Analytic Solution of Ground State for Coulomb Plus Linear Potential *

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Abstract The newly developed single trajectory quadrature method is applied to solve the ground state quantum wave function for Coulomb plus linear potential. The general analytic expressions of the energy and wave function for the ground state are given. The convergence of the solution is also discussed. The method is applied to the ground state of the heavy quarkonium system.

Key words single trajectory quadrature, ground state wave function, Coulomb plus linear potential

Recently a new method has been developed by R. Friedberg, T. D. Lee and W. Q. Zhao[1,2] to solve the N-dimensional low-lying quantum wave functions of Schrödinger equation using quadratures along a single trajectory. Based on the expansion on $1/g$, where g is a scale factor expressing the strength of the potential, Schrödinger equation can be cast into a series of first order partial differential equations, which is further reduced to a series of integrable first order ordinary differential equations by single-trajectory quadratures. New perturbation series expansion is also derived based on this method, both for one-dimensional and N-dimensional cases. Some examples for one-dimensional problems have been illustrated in Refs.[1,2].

Coulomb plus linear potential has been widely applied to describe the heavy quarkonium state. However, it is difficult to obtain an analytic expression of the energy and wave function. Here the single trajectory quadrature method is applied to solve the ground state for Coulomb plus linear potential. The general analytic expressions of the energy and wave function for the ground state are given. The convergence of the solution is also discussed. The result is applied to describe the ground state of heavy quarkonia. Some discussions of the limitation of its applicability are given at the end.

Let us consider a unit mass particle moving in a central potential. The Schrödinger equation in the 3-dimensional space is expressed as

$$\left[-\frac{1}{2}\nabla^2 + V(r)\right]\Psi(r) = E\Psi(r).$$

Based on the single trajectory quadrature method[2], following steps should be taken to solve the problem for the ground state.

1. For the potential $V(r)$, the scale factor g is introduced as

$$V(r) = g^k v(r).$$

First, the energy is expanded in terms of $1/g$:

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$$E = g^l E_0 + g^{l-i} E_1 + g^{l-2i} E_2 + \cdots . \tag{3}$$

The highest g-power, l, in the expansion could be fixed by the dimensional consideration in the following way: Assume the behavior of the potential $v(r)$ approaches r^n when $r \to 0$ and E_0 is dimensionless. The dimension of ∇^2 in the first term of the left hand side in Eq.(1) is the same as $[r^{-2}]$. The dimension of each term in Eq.(1), namely $[r^{-2}]$ for ∇^2, $[g^k r^n]$ for $V(r)$ and $[g^l]$ for E should be the same. This then gives

$$l = \frac{2k}{n+2}. \tag{4}$$

The ground state wave function $\Psi(r)$ is expressed as

$$\Psi(r) = e^{-S(r)}. \tag{5}$$

Substituting Eq.(5) into Eq.(1) a equation for $S(r)$ and E is obtained as

$$\frac{1}{2}\nabla^2 S(r) - \frac{1}{2}(\nabla S(r))^2 + V(r) - E = 0. \tag{6}$$

Then $S(r)$ is also expanded in terms of $1/g$:

$$S(r) = g^m S_0(r) + g^{m-j} S_1(r) + g^{m-2j} S_2(r) + \cdots . \tag{7}$$

Substitute Eqs.(3) and (7) into Eq.(6). By equating the coefficients of each g^{-n}, a series of first order differential equations could be obtained. Now the highest g-power, m, in the S-expansion should be determined. The highest power of g^{2m} comes from the second term $-\frac{1}{2}(\nabla S)^2$ of the left hand side of Eq.(6), which gives $-\frac{1}{2}(\nabla S_0(r))^2$ to the first one of the series of equations. The highest g-powers in $V(r)$ and E are g^k and g^l respectively. When $k > l$, $v(r)$ should enter the first equation, as in the case of Harmonic Oscillator potential[1,2], which requires $2m = k$; in the second equation for g^{2m-j} power, $\nabla S_0 \cdot \nabla S_1$ should be related to E_0, which gives $2m - j = l$. On the other hand, when $k < l$, the first equation should be related to E_0 and this gives $2m = l$; the second equation with the power g^{2m-j} is then related to $v(r)$, which gives $2m - j = k$. From the above discussion we derive the following condition:

$$2m = k,\quad 2m - j = l,\quad \text{for } l < k, 2m = l,\quad 2m - j = k,\quad \text{for } l > k. \tag{8}$$

To ensure that $\{E_i\}$ and $\{S_i\}$ enter the equations successively, we have $i = j$ in Eqs.(3) and (7).

For the Coulomb plus linear potential

$$V(r) = g^2\left(-\frac{1}{r} + \mu r\right), \tag{9}$$

$k = 2$ and

$$v(r) = -\frac{1}{r} + \mu r, \tag{10}$$

which gives $n = -1$. From Eqs.(4) and (8) this gives $l = 4 > k$, which leads to $m = 2$ and $i = j = 2$. Therefore, for this potential Eqs.(3) and (7) are expressed as

$$\begin{aligned} E &= g^4 E_0 + g^2 E_1 + E_2 + g^{-2} E_3 + \cdots + g^{-(2n-4)} E_n + \cdots, \\ S &= g^2 S_0 + S_1 + g^{-2} S_2 + g^{-4} S_3 + \cdots + g^{-(2n-2)} S_n + \cdots . \end{aligned} \tag{11}$$

Substituting Eq.(11) into Eq.(6), equating the coefficients of each g^n term, the following series of equations are obtained:

$$(\nabla S_0)^2 = -2E_0, \tag{12}$$

$$\nabla S_0 \cdot \nabla S_1 = \frac{1}{2}\nabla^2 S_0 + v(r) - E_1, \tag{13}$$

$$\nabla S_0 \cdot \nabla S_2 = \frac{1}{2}\nabla^2 S_1 - \frac{1}{2}(\nabla S_1)^2 - E_2, \tag{14}$$

$$\vdots$$

$$\nabla S_0 \cdot \nabla S_n = \frac{1}{2}\nabla^2 S_{n-1} - \frac{1}{2}\sum_{m=1}^{n-1}\nabla S_m \cdot \nabla S_{n-m} - E_n , \tag{15}$$

$$\vdots$$

2. Following Ref.[2] the series of Eqs.(12—15) could be solved easily. Considering $\nabla S_0 = \frac{dS_0}{dr}$ Eq.(12) gives

$$S_0(r) = \sqrt{-2E_0}\, r . \tag{16}$$

Substituting Eq.(16) into Eq.(13) for E_1 and $S_1(r)$, considering $\nabla^2 r = \frac{2}{r}$, we have

$$\sqrt{-2E_0}\frac{dS_1}{dr} = \frac{1}{2}\left(\sqrt{-2E_0}\frac{2}{r}\right) + \left(-\frac{1}{r} + \mu r\right) - E_1 . \tag{17}$$

To keep $S_1(r)$ regular at $r = 0$ we have

$$E_0 = -\frac{1}{2} \text{ and } S_0(r) = r . \tag{18}$$

This gives

$$\nabla S_0 \cdot \nabla S_1 = \frac{dS_1}{dr} = \mu r - E_1 , S_1(r) = \frac{1}{2}\mu r^2 - E_1 r . \tag{19}$$

Substituting Eq.(19) into Eq.(14) for $S_2(r)$ and E_2, considering $\nabla^2 r^2 = 6$, we have

$$\frac{dS_2}{dr} = \frac{1}{2}\left(\frac{1}{2}\mu \cdot 6 - \frac{2E_1}{r}\right) - \frac{1}{2}(\mu r - E_1)^2 - E_2 . \tag{20}$$

In order that $S_2(r)$ be regular at $r = 0$, we have

$$E_1 = 0,\ S_1(r) = \frac{1}{2}\mu r^2 . \tag{21}$$

This gives

$$\frac{dS_2}{dr} = -\frac{1}{2}\mu^2 r^2 + \frac{3}{2}\mu - E_2 ,\quad S_2(r) = -\frac{1}{6}\mu^2 r^3 + \frac{3}{2}\mu r - E_2 r . \tag{22}$$

Following similar procedure for $S_3(r)$ and E_3, introducing $\nabla^2 r^3 = 12r$, we have

$$\frac{dS_3}{dr} = \frac{1}{2}\left(-2\mu^2 r + \frac{3\mu}{r} - \frac{2E_2}{r}\right) - \mu r\left(-\frac{1}{2}\mu^2 r^2 + \frac{3\mu}{2} - E_2\right) - \tag{23}$$

In order to have $S_3(r)$ also regular

$$E_2 = \frac{3}{2}\mu ,\quad S_2(r) = -\frac{1}{6}\mu^2 r^3 \tag{24}$$

Then Eq.(23) becomes

$$\frac{dS_3}{dr} = -\mu^2 r + \frac{1}{2}\mu^2 r^3 - E_3 ,\quad S_3(r) = -\frac{1}{2}\mu^2 r^2 + \frac{1}{8}\mu^2 r^4 - \tag{25}$$

Continuing similar argument we could reach

$$\begin{aligned} E_3 &= 0, & S_3(r) &= -\frac{1}{2}\mu^2 + \frac{1}{8}\mu^3 r^4 \\ E_4 &= \frac{3}{2}\mu^2 , & S_4(r) &= -\frac{3}{4}\mu^3 r^3 - \frac{3}{20}\mu^4 r^5 \\ & & &\vdots \end{aligned} \tag{26}$$

Now we introduce the general expression

$$\frac{dS_n}{dr} = \sum_{0 \le l < \frac{n}{2}} a_l^{(n)} r^{n-2l}\mu^{n-l} , \tag{27}$$

then we have

$$\sum_{0\leqslant l<\frac{n}{2}} \frac{1}{(n+1-2l)} \alpha_l^{(n)} r^{n+1-2l} \mu^{n-l}$$

$$\nabla^2 S_n = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r} r^2 \frac{\mathrm{d}S_n}{\mathrm{d}r} = \sum_{0\leqslant l<\frac{n}{2}} \alpha_l^{(n)}(n+2-2l) r^{n-1-2l} \mu^{n-l} \tag{28}$$

Substituting Eqs.(27) and (28) into Eq.(15) and comparing the coefficients of the same power of r we obtain

$$\alpha_l^{(n)} = \frac{1}{2}\alpha_{l-1}^{(n-1)}(n+3-2l) - \frac{1}{2}\sum_{m=1}^{n-1}\sum_{i\geqslant 0, i>l-\frac{n-m}{2}}^{i\leqslant l, i<\frac{m}{2}} \alpha_i^{(m)}\alpha_{l-i}^{(n-m)}. \tag{29}$$

To keep S_{2n} or S_{2n+1} regular at $r=0$ we have

$$E_{2n-1} = 0,\ E_{2n} = \frac{3}{2}\alpha_{n-1}^{(2n-1)}\mu^n. \tag{30}$$

From $\alpha_0^{(1)} = 1$ all the $\alpha_l^{(n)}$ and E_n for $n>0$ could be derived based on Eqs.(29) and (30). Combining with $E_0 = -\frac{1}{2}$ and $S_0 = r$ we obtain

$$E = g^4\left(-\frac{1}{2} + \frac{3}{2}\frac{\mu}{g^4} - \frac{3}{2}\left(\frac{\mu}{g^4}\right)^2 + \cdots\right), \tag{31}$$

$$\Psi(r) = \exp\Big[-g^2 r - \frac{1}{2}\mu r^2 + \frac{1}{6g^2}\mu^2 r^3 + \frac{1}{2g^4}\mu^2 r^2 - \frac{1}{8g^4}\mu^3 r^4 - \frac{3}{4g^6}\mu^3 r^3 + \frac{1}{8g^6}\mu^4 r^5 + \cdots\Big]. \tag{32}$$

The same procedure can be performed if we define

$$\varepsilon = g^2\mu \tag{33}$$

and solve Eq.(1) for the potential

$$V(r) = -\frac{g^2}{r} + \varepsilon r. \tag{34}$$

The derivation has been given in Ref.[2] and the result is exactly the same as Eqs.(31) and (32). Considering Eq.(33) we have

$$E = g^4\left(-\frac{1}{2} + \frac{3}{2}\frac{\varepsilon}{g^6} - \frac{3}{2}\left(\frac{\varepsilon}{g^6}\right)^2 + \cdots\right), \tag{35}$$

$$\Psi(r) = \exp\Big[-g^2 r - \frac{1}{2g^2}\varepsilon r^2 + \frac{1}{6g^6}\varepsilon^2 r^3 + \frac{1}{2g^8}\varepsilon^2 r^2 - \frac{1}{8g^{10}}\varepsilon^3 r^4 - \frac{3}{4g^{12}}\varepsilon^3 r^3 + \frac{1}{8g^{14}}\varepsilon^4 r^5 + \cdots\Big]. \tag{36}$$

Introducing a parameter $\lambda = \varepsilon/g^6 = \mu/g^4$ the energy could be expressed as

$$E = g^4\left(-\frac{1}{2} + \sum_{n\geqslant 1}\frac{3}{2}\alpha_{n-1}^{(2n-1)}\lambda^n\right). \tag{37}$$

Introducing $e_n = \frac{3}{2}\alpha_{n-1}^{(2n-1)}$ the energy could be expressed as

$$E = g^4\left(-\frac{1}{2} + \sum_{n\geqslant 1} e_n\lambda^n\right). \tag{38}$$

This method can easily give the energy expansion series up to any order of n. It gives the possibility to analyze the convergence of the series in details. The convergence of the expansion series of the energy E depends on the parameter λ. In fact, this series is an asymptotic one. For certain value of

λ we could only reach a certain accuracy of the energy In Table 1 the ratio of $R_n = |e_n/e_{n-1}|$ for different n is listed. From Eq.(37) we know that the ratio of the successive terms in the energy expansion series is $\lambda|e_n/e_{n-1}| = \lambda R_n$. It can be seen that the series would be meaningful only when $\lambda < \frac{1}{R_n}$. This gives the limitation of the applicability of this method. It also tells us how accurate the final result could reach for a fixed value of λ. In table 2 for each order n, the λ corresponding to $\lambda R_n \approx 1$ is listed. The obtained energy E/g^4 for this special value of λ at each order of n is also given, together with the reached accuracy $e_n\lambda^n$. For example, when $n = 11$ the corresponding $\lambda = 0.052$ gives $\lambda R_{11} \approx 1$. It means that the correction term increases when $n > 11$ and increases further. This would finally give a divergent result. Up to $n = 11$ the obtained $e_n\lambda^n = 0.00004$ which gives the accuracy of the obtained energy $E/g^4 \approx -0.43443$ for. $\lambda = 0.052$.

Table 1. $R_n = |e_n/e_{n-1}|$

n	3	4	5	6	7	8	9	10	11	12
R_n	4.5	7.36	9.67	11.62	13.35	14.95	16.45	17.90	19.32	20.72
$1/R_n$	0.22	0.14	0.10	0.086	0.075	0.067	0.061	0.056	0.052	0.048

Table 2. The obtained energy and its accuracy for different λ

n	3	4	5	6	7	8	9	10	11	12
λ	0.22	0.14	0.10	0.086	0.075	0.067	0.061	0.056	0.052	0.048
E/g^4	0.20	0.31	0.361	0.379	0.394	0.4048	0.4130	0.4198	0.42535	0.43089
$\|e_n\|\lambda^n$	0.07	0.02	0.006	0.002	0.001	0.0004	0.0002	0.0001	0.00004	0.00002

Now consider a pair of heavy quark and antiquark with equal mass m and color charge q and $-q$, moving in a Coulomb plus linear potential. In the non-relativistic approximation the wave function $\psi(r)$ to describe the relative motion of the quark pair satisfies the following Schrödinger equation:

$$\left[-\frac{1}{2\cdot m/2}\nabla^2 - \frac{q^2}{r} + \kappa r\right]\psi(r) = \mathscr{E}\psi(r), \tag{39}$$

where κ is the strength of the linear potential. After a simple transformation, Eq.(39) becomes Eq.(1) and with

$$\frac{1}{2}mq^2 = g^2, \quad \frac{1}{2}m\mathscr{E} = E, \quad \frac{1}{2g^2}m\kappa = \mu \text{ or } \frac{1}{2}m\kappa = \varepsilon, \quad \lambda \equiv \frac{4\kappa}{m^2q^6} = \frac{\varepsilon}{g^6} = \frac{\mu}{g^4}. \tag{40}$$

Substituting Eq.(40) into Eq.(37) the ground state energy $\mathscr{E}$ of the two quark system can be expressed as

$$\mathscr{E} = \frac{1}{2}mq^4\left(-\frac{1}{2} + \sum_{n\geq 1}\frac{3}{2}\alpha_{n-1}^{(2n-1)}\left(\frac{4\kappa}{m^2q^6}\right)^n\right) = \frac{1}{2}mq^4\left(-\frac{1}{2} + \frac{3}{2}\lambda - \frac{3}{2}\lambda^2 + \frac{27}{4}\lambda^3\cdots\right). \tag{41}$$

For the quarkonium system the color charge q^2 can be related to the strong coupling constant α_s as[3]

$$q^2 = \frac{4}{3}\alpha_s.$$

For example, taking $q^2 = 0.5$ and $\kappa = 1\text{GeV/fm}$ we have

$$\lambda = \frac{4\kappa}{m^2q^6} \approx 6.4\text{GeV}^2 \times \frac{1}{m^2}.$$

Taking the mass of the charm quark $m_c = 1.6\text{GeV}$, we have

$$\lambda_c = \frac{4\kappa}{m_c^2 q^6} \approx 2.5, \tag{44}$$

which is too large for the application of this method. If we look at the expansion series, for the ground state of J/ψ we have

$$m_{J/\psi} = 1.6\text{GeV} \times 2 + \mathscr{E}_c \approx 3.2\text{GeV} + 0.2\text{GeV}(-0.5 + 3.7 - 9.4 + 105 + \cdots). \tag{45}$$

Obviously, this expansion has a very bad behaviour. Therefore this method could not be applied to charmnium since the charm quark mass is not heavy enough. For bottom and top quark, taking $m_b \approx 4.5$GeV and $m_t \approx 170$GeV, we could obtain

$$\lambda_b = \frac{4\kappa}{m_b^2 q^6} \approx 0.32 \quad \text{and} \quad \lambda_t \quad \frac{4\kappa}{m_t^2 q^6} \approx 2.2 \times 10^{-4} \tag{46}$$

For the ground state of the bottom and top quarkonium we have

$$\begin{aligned} m_{b\bar{b}} &= 4.5\text{GeV} \times 2 + \mathscr{E}_b \approx \\ &9.0\text{GeV} + 0.56\text{GeV}(-0.5 + 0.48 - 0.15 + 0.22 + \cdots), \end{aligned} \tag{47}$$

$$\begin{aligned} m_{t\bar{t}} &= 170\text{GeV} \times 2 + \mathscr{E}_b \approx \\ &340\text{GeV} + 21.3\text{GeV}(-0.5 + 3.3 \times 10^{-4} - 7.3 \times 10^{-8} + \cdots). \end{aligned} \tag{48}$$

From Eqs.(47) and (48) it can be seen that in the case of bottom quarkonium $\lambda_b \approx 0.32$, the accuracy of the derived energy has been improved and this method could well be applied to the top quarkonium.

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库仑加线性位基态解析解*

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摘要 应用新发展的单一轨迹积分方法求解库仑加线性位的基态量子波函数,得到基态能量和波函数的一般解析表达式,并讨论了解的收敛性.应用此方法讨论了重夸克偶素系统.

关键词 单一轨迹积分 基态波函数 库仑加线性位

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