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Nonlinear Analysis for the Wein Filter with Lie Algebraic Method *

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Abstract In this paper, we analyze the nonlinear transport of the particle motions in the Wein filter up to the third order with the Lie algebraic method. First, we set up the Hamiltonian for the Wein filter; then expand the Hamiltonian into a sum of homogeneous polynomials of different degrees; finally, calculate the particle's nonlinear trajectories up to the third order. Higher orders could be obtained if necessary.

Key words Lie algebraic method, nonlinear, Wein filter

1 Introduction

The Wein filter consists of a pair of parallel electrode plates and a pair of magnetic poles. It is always used to select the low energy charged particle beam of different mass or velocity. Because the beam energy is low, the nonlinear terms of the particle trajectories are not so small. So, to make the calculations more accurate, we must take the nonlinear effects into account. The linear transfer matrix of the Wein filter is known[1]. Here we use the Lie algebraic method[2] to analyze the nonlinear particle trajectories.

2 Lie map

The Lie algebraic method is a good tool for the analysis, because the analyzing procedures are much more similar than that of solving the ordinary differential equations, and all the information of the particle trajectories is contained in the Lie transformation which is associated with the Hamiltonian. We give a brief description of the Lie algebraic method in the following:

The charged particle motions in the accelerator is regarded as the motion in the six-dimensional phase space $\zeta = (x, p_x, y, p_y, \tau, p_\tau)$. The relationship between the initial point ζ^{in} and the final point ζ^{fin} is regarded as the Lie map $\boldsymbol{M}$, which is written as the integral of the Hamiltonian H:

$$\begin{aligned}\boldsymbol{M} &= \exp[-:\int_{z_0}^{z_f} H\mathrm{d}z:] = \cdots \boldsymbol{M}_4\boldsymbol{M}_3\boldsymbol{M}_2 = \\ &\cdots(1 + :f_4: + \frac{1}{2}:f_4:^2 + \cdots) \\ &(1 + :f_3: + \frac{1}{2}:f_3:^2 + \cdots)\boldsymbol{M}_2 = \\ &M_2 + :f_3:M_2 + (:f_4: + \frac{1}{2}:f_3:^2)M_2. \end{aligned} \quad (1)$$

Where:

$$f_2 = -\int_{z_0}^{z_f} H_2\mathrm{d}z, f_3 = -\int_{z_0}^{z_f} h_3^{\text{int}}\mathrm{d}z,$$

$$f_4 = -\int_{z_0}^{z_f} h_4^{\text{int}}\mathrm{d}z + \frac{1}{2}\int_{z_0}^{z_f}\mathrm{d}z_1\int_{z_0}^{z_f}\mathrm{d}z_2[-h_3^{\text{int}}(z_2), -h_3^{\text{int}}(z_1)], \quad (2)$$

where

$$h_n^{\text{int}}(z) = \boldsymbol{M}_2 H_n. \quad (3)$$

Where H_n is the n degree of homogeneous polynomial of the Taylor series of the Hamiltonian H. In Eq.(1) the factorization theorem[3] is used, and

$$\boldsymbol{M}_2 = \exp(:f_2:), \boldsymbol{M}_3 = \exp(:f_3:), \boldsymbol{M}_4 = \exp(:f_4:), \quad (4)$$

and the symbol :: stands for the Poisson bracket, which is defined as:

$$:f:g = [f, g] = \sum_i[(\partial f/\partial q_i)(\partial g/\partial P_i) - (\partial g/\partial q_i)(\partial f/\partial P_i)]. \quad (5)$$

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The Lie transformation can be expressed as a Taylor series[4]:

$$\exp(:f:)g = g + [f,g] + [f,[f,g]]/2! + \ldots \quad (6)$$

Let the map $\boldsymbol{M}$ act on the canonical coordinate ζ^{in}, we obtain each order of the nonlinear result:

$$\zeta_1 = \exp(:f_2:)\zeta^{\text{in}}, \quad \text{(first order)}$$

$$\zeta_2 = :f_3:\zeta_1, \quad \text{(second order)}$$

$$\zeta_3 = :f_4:\zeta_1 + \frac{1}{2}:f_3:^2\zeta_1, \text{(third order)} \quad (7)$$

Here, the subscripts denote the order of the approximation.

3 Hamiltonian and its expanded homogeneous polynomial

In order to calculate the nonlinear trajectories, first we must set up the Hamiltonian, which describes the motions of the charged particles in the Wein filter. Let $\boldsymbol{E}$ stands for the electric field and $\boldsymbol{B}$ the magnetic field, and v_0 the velocity of the reference particle. Then the following equation must be valid:

$$E = v_0 B. \quad (8)$$

Suppose the directions of the vector $\boldsymbol{E}$ and $\boldsymbol{B}$ are along the positive directions of x and y axis respectively. The potential functions of the electric field and the magnetic field are:

$$\psi = -v_0 Bx, \boldsymbol{A} = A_x\boldsymbol{e}_x + A_y\boldsymbol{e}_y + A_z\boldsymbol{e}_z = -Bx\boldsymbol{e}_z. \quad (9)$$

So the Hamiltonian in the Cartesian coordinates with time as an independent variable can be gotten:

$$H_t = \sqrt{m_0^2c^4 + p_x^2c^2 + p_y^2c^2 + (p_z + qBx)^2c^2} - qv_0Bx. \quad (10)$$

Suppose $p_t = -H_t$, and solve p_z from it, we get:

$$p_z = -K(t,x,y,p_t,p_x,p_t) = \sqrt{(p_t - qv_0Bx)^2/c^2 - p_x^2 - p_y^2 - m_0^2c^2} - qBx. \quad (11)$$

Eq.(11) is the Hamiltonian corresponding to the canonical variables x, p_x, y, p_y, t, and p_t with z as the independent variable. Let

$$x = x, p_x = p_x, y = y, p_y = p_y,$$

$$\tau = t - z/v_0, p_\tau = p_t - p_t^0. \quad (12)$$

Where, p_t^0 is the value of p_t of the reference particle. In the phase space $\zeta = (x, p_x, y, p_y, \tau, p_\tau)$, the coordinate of reference particle always keeps zero. According to Eq.(10), we have:

$$p_t^0 = -H_t|_{\text{reference particle}} = -\sqrt{m_0^2c^4 + p_0^2c^2} = -m_0\gamma_0c^2, \quad (13)$$

here, p_0 is the momentum of the reference particle, $\gamma_0 = \sqrt{1-(v_0/c)^2}$.

Under the canonical coordinate transformation of Eq.(12), the generating function is:

$$F_2 = \sum_{i=1}^{3} Q_i p_i = xp_x + yp_y + \tau p_t = xp_x + yp_y + (t - z/v_0)p_t. \quad (14)$$

The new Hamiltonian in the new phase space ζ is:

$$H = -\sqrt{(p_\tau - m_0\gamma_0c^2 - qv_0Bx)^2/c^2 - p_x^2 - p_y^2 - (m_0c)^2} + qBx - (p_\tau - m_0\gamma_0c^2)/v_0. \quad (15)$$

Expand the Hamiltonian H into Taylor series, we have

$$H = \sum_{n=0}^{\infty} H_n. \quad (16)$$

Where H_n is the homogeneous polynomial of n-th order in the phase space ζ. The first five items are: $H_0 = p_0(\beta_0^{-2} - 1)$, (here $p_0 = m_0\gamma_0v_0$)

$$H_1 = 0,$$

$$H_2 = \frac{B^2q^2}{2p_0\gamma_0^2}x^2 + \frac{1}{2p_0}(p_x^2 + p_y^2) + \frac{1}{2p_0v_0^2\gamma_0^2}p_\tau^2 - \frac{Bq}{p_0v_0\gamma_0^2}xp_\tau,$$

$$H_3 = -\frac{B^3q^3}{2p_0^2\gamma_0^2}x^3 - \frac{Bq}{2p_0^2}x(p_x^2 + p_y^2) + \frac{3B^2q^2}{2p_0^2\gamma_0^2v_0}x^2p\tau - \frac{3Bq}{2p_0^2v_0^2\gamma_0^2}xp_\tau^2 + \frac{1}{2p_0^2\gamma_0^2v_0^3}p_\tau^3 + \frac{1}{2p_0^2v_0}(p_x^2 + p_y^2)p_\tau,$$

$$H_4 = \frac{B^4q^4(5-\beta_0^2)}{8p_0^3\gamma_0^2}x^4 + \frac{B^2q^2(3-\beta_0^2)}{4p_0^3}x^2(p_x^2 + p_y^2) + \frac{p_x^2p_y^2}{4p_0^3} - \frac{(5-\beta_0^2)B^3q^3}{2p_0^3v_0\gamma_0^2}x^3p_\tau - \frac{Bq(3-\beta_0^2)}{2p_0^3v_0}xp_\tau \times (p_x^2 + p_y^2) + \frac{3(5-\beta_0^2)B^2q^2}{4p_0^3\gamma_0^2v_0^2}x^2p_\tau^2 - \frac{(5-\beta_0^2)Bq}{2p_0^3\gamma_0^2v_0^3}xp_\tau^3 + \frac{5-\beta_0^2}{8p_0^3\gamma_0^2v_0^4}p_\tau^4 + \frac{3-\beta_0^2}{4p_0^3v_0^2}(p_x^2 + p_y^2)p_\tau^2 + \frac{p_x^4 + p_y^4}{8p_0^3}. \quad (17)$$

4 Particle trajectory calculations

4.1 First order

Because H_2 in Eq.(17) does not depend on z, carrying out the first integration of Eq.(2), we obtain

$$f_2 = -lH_2. \quad (18)$$

Here l is the length of the Wein filter. According to Eq.(6) and Eq.(18), the first expression of Eq.(2) and Eq.(7), and the third expression of Eq.(17), we obtain the particle trajectories of the first order approximation:

$$\begin{bmatrix} x_1 \\ p_{x_1} \\ y_1 \\ p_{y_1} \\ \tau_1 \\ p_{\tau_1} \end{bmatrix} = \begin{bmatrix} \cos(kl) & \frac{1}{kp_0}\sin(kl) & 0 & 0 & 0 & \frac{1-\cos(kl)}{kp_0\gamma_0 v_0} \\ -kp_0\sin(kl) & \cos(kl) & 0 & 0 & 0 & \frac{\sin(kl)}{v_0\gamma_0} \\ 0 & 0 & 1 & \frac{l}{p_0} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{-\sin(kl)}{v_0\gamma_0} & \frac{\cos(kl)-1}{kp_0\gamma_0 v_0} & 0 & 0 & 1 & \frac{\sin(kl)}{kp_0\gamma_0^2 v_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ p_x \\ y \\ p_y \\ \tau \\ p_\tau \end{bmatrix}. \tag{19}$$

Here $(x, p_x, y, p_y, \tau, p_\tau)$ are the initial canonical variables in the phase space, and

$$k = \frac{Bq}{p_0\gamma_0}. \tag{20}$$

4.2 Second order

According to Eq.(3) and Eq.(19), the second expression of Eq.(2) and Eq.(7) and the fourth expression of Eq.(17), one obtains the second order terms of the trajectories:

$$x_2 = [2-\cos(kl)-\cos(2kl)]\frac{kx^2\gamma_0}{2} + [\sin(kl) - \sin(2kl)]\frac{xp_x\gamma_0}{p_0} - [2-\cos(kl)-\cos(2kl)]\frac{xp_\tau}{p_0 v_0} - [2-3\cos(kl)+\cos(2kl)]\frac{p_x^2\gamma_0}{2kp_0^2} - [\sin(kl) - \sin(2kl)]\frac{p_xp_\tau}{kp_0^2 v_0} + [1-\cos(kl)]\frac{p_y^2\gamma_0}{2kp_0^2} + [2-\cos(kl)-\cos(2kl)]\frac{p_\tau^2}{2kp_0^2 v_0^2\gamma_0}, \tag{21}$$

$$p_{x_2} = [\sin(kl)+\sin(2kl)]\frac{k^2p_0x^2\gamma_0}{2} + [\cos(kl) - \cos(2kl)]kxp_x\gamma_0 - [\sin(kl)+\sin(2kl)]\frac{kxp_\tau}{v_0} + [3\sin(kl)-\sin(2kl)]\frac{\gamma_0 p_x^2}{2p_0} - [\cos(kl)-\cos(2kl)]\frac{p_xp_\tau}{p_0v_0} + \frac{\sin(kl)\gamma_0 p_y^2}{2p_0} + [\sin(kl)+\sin(2kl)]\frac{p_\tau^2}{2p_0v_0^2\gamma_0}, \tag{22}$$

$$y_2 = -\frac{xp_y\gamma_0\sin(kl)}{p_0} - \frac{p_xp_y\gamma_0[1-\cos(kl)]}{kp_0^2} + \frac{\sin(kl)p_yp_\tau}{kp_0^2v_0}, \tag{23}$$

$$p_{y_2} = 0, \tag{24}$$

$$\tau_2 = [\sin(kl)+\sin(2kl)]\frac{kx^2}{2v_0} + [\cos(kl)-\cos(2kl)] \times \frac{xp_x}{p_0v_0} - [\sin(kl)+\sin(2kl)]\frac{xp_\tau}{p_0v_0^2\gamma_0} + [3\sin(kl) - \sin(2kl)]\frac{p_x^2}{2kp_0^2v_0} - [\cos(kl)-\cos(2kl)]\frac{p_xp_\tau}{kp_0^2v_0^2\gamma_0} + \frac{p_y^2\sin(kl)}{2kp_0^2v_0} + [\sin(kl)+\sin(2kl)]\frac{p_\tau^2}{2kp_0^2v_0^3\gamma_0^2}, \tag{25}$$

$$p_{\tau_2} = 0. \tag{26}$$

4.3 Third order

In the similar way, according to Eq.(3), Eq.(19), Eq.(21) ~ Eq.(26), the third expression of the Eq.(2) and Eq.(7), and the fifth expression of Eq.(17), we obtain the third order terms of the trajectories

$$x_3 = [4kl\beta_0^2\sin(kl) - 8 + \cos(kl) + 4\cos(2kl) + 3\cos(3kl)]\frac{k^2x^3\gamma_0^2}{8} - [4kl\beta_0^2\cos(kl) + 7\sin(kl) + 4\sin(2kl) - 9\sin(3kl)]\frac{kx^2p_x\gamma_0^2}{8p_0} - [4kl\beta_0^2\sin(kl) - 8 + \cos(kl) + 4\cos(2kl) + 3\cos(3kl)]\frac{3kx^2p_\tau\gamma_0}{8p_0v_0} + [4kl\beta_0^2\sin(kl) - 8 - 3\cos(kl) + 20\cos(2kl) - 9\cos(3kl)]\frac{xp_x^2\gamma_0^2}{8p_0^2} + [4kl\beta_0^2\cos(kl) + 7\sin(kl) + 4\sin(2kl) - 9\sin(3kl)]\frac{xp_xp_\tau\gamma_0}{4p_0^2v_0} + [kl\beta_0^2\sin(kl) - 2 + \cos(kl) + \cos(2kl)]\frac{xp_y^2\gamma_0^2}{2p_0^2} + [4kl\beta_0^2\sin(kl) - 8 + \cos(kl) + 4\cos(2kl) + 3\cos(3kl)]\frac{3xp_\tau^2}{8p_0^2v_0^2} - [4kl\beta_0^2\cos(kl) + 11\sin(kl) - 12\sin(2kl) + 3\sin(3kl)] \times \frac{p_x^3\gamma_0^2}{8kp_0^3} - [4kl\beta_0^2\sin(kl) - 8 - 3\cos(kl) + 20\cos(2kl) - 9\cos(3kl)]\frac{p_x^2p_\tau\gamma_0}{8kp_0^3v_0} - [kl\beta_0^2\cos(kl) + \sin(kl) -$$

$$\sin(2kl)]\frac{p_x p_y^2 \gamma_0^2}{2kp_0^3} - [4kl\beta_0^2\cos(kl) + 7\sin(kl) + 4\sin(2kl) - 9\sin(3kl)]\frac{p_x p_\tau^2}{8kp_0^3 v_0^2} - [kl\beta_0^2\sin(kl) - 2 + \cos(kl) + \cos(2kl)]\frac{p_y^2 p_\tau \gamma_0}{2kp_0^3 v_0} - [4kl\beta_0^2\sin(kl) - 8 + \cos(kl) + 4\cos(2kl) + 3\cos(3kl)]\frac{p_\tau^3}{8kp_0^3 v_0^3 \gamma_0}, \quad (27)$$

Because of the page limit, the third terms of p_{x_3}, y_3, p_{y_3}, τ_3 and p_{τ_3} are not listed here.

5 Conclusion

Kuroda[5] analyzed the third order particle trajectories in the 4-dimensional phase space(x, p_x, y, p_y) for the Wein filter by solving the differential equations. The results can be used in the dc beam nonlinear transport. If we want to calculate the nonlinear transport of pulsed beams in the Wein filter, we have to extend the 4-D phase space into 6-D phase space. That's why we analyze the third order trajectories in the 6-D phase space($x, p_x, y, p_y, \tau, p_\tau$). The results will be put into a computer program.

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电磁交叉场分析器中非线性传输的 Lie 代数分析 *

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摘要 用李代数方法分分析了带电粒子在电磁交叉场分析器中的非线性传输,结果近似到三级近似.分析过程为:首先建立粒子在电磁交叉场分析器中的运动的 Hamilton 函数,然后将 Hamilton 函数在平衡轨道附近展开成幂级数,最后计算粒子的非线性轨迹到三级近似.根据需要,还可以推导出更高级的近似解.

关键词 Lie 代数方法 非线性 电磁交叉场分析器

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