Statistical Properties of Radiation Fields in a Compact Space[*]

CHANG Zhe[1;1)]　　GUAN Cheng-Bo[2;2)]

1 (Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China)
2 (Department of Physics, Southeast University, Nanjing 210096, China)

Abstract　In this paper, we discuss radiation fields in a compact space of finite size instead of that in a cavity for investigating the coupled atom-radiation field system. Representations of $T(1) \times SO(4)$ group are used to give a formulation for kinematics of the radiation fields. The explicit geometrical parameter dependence of statistical properties of radiation fields is obtained. Our results show remarkable differences from those of the black-body radiation system in free space.

Key words　cavity QED, radiation fields, compact manifold

1　Introduction

The development of a single-atom maser allows a detailed study of the atom-field interaction[1—4]. The realization of a single-atom maser has been made possible due to the enormous progress in the construction of superconducting cavities together with the laser preparation of highly excited atoms called Rydberg atoms. Resonant effects[5] related to the coupling of atoms with strong radiofrequency field has been observed. In particular, it has been demonstrated that the spontaneous emission rate of an atom inside the cavity is different from its value in free space[6—10]. This effect can be discussed from several different approaches, e.g., attributing it to a change of the spectral density of the modes of the vacuum radiation field due to the cavity's resonating structure. The theoretical understanding of these effects by making use of perturbation theory requires the calculation of very high-order terms. It makes the standard Feynman diagram technique practically unreliable in the case. On the other hand, the nonlinear character of the problem involved in realistic situations cannot be ignored simply. A naive solution of this difficulty may be to assume that under certain conditions, the coupled atom-radiation field system can be approximated by a system composed of a harmonic oscillator coupled linearly to the radiation field through some effective coupling constants. Thus, a significant number of work has been sparked on the study of cavity QED.

Another motivation to study the radiation field system in cavity comes from cosmology. Astronomical observations[11—13] have provided plenty of support to the cosmological principle[14, 15], which states that the universe is spatially homogeneous and isotropic on large scales. This principle implies that we can build up a comoving coordinate system, in which the spatial part has a maximal symmetry. In general, there are three types of global symmetry for the three-dimensional maximally symmetric space, i.e., $ISO(3)$, $SO(4)$ and $SO(1,3)$. Usually, they are called the flat, closed and open universe, respectively. In order to understand the evolution of the universe, the whole distribution and activity of matters (including radiations) in space have to be taken into account. And we would like to point out that, if properties of

matters in space as a whole are to be studied, the difference of global symmetries should not be neglected. In particular, we know that the early epoch of the universe was governed by the radiation fields. Therefore, a careful study of radiation fields in a compact space is necessary.

In free space, it is well-known that the radiation fields can be described by an ideal gas model (a thermodynamic system consisting of free massless particles). However, one cannot extend this idea to the case of compact space straightforwardly. In order to investigate the quantum statistical properties of radiation fields in a compact space, we begin with constructing of the representation of one-particle states. We notice that the space-time $\mathbb{R}^1 \times S^3$ is a conformally deformed de Sitter space-time. This space-time has a translation symmetry in the time direction and six rotation symmetries in the spatial part. That is to say, the isometry group of the space-time $\mathbb{R}^1 \times S^3$ is $T(1) \times SO(4)$. By making use of the representation theory of this group, we get a representation of one-particle states as well as the corresponding dispersion relation for the radiation field. The density of state with respect to the energy spectrum is continuous in the case of the compact space with a large radius. It should be pointed out that the obtained density of state is different from that of a black body in free space. The geometrical parameter dependence of statistical properties of the radiation field system in a compact space can be obtained explicitly. It has been a goal of cavity QED for a long time.

This paper is organized as follows. In Section 2, we will discuss the kinematics of the radiation fields in the compact space S^3. The Hilbert space of free photons as well as the corresponding dispersion relations are presented. The Bose-Einstein statistics of the radiation fields is intensively studied in Section 3. Statistical properties of the radiation fields, which are remarkably different from those in free space, are shown. We then give the conclusions and remarks in Section 4.

2　The kinematics of radiation fields

The metric of the compact space-time $\mathbb{R}^1 \times S^3$ is

of the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2\mathrm{d}\Omega_3^2 , \tag{1}$$

where R denotes the radius of the sphere S^3. The spatial part of the metric, $\mathrm{d}\Omega_3^2$, can be expanded with Euler angular variables

$$\mathrm{d}\Omega_3^2 = \mathrm{d}r^2 + \sin^2 r(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) , \tag{2}$$

or with a unit vector in the four-dimensional Euclidean space

$$\mathrm{d}\Omega_3^2 = \mathrm{d}x^i\mathrm{d}x^i ,$$
$$\sum_i x^i x^i = 1, \quad (i = 1, 2, 3, 4) . \tag{3}$$

The isometry group of the space-time is $T(1) \times SO(4)$. Infinitesimal elements of the isometry group can be expressed in terms of the following generators

$$\hat{H} = -\mathrm{i}\partial_t , \quad \hat{M}^{ij} = -\mathrm{i}(x^i\partial_j - x^j\partial_i) . \tag{4}$$

Here $\hat{H}$ denotes the Hamiltonian and $\hat{M}^{ij}$ are the angular momentums. The motion of a free photon is described by the Klein-Gordon equation

$$\Box A^\lambda = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)A^\lambda = 0 . \tag{5}$$

Here the d'Alembertian operator $\Box$ is an invariant of $T(1) \times SO(4)$

$$\Box = -(\hat{H}^2 - \frac{1}{2}\hat{M}^{ij}\hat{M}_{ij}R^{-2}) . \tag{6}$$

It is convenient to introduce the set of operators $\hat{L}_\alpha$ and $\hat{P}_\alpha$ as[16]

$$\hat{L}_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}\hat{M}^{\beta\gamma} ,$$
$$\hat{P}_\alpha = \hat{M}^{4\alpha} , \quad (\alpha, \beta, \gamma = 1, 2, 3) , \tag{7}$$

and their linear combinations,

$$\hat{J}_\alpha = \frac{1}{2}(\hat{L}_\alpha + \hat{P}_\alpha) ,$$
$$\hat{S}_\alpha = \frac{1}{2}(\hat{L}_\alpha - \hat{P}_\alpha) . \tag{8}$$

Then, one can separate the $SO(4)$ Lie algebra $\{\hat{M}^{ij}\}$ into two parts, which commutates with each other

$$[\hat{J}_\alpha, \hat{S}_\beta] = 0 ,$$
$$[\hat{J}_\alpha, \hat{J}_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}\hat{J}_\gamma ,$$
$$[\hat{S}_\alpha, \hat{S}_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}\hat{S}_\gamma , \tag{9}$$
$$\frac{1}{2}\hat{M}^{ij}\hat{M}_{ij} = \hat{P}^2 + \hat{L}^2 = 2(\hat{J}^2 + \hat{S}^2) .$$

In other words, the $SO(4)$ group can be expressed locally as a direct product of two $SO(3)$ groups. Nevertheless, it is easy to see that,

$$\hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{P}} = 0 ,$$
$$\hat{J}^2 = \hat{S}^2 . \tag{10}$$

And then, we obtain the representation of the $SO(4)$ universal Lie algebra (8)

$$\hat{J}^2|j,j_3,s_3\rangle = j(j+1)|j,j_3,s_3\rangle , \tag{11}$$

$$\hat{S}^2|j,j_3,s_3\rangle = j(j+1)|j,j_3,s_3\rangle , \tag{12}$$

$$\hat{J}_3|j,j_3,s_3\rangle = j_3|j,j_3,s_3\rangle , \tag{13}$$

$$\hat{S}_3|j,j_3,s_3\rangle = s_3|j,j_3,s_3\rangle , \tag{14}$$

$$\frac{1}{2}\hat{M}^{ij}\hat{M}_{ij}|j,j_3,s_3\rangle = 4j(j+1)|j,j_3,s_3\rangle ,$$

where

$$j = 0,1,2,\cdots,$$
$$j_3 = -j,\cdots,0,\cdots,j,$$
$$s_3 = -j,\cdots,0,\cdots,j .$$

The representation of the translation group $T(1)$ is of the form

$$\hat{H}|\varepsilon\rangle = \varepsilon|\varepsilon\rangle . \tag{15}$$

Now we can write the wave-function A^λ into the form

$$|\varepsilon; j,j_3,s_3;\sigma\rangle , \tag{16}$$

where $\sigma = \pm 1$ denotes the degrees of freedom of spin. The corresponding dispersion relation for this eigenstate is given by Eqs. (5) and (6)

$$\varepsilon^2 = 4j(j+1)R^{-2} . \tag{17}$$

The Hilbert space of a free photon can therefore be parameterized as

$$|j,j_3,s_3;\sigma\rangle . \tag{18}$$

3　Statistical properties of radiation fields

To conveniently study the statistical properties of the radiation fields in a compact space, one should compute the density of state for energy spectrum first. The dispersion relation (17) can be rewritten into the following form

$$\varepsilon^2 = [(2j+1)^2 - 1] R^{-2} = K^2 - R^{-2} , \tag{19}$$

where we have introduced $K \equiv (2j+1)R^{-1}$ for convenience.

For a given energy level ε, the value of K (or j) is determined. There exist $2K^2R^2$ eigen-states at this energy level resulted by summing over the index parameters (j_3, s_3) and σ. If the scale of the radius R is large enough, one can regard the spectrum of energy and K as a continuous distribution. Then, we get a measure of states with respect to the energy spectrum,

$$\mathrm{d}N_\varepsilon = 2R^3 K^2 \mathrm{d}K = 2R^3(\varepsilon^2 + R^{-2})^{1/2}\varepsilon\mathrm{d}\varepsilon .$$

We know that the volume of the 3-dimensional sphere with radius R is $2\pi^2 R^3$. Hence the density of state is

$$f(\varepsilon) = \frac{2R^3(\varepsilon^2 + R^{-2})^{1/2}\varepsilon}{V} = \frac{(\varepsilon^2 + R^{-2})^{1/2}\varepsilon}{\pi^2} . \tag{20}$$

This formula is obviously different from that of the ordinary black-body radiation system in free space, which is proportional to the square of energy (ε^2).

The photons obey the Bose-Einstein statistics. The mean number of photons on an eigen-state with energy ε is[17]

$$\frac{1}{\mathrm{e}^{\varepsilon/T} - 1} ,$$

where T is the temperature of this system[1]. Immediately, we obtain a modified Planck distribution for the radiation fields in the compact space S^3,

$$\frac{\mathrm{d}U}{V} = \frac{\varepsilon\mathrm{d}N_\varepsilon}{V(\mathrm{e}^{\varepsilon/T} - 1)} = \frac{(\varepsilon^2 + R^{-2})^{1/2}\varepsilon^2}{\pi^2(\mathrm{e}^{\varepsilon/T} - 1)}\mathrm{d}\varepsilon. \tag{21}$$

The distribution function represents the amount of radiation energy in the spectral interval $\varepsilon \sim (\varepsilon + \mathrm{d}\varepsilon)$ and a unit volume. In Fig. 1, we give a sketch for this modified Planck distribution comparing with the ordinary one in free space.

The total number of photons of the radiation system can be expressed as

$$N = V\int_0^\infty \frac{f(\varepsilon)}{\mathrm{e}^{\varepsilon/T} - 1}\mathrm{d}\varepsilon =$$
$$2R^3 T^3 \int_0^\infty \frac{(x^2 + (RT)^{-2})^{1/2}x}{\mathrm{e}^x - 1}\mathrm{d}x. \tag{22}$$

When the parameter RT runs to infinity, we recover the familiar result of the black-body radiation system

---

1) For simplicity, we have taken the chemical potential μ to be zero and hence $F = \Omega + \mu N = \Omega$

in free space. In the following, we will calculate quantities by expanding them in terms of $(RT)^{-1}$ to see clearly the deviation from those got in free space.
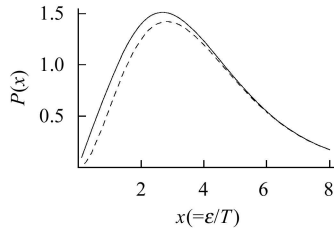

Fig. 1. Planck distribution of radiation fields on S^3 (real line, $RT = 1$) and the ordinary distribution of a black-body radiation system in free space (dashed line).

The thermodynamic quantity of total energy is

$$
U = V \int_0^\infty \frac{f(\varepsilon)\varepsilon}{e^{\varepsilon/T} - 1} d\varepsilon \simeq
12\zeta(4)R^3T^4 + \zeta(2)RT^2 , \tag{23}
$$

where $\zeta(m)$ $(\equiv \sum_{n=1}^{\infty} 1/n^m)$ is the Riemann zeta-function.

The free energy and thermodynamic potential can also be calculated easily,

$$
\begin{aligned}
F = \Omega = -PV = \\
VT \int_0^\infty f(\varepsilon) \log(1 - e^{-\varepsilon/T}) d\varepsilon \simeq \\
-(4\zeta(4)R^3T^4 + \zeta(2)RT^2) .
\end{aligned} \tag{24}
$$

Hence the radiation pressure is

$$
P = -\frac{F}{V} \simeq \frac{1}{2\pi^2} (4\zeta(4)T^4 + \zeta(2)R^{-2}T^2) . \tag{25}
$$

Furthermore, by making use of the relations among thermodynamic quantities

$$
U = F + TS \quad \text{or} \quad dF = -SdT - PdV , \tag{26}
$$

we can obtain the entropy of radiation fields

$$
S \simeq 16\zeta(4)R^3T^3 + 2\zeta(2)RT . \tag{27}
$$

In an adiabatic process experienced by the radiation fields, the entropy doesn't change. So the product RT can be viewed as a constant in the process. The total number of photons is also kept invariant in the adiabatic process. But the other thermodynamic quantities will vary with the geometrical parameter R as follows,

$$
U, F, \Omega \propto R^{-1}, \quad P \propto R^{-4} . \tag{28}
$$

To discuss the equation of state of this radiation system, we now calculate the density of thermal energy,

$$
\rho = \frac{U}{V} \simeq \frac{1}{2\pi^2} (12\zeta(4)T^4 + \zeta(2)R^{-2}T^2) . \tag{29}
$$

The equation of state of the radiation fields in the compact space can be read out from Eqs. (25) and (29),

$$
P \simeq \frac{\rho}{3} + \frac{\zeta(2)}{3\pi\sqrt{6\zeta(4)}R^2} \rho^{1/2} . \tag{30}
$$

It is obvious that, in the case of a large radius of the compact space, the ordinary equation of state for the radiation fields in free space can be recovered.

4　Conclusions

A good understanding of the resonant effect and spontaneous emission rate difference in cavity requires first to know the full space dependence of the cavity-induced damping and level shifts. However, we know that it is a really difficult mathematical task. Thus, many useful approximation methods have been developed. As the first step on the way of getting an exact solution of the problem, we suggest working on a compact space of finite size instead of a cavity. By making use of the representation theory of the $T(1) \times SO(4)$ group, we study the kinematics for a free photon propagating in the space-time $\mathbb{R}^1 \times S^3$. The explicit geometrical parameter dependence of statistical properties of the radiation fields is presented. The results show remarkable differences from those of the black-body radiation in free space.

The resonant effects related to the coupling of atoms with strong radiofrequency field as well as the spontaneous emission rate can also be computed in a straightforward way. It is expected that the non-linear character of the coupled atom-radiation field may be attributed to the geometrical parameters of the compact space. These results will be published in a forthcoming paper.

In fact, the statistical properties of radiation fields in a compact space are crucial to the evolution of the early universe. In the early universe, radiation field is the dominant matter. A bit difference of statistical properties of the radiation system can generate observable effects on our present cosmology. All of

these studies will appear in our next work.

One of us (C.B.G) would like to thank Dr. Yu-

Qi Li and Xiao-Jun Wang for useful discussion and suggestions.

References

1 Meschede D, Walther H, Müller G. Phys. Rev. Lett., 1985, **54**: 551

2 Rempe G, Walther H, Klein N. Phys. Rev. Lett., 1987, **58**: 353

3 Filipowicz P, Javanainen J, Meystre P. Phys. Rev., 1986, **A34**: 3077

4 Lugiato L A, Scully M O, Walther H. Phys. Rev., 1987, **A36**: 740

5 Winter J M. Ann. Phys. (Paris), 1959, **4**: 745

6 Drexhage K H. In: Progress in Optics XII. Edited by Wolf E. Amsterdam: North-Holland Publishing Comp., 1974. 165

7 Kleppner D. Phys. Rev. Lett., 1981, **47**: 233

8 Goy P, Raimond J M, Gross M et al. Phys. Rev. Lett., 1983, **50**: 1903

9 ZHU Yi-Fu, Lezama A, Mossberg T W et al. Phys. Rev. Lett., 1988, **61**: 1946

10 Morin S E, YU C C, Mossberg T W. Phys. Rev. Lett., 1994, **73**: 1489

11 Penzias A A, Wilson R W. Astrophys. J., 1965, **142**: 419

12 Bennett C L et al. Astrophys. J. Lett., 1996, **464**: L1

13 Bennett C L et al. Astrophys. J., 2003, **148**(Suppl.): 1

14 Friedmann A. Z. Phys., 1922, **10**: 377; Z. Phys., 1924, **21**: 326; Robertson H P. Astrophys. J., 1935, **82**: 284; Astrophys. J., 1936, **83**: 187, 257; Walker A G. Proc. Lond. Math. Soc., 1936, **42**(2): 90

15 Weinberg S. Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. New York: John Wiley & Sons, 1972

16 Thomas L H. Ann. of Math., 1941, **42**: 113

17 Landau L D, Lifshitz E M. Statistical Physics, Part 1. 3rd Edition. Trans. by Sykes J B, Kearsley M J. Butterworth-Heinemann, Beijing World Publishing Corp., 1999

紧致空间中辐射场的统计性质[*]

常哲[1;1)]   关成波[2;2)]

1 (中国科学院高能物理研究所   北京   100049)
2 (东南大学物理系   南京   210096)

摘要   通过对有限紧致空间中辐射场的研究来讨论腔中的原子-辐射场耦合系统. 利用 $T(1) \times SO(4)$ 群的表示，给出了辐射场的单粒子波函数以及相应的色散关系. 由此详细讨论了紧致空间中辐射场的玻色-爱因斯坦统计. 发现其性质与空间的几何性质 (曲率半径) 有显著的依赖关系，并表现与通常黑体辐射系统的显著差异.

关键词   辐射场   紧致流形   腔量子电动力学