(\frac{1}{252}k^4l^7 + \frac{l}{2}\right)y'^3 + \frac{k^2l^2}{\beta_0}xy\tau' + \frac{2k^2l^3}{3\beta_0}xy'\tau' + \frac{2k^2l^3}{3\beta_0}x'y\tau' + \\
& \frac{k^2l^4}{2\beta_0}x'y'\tau' + \frac{l}{2}\left(\frac{3}{\beta_0^2} - 1\right)y'\tau'^2, \\
y'_3 =& -\frac{1}{3}k^4l^3x^2y - \frac{5}{12}k^4l^4x^2y' - \frac{1}{6}k^4l^4xx'y - \frac{1}{2}k^4l^5xx'y' + \frac{1}{15}k^4l^5x'^2y - \\
& \frac{1}{18}k^4l^6x'^2y' - \frac{1}{3}k^4l^3y^3 - \frac{7}{12}k^4l^4y^2y' - \frac{1}{3}k^4l^5yy'^2 - \\
& \frac{1}{18}k^4l^6y'^3 + \frac{k^2l^2}{\beta_0}xy'\tau' + \frac{k^2l^2}{\beta_0}x'y\tau' + \frac{2k^3l^3}{3\beta_0}x'y'\tau', \\
\tau_3 =& -\frac{1}{2\beta_0}k^2l^2x^2x' - \frac{1}{3\beta_0}k^2l^3xx'^2 + \frac{1}{\beta_0}k^2l^2xyy' + \frac{1}{3\beta_0}k^2l^3xy'^2 - \\
& \frac{1}{12\beta_0}k^2l^4x'^3 + \frac{1}{2\beta_0}k^2l^2x'y^2 + \frac{2}{3\beta_0}k^2l^3x'yy' + \frac{1}{4\beta_0}k^2l^4x'y'^2 + \\
& \frac{1}{2}l\left(\frac{3}{\beta_0^2} - 1\right)x'^2\tau' + \frac{1}{2}l\left(\frac{3}{\beta_0^2} - 1\right)y'^2\tau' + \frac{1}{2}l\left(\frac{5}{\beta_0^4\gamma_0^2} - \frac{1}{\beta_0^2\gamma_0^2}\right)\tau'^3
\end{aligned} \tag{18}$$

$$\tau_3' = 0.$$

6 讨论

本文用Lie代数方法导出了六维相空间($x, x', y, y', \tau, \tau'$)中相对论粒子在六极透镜中的三级轨迹. 其横向分量(x, x', y, y')与通常束流光学中所采用的横向分量的意义相同. 但需要对纵向分量(τ, τ')的意义作一些说明. 根据式(4)中对 τ 和 τ' 的定义, 将 τ 除以 c 得

$$\tau / c = z(1/v - 1/v_0) = \Delta t, \tag{19}$$

这就是任意粒子与参考粒子之间的时间差. 同样, 将 τ' 乘以 $p_0 c$ 得

$$\tau' p_0 c = p_t - p_t^0 = -\Delta E, \tag{20}$$

ΔE 就是任意粒子与参考粒子之间的能量差. 因此 τ 和 τ' 本质上反映了任意粒子与参考粒子之间在时间和能量上的差别.

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Third Order Lie Map for the Sextupole Magnets

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Abstract Sextupole magnets are usually used in beam analyzing systems with high resolving power, micro-beam systems and storage rings of high energy accelerators. It is well known that sextupoles are equivalent to the drift spaces under first order approximation. Therefore, this kind of optical elements is often used to correct the second order aberrations of beam optics systems. When it is necessary to calculate the third order aberrations of a system, or to correct the third order aberrations with octupole magnets, one should know the analytical expressions of the third order terms of sextupoles.

In this paper, Lie algebraic methods were used in the analysis of relativistic particle trajectories in the sextupole magnets, and the solutions of third order approximation in the six dimensional phase spaces were obtained.

Key words sextupole, Lie map, third order approximation

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