第 27 卷 第 1 期
2003 年 1 月
高能物理与核物理
HIGH ENERGY PHYSICS AND NUCLEAR PHYSICS
Vol.27, No.1
Jan., 2003

Lie Algebraic Analysis for the Nonlinear Transport of Intense Pulsed Beams in Electrostatic Lenses*

LÜ Jian-Qin LI Chao-Long LI Jin-Hai
(Institute of Heavy Ion Physics, Peking University, Beijing 100871, China)

Abstract The nonlinear transport of intense pulsed beams is analyzed with the Lie algebraic method in the case of K-V distribution, and the particle trajectories of second order approximation in the 6-D phase space ($x, x', y, y', \tau, p_\tau$) are obtained. The beams could be axial-symmetrical or non-axial-symmetrical in the transverse direction. In the analysis, the effective fields of lens are divided into several small intervals. Each interval is treated as a uniform accelerating field and each dividing point is considered as a thin lens. Let the Lie map act on each uniform accelerating field and thin lens, the nonlinear particle trajectories can be obtained.

Key words electrostatic lenses, intense pulsed beams, Lie map, Nonlinearity

1 Introduction

It is a very complicated problem to calculate the nonlinear transport of intense beams, because the particle trajectories depend on the electric potentials excited by the particle beams, and the electric potentials of the beams depend on the particle trajectories and the particle distributions in the phase spaces. So, it is necessary to solve the problem by iteration to get self-consistent solutions. This paper presents the analytical calculations with Lie map for the nonlinear transport of intense beams in 6-D phase space in the case of K-V type distribution.

2 Hamiltonian and Lie map

According to the principle of classical mechanics[1], when a charge particle moves in the electromagnetic fields from position z to z_f, the Hamiltonian with time t as an independent variable is

$$H_t = (m_0^2c^4 + p_x^2c^2 + p_y^2c^2 + p_z^2c^2)^{\frac{1}{2}} + q\psi, \quad (1)$$

where m_0 is the rest mass of the particle, q the charge, p_x, p_y and p_z the particle momenta in the x, y and z directions, respectively, ψ the electric potential, which consists of two parts: the external potential ψ_e and the potential excited by the beam itself ψ_s. ψ_e and ψ_s are expressed as the following:

$$\psi_e = \phi(z) - \frac{1}{2}\phi''(z)(x^2+y^2) + \frac{1}{64}\phi^{(4)}(z)(x^4+x^2y^2+y^4) - \frac{1}{2304}\phi^{(6)}(z)\times (x^6+3x^4y^2+3x^2y^4+y^6)+\cdots, \quad (2)$$

$$\psi_s = -\frac{3IT_{rf}}{8\pi\varepsilon_0\gamma_0 XYZ}(\mu_x x^2+\mu_y y^2+\mu_z z_r^2), \quad (3)$$

where I is the average beam current, T_{rf} the period of the beam pulses, X, Y and Z the beam dimensions, z_r the longitudinal position of the arbitrary particle relative to the reference particle, μ_x, μ_y and μ_z the shape factors of the beam as the following:

$$\mu_x = \frac{XYZ\gamma}{2}\int_0^\infty \frac{d\xi}{(X^2+\xi)\sqrt{(X^2+\xi)(Y^2+\xi)(Z^2\gamma^2+\xi)}},$$
$$\mu_y = \frac{XYZ\gamma}{2}\int_0^\infty \frac{d\xi}{(Y^2+\xi)\sqrt{(X^2+\xi)(Y^2+\xi)(Z^2\gamma^2+\xi)}},$$
$$\mu_z = \frac{XYZ\gamma}{2}\int_0^\infty \frac{d\xi}{(Z^2\gamma^2+\xi)\sqrt{(X^2+\xi)(Y^2+\xi)(Z^2\gamma^2+\xi)}}. \quad (4)$$

Define the function p_t as

$$p_t = -H_t(x, p_x, y, p_y, z, p_z).$$

Received 17 April 2002
* Supported by National Natural Science Foundation of China (10075005)

Solve p_z from Eq. (5), one obtains

$$p_z = -K(t, x, p_x, y, p_y, z, p_z), \quad (6)$$

where

$$K = -p_z = -[(p_t + q\phi)^2/c^2 - p_x^2 - p_y^2 - m_0^2c^2]^{\frac{1}{2}}. \quad (7)$$

Equation (7) is the Hamiltonian with (x, p_x, y, p_y, t, p_t) the canonical variables and z an independent variable. We can define another set of canonical variables $\zeta = (x, p_x, y, p_y, \tau, p_\tau)$ as

$$x = x, p_x = p_x, y = y, p_y = p_y, \quad \tau = t - z/v_0, p_\tau = p_t - p_t^0, \quad (8)$$

where v_0 is the velocity of reference particle, p_t^0 the value of p_t for reference particle. We can see from Eq. (8) that in the phase space ζ, the coordinates of reference particle are always zero. According to Eq. (5), we have

$$p_t^0 = -H_t|_{\text{reference orbit}} = -(m_0^2c^4 + p_0^2c^2)^{\frac{1}{2}} - q\phi = -m_0\gamma_0c^2 - q\phi, \quad (9)$$

where p_0 is the momentum of reference particle, and $\gamma_0 = [1-(v_0/c)^2]^{-1/2}$.

Corresponding to the phase space ζ, the new Hamiltonian is

$$H = -[(p_\tau + q(\psi_\tau - \psi_s) - m_0\gamma_0c^2)^2/c^2 - p_x^2 - p_y^2 - (m_0c)^2]^{\frac{1}{2}} - (p_\tau - m_0\lambda_0c^2)/v_0. \quad (10)$$

Now, expanding the Hamiltonian H into Taylor series, one obtains

$$H = \sum_{n=0}^{\infty} H_n,$$

where

$$H_0 = p_0\left(\beta_0^{-2}\gamma_0^{-2} - \frac{1}{2}\right),$$

$$H_1 = 0,$$

$$H_2 = -\frac{1}{2\beta_0 c}\left(\frac{q\phi''}{2} + 2q\mu_x\right)(x^2 + y^2) + \frac{(p_x^2 + p_y^2)}{2m_0\lambda_0\beta_0 c} - Q\mu_z v_0\tau^2 + \frac{1}{2m_0\beta_0^3\gamma_0^3c^3}p_\tau^2,$$

$$H_3 = -\frac{1}{2m_0\gamma_0^3\beta_0^2c^3}\left(\frac{q\phi''}{2} + 2Q\mu_x\right)x^2p_\tau + \frac{1}{2m_0^2\gamma_0^2\beta_0^3c^3}p_x^2p_\tau - \frac{1}{2m_0\gamma_0^3\beta_0^2c^3}\left(\frac{q\phi''}{2} + 2Q\mu_y\right)y^2p_\tau + \frac{1}{2m_0^2\gamma_0^2\beta_0^3c^3}p_y^2p_\tau + \frac{Q\mu_z v_0^2}{m_0\gamma_s\beta_0c^2}\tau^2p_\tau - \frac{1}{m_0^2\gamma_0^4\beta_0^5c^5}p_\tau^3,$$

$$\cdots$$

and

$$Q = \frac{3qIT_{rf}}{8\pi\varepsilon_0\gamma_0 XYZ}.$$

3 Lie map

The Lie map $\boldsymbol{M}$ associate with H is

$$\boldsymbol{M} = \exp\left[-:\int_{z_0}^{z_f} H\mathrm{d}z:\right] = \cdots \boldsymbol{M}_4\boldsymbol{M}_3\boldsymbol{M}_2, \quad (14)$$

where

$$\boldsymbol{M}_2 = \exp(:f_2:), \boldsymbol{M}_3 = \exp(:f_3:), \boldsymbol{M}_4 = \exp(:f_4:), \cdots,$$

and

$$f_2 = -\int_{z_0}^{z_f} H_2\mathrm{d}z, f_3 = -\int_{z_0}^{z_f} h_3^{\text{int}}\mathrm{d}z, \cdots, \quad (16)$$

here

$$h_n^{\text{int}}(z) = \boldsymbol{M}_2 H_n. \quad (17)$$

Let the map $\boldsymbol{M}$ acts on the initial coordinate ζ, and the subscript "i" of ζ express the order of the approximation, we have

$$\zeta_1 \quad \exp(:f_2:)\zeta \quad \text{(first order)}, \qquad \zeta_2 = :f_3:\zeta_1 \quad \text{(second order)}\cdots. \quad (18)$$

4 First order

Similar to the analysis described in the Ref. [2], we divided the whole lens into some small segments. Each segment is treated as a uniform accelerating field, and each dividing point is considered as a thin lens. The following is the analysis for each segment and each point.

A. In the interval (z_{i-1}, z_i)

In the interval (z_{i-1}, z_i), $i = 1, 2, \ldots, n$, let $L_i = z_i - z_{i-1}$, the subscript "$i-1$" stands for the initial value of the ith interval, the subscript "i" for the final value of it, and the subscript "0" for the values related with the reference particle. In each interval, the electric field is treated as an uniform one, which is $\phi'' \approx 0$. According to the first expressions of Eq. (15) and Eq. (16), one obtains

$$\begin{bmatrix} x_1 \\ p_x \\ y_1 \\ p_y \\ \tau_1 \\ p_\tau \end{bmatrix} = \begin{bmatrix} \cosh(k_x\ell_{ie}) & \frac{\sinh(k_x\ell_{ie})}{p_{0i-1}k_x} & 0 & 0 & 0 & 0 \\ p_{0i-1}k_x\sinh(k_x\ell_{ie}) & \cosh(k_x\ell_{ie}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(k_y\ell_{ie}) & \frac{\sinh(k_y\ell_{ie})}{p_{0i-1}k_y} & 0 & 0 \\ 0 & 0 & p_{0i-1}k_y\sinh(k_y\ell_{ie}) & \cosh(k_y\ell_{ie}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cosh(k_\tau\ell_{ie}) & \frac{\sinh(k_\tau\ell_{ie})}{m_0\beta_{0i-1}^3\bar{\gamma}_{0i}^3c^3\eta_ik_\tau} \\ 0 & 0 & 0 & 0 & m_0\beta_{0i-1}^3\bar{\gamma}_{0i}^3c^3\eta_ik_\tau\sinh(k_\tau\ell_{ie}) & \cosh(k_\tau\ell_{ie}) \end{bmatrix} \begin{bmatrix} x \\ p_x \\ y \\ p_y \\ \tau \\ p_\tau \end{bmatrix}, \tag{19}$$

where $k_x^2 = \frac{2Q\mu_x}{p_{0i-1}v_{0i-1}}$, $k_y^2 = \frac{2Q\mu_y}{p_{0i-1}v_{0i-1}}$, $k_\tau^2 = \frac{2Q\mu_z(\eta_i^2+\eta_i+1)}{3m_0\beta_{0i-1}^2\bar{\gamma}_{0i}^2c^2\eta_i}$, $\eta_i^2 = V_i/V_{i-1}$, V_i and V_{i-1} are the energy gauged potential of the reference particle at the position z_i and z_{i-1}, respectively, $\ell_{ie} = 2L_i/(\eta_i+1)$ is called effective length of L_i, v_{0i-1} and β_{0i-1} locity and relative velocity of reference particle, respectively, at the beginning of the interval. In Eq.(19), if we expand each matrix element into Taylor series, the first term of each series will be the element of the matrixwithout space charge forces.

B. At the dividing point z_i

We treated each dividing point $z_i\,(i=1,\ldots,n)$ as a thin lens. Let the subscript "1" stand for the first order of particle trajectories, "i" for the values at the point z_i, "$i-$" for the values on the left of z_i, "$i+$" for the values on the right of z_i, and note that when the particle passes cross the point z_i, its velocity and related values are all constant. After many calculations, one obtains

$$\begin{bmatrix} x_{1i+} \\ p_{x_{1i+}} \\ y_{1i+} \\ p_{y_{1i+}} \\ \tau_{1i+} \\ p_{\tau_{1i+}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -p_{0i}\frac{\eta_{i+1}^2\eta_i^2-2\eta_i^2+1}{4\eta_i^2L_i} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -p_{0i}\frac{\eta_{i+1}^2\eta_i^2-2\eta_i^2+1}{4\eta_i^2L_i} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ p_{x_{i-}} \\ y_{i-} \\ p_{y_{i-}} \\ \tau_{i-} \\ p_{\tau_{i-}} \end{bmatrix}, \tag{20}$$

where p_{0i} is the momentum of the reference particle at the point z_i.

5 Second order

A. In the interval (z_{i-1}, z_i)

Let the subscript "2" stand for the second order terms of the orbit. According to the second expressions of Eq.(15), Eq.(16) and Eq.(18), one obtains

$$x_2 = \frac{x\tau\eta_i(\eta_i+1)}{4k_x^2-k_\tau^2}\Big\{\frac{Q\mu_x}{p_{0i-1}k_x}[-(2k_x^2-k_\tau^2)\sinh(k_x\ell_{ie}) +$$
$$2k_x^2\sinh(k_x\ell_{ie})\cosh(k_\tau\ell_{ie}) -$$
$$k_xk_\tau\cosh(k_x\ell_{ie})\sinh(k_\tau\ell_{ie})] +$$
$$\frac{\bar{\gamma}_0p_{0i-1}k_x}{2m_0}[-2k_x^2\sinh(k_x\ell_{ie}) +$$
$$(2k_x^2-k_\tau^2)\sinh(k_x\ell_{ie})\cosh(k_\tau\ell_{ie}) +$$
$$k_xk_\tau\cosh(k_x\ell_{ie})\sinh(k_\tau\ell_{ie})]\Big\} +$$
$$-\frac{xp_\tau(\eta_i+1)}{m_0\bar{\gamma}_0^2\bar{\beta}_0^3c^3(4k_x^2-k_\tau^2)k_\tau}\Big\{\frac{Q\mu_x}{\gamma_0p_{0i-1}}[k_\tau\cosh(k_x\ell_{ie}) +$$
$$2k_x\sinh(k_x\ell_{ie})\sinh(k_\tau\ell_{ie}) -$$
$$k_\tau\cosh(k_x\ell_{ie})\cosh(k_\tau\ell_{ie})] +$$
$$\frac{p_{0i-1}k_x}{2m_0}[-k_xk_\tau\cosh(k_x\ell_{ie}) +$$
$$(2k_x^2-k_\tau^2)\sinh(k_x\ell_{ie})\sinh(k_\tau\ell_{ie}) +$$
$$k_xk_\tau\cosh(k_x\ell_{ie})\cosh(k_\tau\ell_{ie})] +$$
$$\frac{p_x\tau\eta_i(\eta_i+1)}{(4k_x^2-k_\tau^2)}\Big\{\frac{Q\mu_x}{p_{0i-1}^2k_x}[-2k_x\cosh(k_x\ell_{ie}) +$$
$$2k_x\cosh(k_x\ell_{ie})\cosh(k_\tau\ell_{ie}) -$$
$$k_\tau\sinh(k_x\ell_{ie})\sinh(k_\tau\ell_{ie})] +$$
$$\frac{\bar{\gamma}_0}{2m_0}[-(2k_x^2-k_\tau^2)\cosh(k_x\ell_{ie}) +$$
$$(2k_x^2-k_\tau^2)\cosh(k_x\ell_{ie})\cosh(k_\tau\ell_{ie}) +$$

$$k_x k_\tau \sinh(k_x l_{ie})\sinh(k_\tau l_{ie})]\} +$$

$$\frac{p_x p_\tau(\eta_i+1)}{m_0\bar{\gamma}_0^2\bar{\beta}_0^3 c^3(4k_x^2-k_\tau^2)}\{\frac{Q\mu_x}{\bar{\gamma}_0 p_{0i-1}^2 k_x k_\tau}$$

$$[-k_\tau \sinh(k_x l_{ie}) +$$

$$2k_x\cosh(k_x l_{ie})\sinh(k_\tau l_{ie}) -$$

$$k_\tau \sinh(k_x l_{ie})\cosh(k_\tau l_{ie})] +$$

$$\frac{1}{2m_0}[k_x k_\tau \sinh(k_x l_{ie}) +$$

$$(2k_x^2-k_\tau^2)\cosh(k_x l_{ie})\sinh(k_\tau l_{ie}) +$$

$$k_x k_\tau \sinh(k_x l_{ie})\cosh(k_\tau l_{ie})]\},$$

$$p_{x_2} = \frac{x\tau\eta_i(\eta_i+1)}{4k_x^2-k_\tau^2}\{Q\mu_x[-(2k_x^2-k_\tau^2)\cosh(k_x l_{ie}) +$$

$$(2k_x^2-k_\tau^2)\cosh(k_x l_{ie})\cosh(k_\tau l_{ie}) +$$

$$k_x k_\tau \sinh(k_x l_{ie})\sinh(k_\tau l_{ie})] +$$

$$\frac{\bar{\gamma}_0 p_{0i-1}^2 k_x^3}{2m_0}[-2k_x\cosh(k_x l_{ie}) +$$

$$2k_x\cosh(k_x l_{ie})\cosh(k_\tau l_{ie}) -$$

$$k_\tau \sinh(k_x l_{ie})\sinh(k_\tau l_{ie})]\} +$$

$$\frac{x p_\tau(\eta_i+1)}{m_0\bar{\gamma}_0^2\bar{\beta}_0^3 c^3(4k_x^2-k_\tau^2)k_\tau}\{\frac{Q\mu_x}{\bar{\gamma}_0}[k_x k_\tau \sinh(k_x l_{ie}) +$$

$$(2k_x^2-k_\tau^2)\cosh(k_x l_{ie})\sinh(k_\tau l_{ie}) +$$

$$k_x k_\tau \sinh(k_x l_{ie})\cosh(k_\tau l_{ie})] +$$

$$\frac{p_{0i-1}^2 k_x^3}{2m_0}[-k_\tau \sinh(k_x l_{ie}) +$$

$$2k_x\cosh(k_x l_{ie})\sinh(k_\tau l_{ie}) -$$

$$k_\tau \sinh(k_x l_{ie})\cosh(k_\tau l_{ie})]\} +$$

$$\frac{p_x\tau\eta_i(\eta_i+1)}{(4k_x^2-k_\tau^2)}\{\frac{Q\mu_x}{p_{0i-1}k_x}[-2k_x^2\sinh(k_x l_{ie}) +$$

$$(2k_x^2-k_\tau^2)\sinh(k_x l_{ie})\cosh(k_\tau l_{ie}) +$$

$$k_x k_\tau \cosh(k_x l_{ie})\sinh(k_\tau l_{ie})] +$$

$$\frac{\bar{\gamma}_0 p_{0i-1}k_x}{2m_0}[-(2k_x^2-k_\tau^2)\sinh(k_x l_{ie}) +$$

$$2k_x^2\sinh(k_x l_{ie})\cosh(k_\tau l_{ie}) -$$

$$k_x k_\tau \cosh(k_x l_{ie})\sinh(k_\tau l_{ie})]\} +$$

$$\frac{p_x p_\tau(\eta_i+1)}{m_0\bar{\gamma}_0^2\bar{\beta}_0^3 c^3(4k_x^2-k_\tau^2)k_\tau}\{\frac{Q\mu_x}{\bar{\gamma}_0 p_{0i-1}k_x}$$

$$[-k_x k_\tau \cosh(k_x l_{ie}) +$$

$$(2k_x^2-k_\tau^2)\sinh(k_x l_{ie})\sinh(k_\tau l_{ie}) +$$

$$k_x k_\tau \cosh(k_x l_{ie})\cosh(k_\tau l_{ie})] +$$

$$\frac{p_{0i-1}k_x^2}{2m_0}[k_\tau \cosh(k_x l_{ie}) +$$

$$2k_x\sinh(k_x l_{ie})\sinh(k_\tau l_{ie}) -$$

$$k_\tau \cosh(k_x l_{ie})\cosh(k_\tau l_{ie})]\},$$

$$y_2 = \frac{y\tau\eta_i(\eta_i+1)}{4k_y^2-k_\tau^2}\{\frac{Q\mu_y}{p_{0i-1}k_y}[-(2k_y^2-k_\tau^2)\sinh(k_y l_{ie}) +$$

$$2k_y^2\sinh(k_y l_{ie})\cosh(k_\tau l_{ie}) -$$

$$k_y k_\tau \cosh(k_y l_{ie})\sinh(k_\tau l_{ie})] +$$

$$\frac{\bar{\gamma}_0 p_{0i-1}k_y}{2m_0}[-2k_y^2\sinh(k_y l_{ie}) +$$

$$(2k_y^2-k_\tau^2)\sinh(k_y l_{ie})\cosh(k_\tau l_{ie}) +$$

$$k_y k_\tau \cosh(k_y l_{ie})\sinh(k_\tau l_{ie})]\} +$$

$$\frac{y p_\tau(\eta_i+1)}{m_0\bar{\gamma}_0^2\bar{\beta}_0^3 c^3(4k_y^2-k_\tau^2)k_\tau}\{\frac{Q\mu_y}{\bar{\gamma}_0 p_{0i-1}}$$

$$[k_\tau \cosh(k_y l_{ie}) + 2k_y\sinh(k_y l_{ie})\sinh(k_\tau l_{ie}) -$$

$$k_\tau \cosh(k_y l_{ie})\cosh(k_\tau l_{ie})] +$$

$$\frac{p_{0i-1}k_y}{2m_0}[-k_y k_\tau \cosh(k_y l_{ie}) +$$

$$(2k_y^2-k_\tau^2)\sinh(k_y l_{ie})\sinh(k_\tau l_{ie}) +$$

$$k_y k_\tau \cosh(k_y l_{ie})\cosh(k_\tau l_{ie})]\} +$$

$$\frac{p_y\tau\eta_i(\eta_i+1)}{(4k_y^2-k_\tau^2)}\{\frac{Q\mu_y}{p_{0i-1}^2 k_y}[-2k_y\cosh(k_y l_{ie}) +$$

$$2k_y\cosh(k_y l_{ie})\cosh(k_\tau l_{ie}) -$$

$$k_\tau \sinh(k_y l_{ie})\sinh(k_\tau l_{ie})] +$$

$$\frac{\bar{\gamma}_0}{2m_0}[-(2k_y^2-k_\tau^2)\cosh(k_y l_{ie}) +$$

$$(2k_y^2-k_\tau^2)\cosh(k_y l_{ie})\cosh(k_\tau l_{ie}) +$$

$$k_y k_\tau \sinh(k_y l_{ie})\sinh(k_\tau l_{ie})]\} +$$

$$\frac{p_y p_\tau(\eta_i+1)}{m_0\bar{\gamma}_0^2\bar{\beta}_0^3 c^3(4k_y^2-k_\tau^2)k_\tau}\{\frac{Q\mu_y}{\bar{\gamma}_0 p_{0i-1}^2 k_y}$$

$$[-k_\tau \sinh(k_y l_{ie}) + 2k_y\cosh(k_y l_{ie})\sinh(k_\tau l_{ie}) -$$

$$k_\tau \sinh(k_y l_{ie})\cosh(k_\tau l_{ie})] +$$

$$\frac{1}{2m_0}[k_y k_\tau \sinh(k_y l_{ie}) +$$

$$(2k_y^2-k_\tau^2)\cosh(k_y l_{ie})\sinh(k_\tau l_{ie}) +$$

$$k_y k_\tau \sinh(k_y l_{ie})\cosh(k_\tau l_{ie})]\},$$

$$p_{y_2} = \frac{y\tau\eta_i(\eta_i+1)}{4k_y^2-k_\tau^2}\{Q\mu_y[-(2k_y^2-k_\tau^2)\cosh(k_y l_{ie}) +$$

$$(2k_y^2-k_\tau^2)\cosh(k_y l_{ie})\cosh(k_\tau l_{ie}) +$$

$$k_y k_\tau \sinh(k_y l_{ie})\sinh(k_\tau l_{ie})] +$$

$$\frac{\tilde{\gamma}_0 p_{0i-1}^2 k_y^3}{2m_0}[-2k_y\cosh(k_y\ell_{ie})+$$

$$2k_y\cosh(k_y\ell_{ie})\cosh(k_\tau\ell_{ie})-$$

$$k_\tau\sinh(k_y\ell_{ie})\sinh(k_\tau\ell_{ie})]\Big\}+$$

$$\frac{yp_\tau(\eta_i+1)}{m_0\tilde{\gamma}_0^2\tilde{\beta}_0^3c^3(4k_y^2-k_\tau^2)k_\tau}\Big\{\frac{Q\mu_y}{\tilde{\gamma}_0}[k_yk_\tau\sinh(k_y\ell_{ie})+$$

$$(2k_y^2-k_\tau^2)\cosh(k_y\ell_{ie})\sinh(k_\tau\ell_{ie})+$$

$$k_yk_\tau\sinh(k_y\ell_{ie})\cosh(k_\tau\ell_{ie})]+$$

$$\frac{p_{0i-1}^2k_y^3}{2m_0}[-k_\tau\sinh(k_y\ell_{ie})+$$

$$2k_y\cosh(k_y\ell_{ie})\sinh(k_\tau\ell_{ie})-$$

$$k_\tau\sinh(k_y\ell_{ie})\cosh(k_\tau\ell_{ie})]\Big\}+$$

$$\frac{p_y\tau\eta_i(\eta_i+1)}{(4k_y^2-k_\tau^2)k_y}\Big\{\frac{Q\mu_y}{p_{0i-1}}[-2k_y^2\sinh(k_y\ell_{ie})+$$

$$(2k_y^2-k_\tau^2)\sinh(k_y\ell_{ie})\cosh(k_\tau\ell_{ie})+$$

$$k_yk_\tau\cosh(k_y\ell_{ie})\sinh(k_\tau\ell_{ie})]+$$

$$\frac{\tilde{\gamma}_0p_{0i-1}k_y}{2m_0}[-(2k_y^2-k_\tau^2)\sinh(k_y\ell_{ie})+$$

$$2k_y^2\sinh(k_y\ell_{ie})\cosh(k_\tau\ell_{ie})-$$

$$k_yk_\tau\cosh(k_y\ell_{ie})\sinh(k_\tau\ell_{ie})]\Big\}+$$

$$\frac{p_yp_\tau(\eta_i+1)}{m_0\tilde{\gamma}_0^2\tilde{\beta}_0^3c^3(4k_y^2-k_\tau^2)k_\tau}\Big\{\frac{Q\mu_y}{\tilde{\gamma}_0p_{0i-1}k_y}$$

$$[-k_yk_\tau\cosh(k_y\ell_{ie})+$$

$$(2k_y^2-k_\tau^2)\sinh(k_y\ell_{ie})\sinh(k_\tau\ell_{ie})+$$

$$k_yk_\tau\cosh(k_y\ell_{ie})\cosh(k_\tau\ell_{ie})]+$$

$$\frac{p_{0i-1}k_y^2}{2m_0}[k_\tau\cosh(k_y\ell_{ie})+$$

$$2k_y\sinh(k_y\ell_{ie})\sinh(k_\tau\ell_{ie})-$$

$$k_\tau\cosh(k_y\ell_{ie})\cosh(k_\tau\ell_{ie})]\Big\},$$

$$\tau_2=\frac{x^2(\eta_i+1)}{2m_0\tilde{\gamma}_0^2\tilde{\beta}_0^3c^3(4k_x^2-k_\tau^2)k_\tau}\Big\{\frac{Q\mu_x}{\tilde{\gamma}_0}[-(2k_x^2-$$

$$k_\tau^2)\sinh(k_\tau\ell_{ie})+k_xk_\tau\sinh(2k_x\ell_{ie})]+$$

$$\frac{p_{0i-1}^2k_x^3}{m_0}[-k_x\sinh(k_\tau\ell_{ie})+$$

$$k_\tau\sinh(k_x\ell_{ie})\cosh(k_x\ell_{ie})]\Big\}+$$

$$\frac{xp_x(\eta_i+1)}{m_0\tilde{\gamma}_0^2\tilde{\beta}_0^3c^3(4k_x^2-k_\tau^2)}\Big\{\Big(\frac{Q\mu_x}{\tilde{\gamma}_0p_{0i-1}}+$$

$$\frac{p_{0i-1}k_x^2}{2m_0}\Big)[\cosh(k_\tau\ell_{ie})-\cosh(2k_x\ell_{ie})]\Big\}+$$

$$\frac{p_x^2(\eta_i+1)}{m_0\tilde{\gamma}_0^2\tilde{\beta}_0^3c^3(4k_x^2-k_\tau^2)k_\tau}\Big\{\frac{2Q\mu_x}{p_{0i-1}^2\tilde{\gamma}_0k_x}\times$$

$$[k_x\sinh(k_\tau\ell_{ie})-k_\tau\sinh(k_x\ell_{ie})\cosh(k_x\ell_{ie})]+$$

$$\frac{1}{4m_0}[(2k_x^2-k_\tau^2)\sinh(k_\tau\ell_{ie})+k_xk_\tau\sinh(2k_x\ell_{ie})]\Big\}+$$

$$\frac{y^2(\eta_i+1)}{2m_0\tilde{\gamma}_0^2\tilde{\beta}_0^3c^3(4k_y^2-k_\tau^2)k_\tau}\Big\{-\frac{Q\mu_y}{\tilde{\gamma}_0}[(2k_y^2-$$

……

$$k_\tau\sinh(k_y\ell_{ie})\cosh(k_y\ell_{ie})]\Big\}+$$

$$\frac{yp_y(\eta_i+1)}{m_0\tilde{\gamma}_0^2\tilde{\beta}_0^3c^3(4k_y^2-k_\tau^2)}\Big\{\Big(\frac{Q\mu_y}{\tilde{\gamma}_0p_{0i-1}}+$$

$$\frac{p_{0i-1}k_y^2}{2m_0}\Big)[\cosh(k_\tau\ell_{ie})-\cosh(2k_y\ell_{ie})]\Big\}+$$

$$\frac{p_y^2(\eta_i+1)}{m_0\tilde{\gamma}_0^2\tilde{\beta}_0^3c^3(4k_y^2-k_\tau^2)k_\tau}\Big\{\frac{2Q\mu_y}{\tilde{\gamma}_0p_{0i-1}^2}[k_y\sinh(k_\tau\ell_{ie})-$$

$$k_\tau\sinh(k_y\ell_{ie})\cosh(k_y\ell_{ie})]+$$

$$\frac{1}{4m_0}[(2k_y^2-k_\tau^2)\sinh(k_\tau\ell_{ie})+k_yk_\tau\sinh(2k_y\ell_{ie})]\Big\}+$$

$$\tau^2(\eta_i+1)\Big\{\frac{Q\mu_xv_{0i-1}^2}{2m_0\tilde{\gamma}_0\tilde{\beta}_0c^3k_\tau}\sinh(k_\tau\ell_{ie})+$$

$$\tilde{\gamma}_0^2\tilde{\beta}_0ck_\tau\eta_i^2[-\sinh(k_\tau\ell_{ie})+\sinh(k_\tau\ell_{ie})\cosh(k_\tau\ell_{ie})]\Big\}+$$

$$\frac{\tau p_\tau\eta_i(\eta_i+1)}{m_0\tilde{\gamma}_0\tilde{\beta}_0^2c^2}[\sinh^2(k_\tau\ell_{ie})+$$

$$\cosh^2(k_\tau\ell_{ie})-\cosh(k_\tau\ell_{ie})]+$$

$$p_\tau^2\frac{\eta_i+1}{2m_0^2\tilde{\gamma}_0^4\tilde{\beta}_0^5c^5k_\tau}[\sinh(k_\tau\ell_{ie})+\sinh(2k_\tau\ell_{ie})],$$

$$p_{\tau_2}=\frac{x^2\eta_i(\eta_i+1)}{4(4k_x^2-k_\tau^2)}\Big\{Q\mu_x[4k_x^2(1-\cosh(k_\tau\ell_{ie}))-$$

$$k_\tau^2(1-2\cosh(k_\tau\ell_{ie})+\cosh(2k_x\ell_{ie}))]+$$

$$\frac{\tilde{\gamma}_0p_{0i-1}^2k_x^2}{2m_0}[4k_x^2(1-\cosh(k_\tau\ell_{ie}))+$$

$$k_\tau^2(\cosh(2k_x\ell_{ie})-1)]\Big\}+\frac{xp_x\eta_i(\eta_i+1)k_\tau}{4k_x^2-k_\tau^2}$$

$$\Big\{\Big(\frac{Q\mu_x}{p_{0i-1}k_x}-\frac{\tilde{\gamma}_0p_{0i-1}k_x}{2m_0}\Big)[k_x\sinh(k_\tau\ell_{ie})-$$

$$k_\tau\sinh(k_x\ell_{ie})\cosh(k_x\ell_{ie})]\Big\}+$$

$$\frac{p_x^2\eta_i(\eta_i+1)}{4(4k_x^2-k_\tau^2)}\Big\{\frac{Q\mu_x}{p_{0i-1}^2k_x^2}[4k_x^2(\cosh(k_\tau\ell_{ie})-1)+$$

$$k_\tau^2(1-\cosh(2k_x\ell_{ie}))]+\frac{\tilde{\gamma}_0}{2m_0}[4k_x^2(\cosh(k_\tau\ell_{ie})-1)+$$

$$
\begin{aligned}
&k_\tau^2(1-2\cosh(k_\tau \ell_{ie})+\cosh(2k_y \ell_{ie}))]\Big\}+\\
&\frac{y^2\eta_i(\eta_i+1)}{4(4k_y^2-k_\tau^2)}\Big\{Q\mu_y[4k_y^2(1-\cosh(k_\tau \ell_{ie}))-\\
&k_\tau^2(1-2\cosh(k_\tau \ell_{ie})+\cosh(2k_y \ell_{ie}))]+\\
&\frac{\tilde{\gamma}_0 p_{0i-1}^2 k_y^2}{2m_0}[4k_y^2(1-\cosh(k_\tau \ell_{ie}))+\\
&k_\tau^2(\cosh(2k_y \ell_{ie})-1)]\Big\}+\\
&\frac{yp_y\eta_i(\eta_i+1)k_\tau}{4k_x^2-k_\tau^2}\Big\{\Big(\frac{Q\mu_y}{p_{0i-1}k_y}-\\
&\frac{\tilde{\gamma}_0 p_{0i-1}k_y}{2m_0}\Big)[k_y\sinh(k_\tau \ell_{ie})-\\
&k_\tau\sinh(k_y \ell_{ie})\cosh(k_y \ell_{ie})]\Big\}+\\
&\frac{p_y^2\eta_i(\eta_i+1)}{4(4k_y^2-k_\tau^2)}\Big\{\frac{Q\mu_y}{p_{0i-1}^2k_y^2}[4k_y^2(\cosh(k_\tau \ell_{ie})-1)+\\
&k_\tau^2(1-\cosh(2k_y \ell_{ie}))]+\frac{\bar{\gamma}_0}{2m_0}\\
&[4k_y^2(\cosh(k_\tau \ell_{ie})-1)+\\
&k_\tau^2(1-2\cosh(k_\tau \ell_{ie})+\cosh(2k_y \ell_{ie}))]\Big\}+\\
&\tau^2\eta_i(\eta_i+1)\Big\{\frac{1}{2}Q\mu_z v_{0i-1}^2\tilde{\gamma}_0^2\tilde{\beta}_0^2[\cosh(k_\tau \ell_{ie})-\\
&\cosh^2(k_\tau \ell_{ie})]+\frac{1}{2}m_0\tilde{\gamma}_0^5\tilde{\beta}_0^4c^4k_\tau^2\eta_i^2[\cosh^2(k_\tau \ell_{ie})-\\
&2\cosh(k_\tau \ell_{ie})+1]\Big\}+\tau p_\tau(\eta_i+1)\\
&\Big\{\Big[-\frac{Q\mu_z v_{0i-1}^2}{m_0\tilde{\gamma}_0\tilde{\beta}_0c^3k_\tau}\sinh(k_\tau \ell_{ie})\cosh(k_\tau \ell_{ie})\Big]+\\
&\tilde{\gamma}_0^2\tilde{\beta}_0ck_\tau\eta_i^2[-\sinh(k_\tau \ell_{ie})+\\
&\sinh(k_\tau \ell_{ie})\cosh(k_\tau \ell_{ie})]\Big\}+\\
&p_\tau^2(\eta_i+1)\Big\{-\frac{Q\mu_z v_{0i-1}^2}{2m_0^2\tilde{\gamma}_0^4\tilde{\beta}_0^4c^6k_\tau^2\eta_i}\sinh^2(k_\tau \ell_{ie})+\\
&\frac{\eta_i}{2m_0\tilde{\gamma}_0^2\tilde{\beta}_0^2c^2}[-2+\cosh(k_\tau \ell_{ie})+\cosh^2(k_\tau \ell_{ie})]\Big\}.
\end{aligned}
\tag{21}
$$

Note that all the $\tilde{\beta}_0=\frac{1}{2}(\beta_{0i}+\beta_{0i+1})$ and $\tilde{\gamma}_0=\frac{1}{2}(\gamma_{0i}+\gamma_{0i+1})$ in Eq.(21) are the average values of β and γ for the reference particle in the interval.

B. At the dividing point z_i

Note that from the left of the point z_i to its right, $\phi''\neq 0$, but $\phi'=0$. After some calculations, we have

$$
\begin{aligned}
p_{x_{2i+}} &= xp_\tau\frac{\eta_{i+1}^2\eta_i^2-2\eta_i^2+1}{4\gamma_0^2\beta_0c\eta_i^2L_i},\\
p_{y_{2i+}} &= yp_\tau\frac{\eta_{i+1}^2\eta_i^2-2\eta_i^2+1}{4\gamma_0^2\beta_0c\eta_i^2L_i},\\
\tau_{2i+} &= -(x^2+y^2)\frac{\eta_{i+1}^2\eta_i^2-2\eta_i^2+1}{8\gamma_0^2\beta_0c\eta_i^2L_i},
\end{aligned}
\tag{22}
$$

here β_0 and γ_0 are the value of β and γ, respectively, for the reference particle at point z_i.

6 Discussion

The second order orbits have been derived in the phase space $(x, p_x, y, p_y, \tau, p_\tau)$ for the electrostatic lenses. It could be extended to higher orders if desired. Usually, people like using the phase space $(x, x', y, y', \Delta t, \Delta E)$ rather than $(x, p_x, y, p_y, \tau, p_\tau)$. Therefore, it is necessary to transfer the variables p_x, p_y and p_τ to x', y', Δt and ΔE. The procedures are as the following:

$$x' = p_x/p_0,\quad y' = p_y/p_0,$$

here p_0 is the momentum of the reference particle in the place where it is concerned.

As to the variables τ and p_τ, we can see from Eq.(8) that

$$\tau = t - z/v_0 = \Delta t,\quad p_\tau = p_t - p_t^0 = -\Delta E,$$

where v_0 is the velocity of the reference particle.

参考文献(References)

1 Goldstein H. Classical Mechanics. 2nd ed., Reading, Massachusetts; Addison-Wesley, 1980

2 LÜ Jian-Qin. Nucl. Instrum. & Methods, 1998, **A416**:229–235

强流脉冲束在静电透镜中非线性传输的 Lie 代数分析*

吕建钦 李超龙 李金海

(北京大学重离子物理研究所 北京 100871)

摘要 用 Lie 代数方法分析了相对论强流脉冲束在轴对称静电透镜中的非线性传输,得到粒子在6维相空间(x,x',y,y',τ,p_τ)中的2级近似轨迹. 在分析中,把静电透镜的场作用区分成若干小段,每个小段看作匀加速段,而每个分点看作薄透镜,依次施加 Lie 变换,就得到轨迹的各级近似. 这里,假定粒子在相空间中为 K-V 分布.

关键词 静电透镜 强流脉冲束 Lie 映射 非线性

2002-04-17 收稿

* 国家自然科学基金(10075005)资助