Lie algebraic analysis for the nonlinear transport of intense bunched beam in electrostatic quadrupoles*

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Abstract In this paper, the nonlinear transport of intense bunched beams in electrostatic quadrupoles is analyzed using the Lie algebraic method, and the results are briefly presented of the linear matrix approximation and the second order correction of particle trajectory in the state space. Beam having K-V distribution and Gaussian distribution approximation are respectively considered. A brief discussion is also given of the total effects of the quadrupole and the space charge forces on the evolution of the beam envelope.

Key words electrostatic quadrupole, space charge effect, Lie algebraic, nonlinearity

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1 Introduction

Lie algebraic method[1] has been successfully implemented in accelerator study. Until now, it has not taken into account the nonlinear space charge effects. One commonly used distribution which causes nonlinear space charge effect is Gaussian distribution. In this paper, we present the polynomials approximation of particle trajectories to the second order. We first treat the K-V distribution. That is take the space-charge forces to be linear, and postulate a beam having a uniform charge distribution in the ellipsoid of bunched beam in real space. Then we consider the case of Gaussian distribution.

2 Hamiltonian and expansion

Let us consider the case of a perfect electrostatic quadrupole of length L and employ the Cartesian coordinates. The relativistic Hamiltonian for the motion of a particle of rest mass m_0 and charge q in the electromagnetic field is given by the expression[2]

$$H=\left(m_0^2c^4+c^2p_x^2+c^2p_y^2+c^2p_z^2\right)^{1/2}+q\Psi\,. \quad (1)$$

Here, x and y denote the two coordinates perpendicular to the design trajectory, along which is z. p_x, p_y, and p_z are the canonical momenta. Ψ is the electric potential, which is a sum of the external potential $\Psi_{\rm e}$ and the potential excited by the beam itself $\Psi_{\rm s}$. For the beam having a K-V distribution, $\Psi_{\rm e}$ and $\Psi_{\rm s}$ are given by

$$\Psi_{\rm e}=\frac{V}{r_0^2}\left(x^2-y^2\right)\,. \quad (2)$$

$$\Psi_{\rm s}=-U\left(\mu_x x^2+\mu_y y^2+\mu_z z_r^2\right)\,. \quad (3)$$

Here V denotes the potential of the electrode and r_0 is the inner radius of the electrostatic quadrupole. U is defined as

$$U=\frac{3IT_{\rm rf}}{8\pi\varepsilon_0\gamma_0 XYZ}\,. \quad (4)$$

I is the average beam current. $T_{\rm rf}$ is the period of the beam pulses. X, Y and Z are the beam dimensions. z_r is the relative longitudinal position of arbitrary particle to the reference particle and is defined by

$$z_r=z-v_0t\,. \quad (5)$$

v_0 is the velocity of reference particle. γ_0 is

$$\gamma_0=\frac{1}{\sqrt{1-\beta_0^2}}\,,\qquad \beta_0=\frac{v_0}{c}\,. \quad (6)$$

μ_x, μ_y and μ_z are the factors related to the shape of

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the beam, and

$$\mu_x = \frac{XYZ\gamma_0}{2}\times \int_0^\infty \frac{1}{(X^2+\xi)\sqrt{(X^2+\xi)(Y^2+\xi)(Z^2\gamma_0^2+\xi)}}\mathrm{d}\xi\,,$$
$$\mu_y = \frac{XYZ\gamma_0}{2}\times \int_0^\infty \frac{1}{(Y^2+\xi)\sqrt{(X^2+\xi)(Y^2+\xi)(Z^2\gamma_0^2+\xi)}}\mathrm{d}\xi\,,$$
$$\mu_z = \frac{XYZ\gamma_0}{2}\times \int_0^\infty \frac{1}{(Z^2\gamma_0^2+\xi)\sqrt{(X^2+\xi)(Y^2+\xi)(Z^2\gamma_0^2+\xi)}}\mathrm{d}\xi\,. \tag{7}$$

Define p_t by writing

$$p_t = -H_t(t,x,y,z,p_x,p_y,p_z)\,. \tag{8}$$

Solving p_z from Eq. (8), one obtains

$$p_z = -K(z,x,y,t,p_x,p_y,p_t)\,. \tag{9}$$

where K is the Hamiltonian with the axis z as an independent variable

$$K = -\frac{1}{c}\sqrt{-m_0^2c^4 - c^2\left(p_x^2+p_y^2\right) + \left(p_t + q\left(\psi_\mathrm{e}+\psi_\mathrm{s}\right)\right)^2}\,. \tag{10}$$

The design orbit passes through the center of the quadrupole and has certain design energy, which can be characterized by writing the equations

$$\begin{aligned} p_x &= 0, x = 0,\\ p_y &= 0, y = 0,\\ p_t &= p_t^0, t(z) = \frac{z}{v_0}. \end{aligned} \tag{11}$$

p_t^0 is a constant value, which is the value of p_t for the reference particle,

$$p_t^0 = -H_t\big|_{\text{reference porbit}} = -\gamma_0 m_0 c^2\,. \tag{12}$$

According to Eq. (11), following the Hamiltonian flow generated by K along the design orbit does not lead to an analytical map. Define “new” variables τ, x, y, p_τ, p_x, $p_y$by the relation

$$\begin{aligned} p_x &= p_x, x = x,\\ p_y &= p_y, y = y,\\ p_t &= p_\tau + p_t^0, t = \tau + \frac{z}{v_0}. \end{aligned} \tag{13}$$

This change of variables is a canonical transformation arising from the transformation function

$$F_2 = xp_x + yp_y + \left(t - \frac{z}{v_0}\right)(p_\tau + p_t^0)\,. \tag{14}$$

In terms of these new variables, the design orbit can be taken to be given by the equations

$$\tau = x = y = p_\tau = p_x = p_y = 0\,. \tag{15}$$

The variables τ, x, y, and their canonical momenta, are measured as the deviation from the design trajectory. Let H denote the Hamiltonian for the new variables. Then one has the relation

$$H = K + \frac{\partial F_2}{\partial z}\,. \tag{16}$$

Carrying out the prescription Eq. (16), one finds the result

$$H = -\frac{p_\tau + p_t^0}{c\beta_0} - \frac{1}{c}\sqrt{-m_0^2c^4 - c^2\left(p_x^2+p_y^2\right) + \left(p_\tau + p_t^0 + q\left(\psi_\mathrm{e}+\psi_\mathrm{s}\right)\right)^2}\,. \tag{17}$$

Expanding the Hamiltonian H into Taylor series, one can find for the first few polynomials the results

$$\begin{aligned} H_0 &= -p_0 - \frac{p_t^0}{\beta_0 c},\\ H_1 &= 0,\\ H_2 &= \frac{p_\tau^2}{2p_0\beta_0^2\gamma_0^2c^2} + \frac{p_x^2}{2p_0} + \frac{p_y^2}{2p_0} + x^2\frac{p_0k_x^2}{2} - y^2\frac{p_0k_y^2}{2} - \tau^2\frac{p_0\gamma_0^2\beta_0^2c^2k_\tau^2}{2},\\ H_3 &= \frac{p_\tau^3}{2p_0^2\gamma_0^2\beta_0^3c^3} + \frac{p_x^2p_\tau}{2p_0^2\beta_0c} + \frac{p_y^2p_\tau}{2p_0^2\beta_0c} + \frac{x^2p_\tau k_x^2}{2\gamma_0^2\beta_0c} - \frac{y^2p_\tau k_y^2}{2\gamma_0^2\beta_0c} - \tau^2p_\tau\frac{\beta_0ck_\tau^2}{2}. \end{aligned} \tag{18}$$

Here p_0 denotes the magnitude of the design relativistic mechanical momentum

$$p_0 = \gamma_0 m_0\beta_0 c\,. \tag{19}$$

The parameters k_x, k_y, and k_τ are defined by

$$\begin{aligned} k_x^2 &= \frac{2q(V - U\mu_x r_0^2)}{\gamma_0 m_0\beta_0^2c^2r_0^2},\\ k_y^2 &= \frac{2q(V + U\mu_y r_0^2)}{\gamma_0 m_0\beta_0^2c^2r_0^2},\\ k_\tau^2 &= \frac{2qU\mu_z}{\gamma_0^3m_0\beta_0^2c^2}. \end{aligned} \tag{20}$$

For the Gaussian distribution beam, Ψ_s can be expressed by[3]

$$\psi_\mathrm{s} = \frac{IT_\mathrm{rf}}{8\pi^{3/2}\varepsilon_0}\times \int_0^\infty \frac{\exp\left[-\left(\frac{x^2}{2X^2+\xi} + \frac{y^2}{2Y^2+\xi} + \frac{z^2\gamma_0^2}{2Z^2\gamma_0^2+\xi}\right)\right]}{\sqrt{(2X^2+\xi)(2Y^2+\xi)(2Z^2\gamma_0^2+\xi)}}\mathrm{d}\xi\,. \tag{21}$$

p_t^0 is

$$p_t^0 = -H_t\big|_{\text{reference porbit}} = -\gamma_0 m_0 c^2 - \frac{IqT_{\text{rf}} F_{\text{elliptic}}\left(\arcsin\left(\frac{\sqrt{-X^2+Z^2\gamma_0^2}}{Z\gamma_0}\right), \frac{Y^2-Z^2\gamma_0^2}{X^2-Z^2\gamma_0^2}\right)}{4\sqrt{2}\pi^{\frac{3}{2}}\varepsilon_0\sqrt{-X^2+Z^2\gamma_0^2}}. \quad (22)$$

The Hamiltonian has the same form of Eq. (17) and the same form of expansion as Eq. (18) but different k_x, k_y, k_τ.

$$k_x^2 = \frac{2qV}{p_0\beta_0 c\gamma_0^2} + \frac{IqT_{\text{rf}}\left((X^2-Y^2)Z\gamma_0 + XY\sqrt{Y^2-Z^2\gamma_0^2}E_{\text{elliptic}}\left(\arcsin\left(\frac{\sqrt{Y^2-Z^2\gamma_0^2}}{Y}\right), \frac{-X^2+Y^2}{Y^2-Z^2\gamma_0^2}\right)\right)}{4\sqrt{2}\pi^{\frac{3}{2}}\varepsilon_0 p_0\beta_0 cXY(X^2-Y^2)(X^2-Z^2\gamma_0^2)} -$$

$$\frac{IqT_{\text{rf}}\left(XY\sqrt{X^2-Z^2\gamma_0^2}F_{\text{elliptic}}\left(\arcsin\left(\frac{\sqrt{X^2-Z^2\gamma_0^2}}{X}\right), \frac{X^2-Y^2}{X^2-Z^2\gamma_0^2}\right)\right)}{4\sqrt{2}\pi^{\frac{3}{2}}\varepsilon_0 p_0\beta_0 cXY(X^2-Y^2)(X^2-Z^2\gamma_0^2)},$$

$$k_y^2 = \frac{2qV}{p_0\beta_0 c\gamma_0^2} - \frac{IqT_{\text{rf}}\left(\sqrt{-X^2+Y^2}Z\gamma_0 - \mathrm{i}XYE_{\text{elliptic}}\left(\arcsin\left(\frac{\sqrt{X^2-Y^2}}{X}\right), \frac{X^2-Z^2\gamma_0^2}{X^2-Y^2}\right)\right)}{4\sqrt{2}\pi^{\frac{3}{2}}\varepsilon_0 p_0\beta_0 cXY\sqrt{-X^2+Y^2}(Y^2-Z^2\gamma_0^2)}, \quad (23)$$

$$k_\tau^2 = \frac{IqT_{\text{rf}}}{p_0\gamma_0\beta_0 c}\left(\frac{Y\sqrt{-X^2+Y^2}(X^2-Z^2\gamma_0^2) - \mathrm{i}XZ\gamma_0(X^2-Y^2)E_{\text{elliptic}}\left(\arcsin\left(\frac{\sqrt{X^2-Y^2}}{X}\right), \frac{X^2-Z^2\gamma_0^2}{X^2-Y^2}\right)}{4\sqrt{2}\pi^{\frac{3}{2}}\varepsilon_0 XZ\sqrt{-X^2+Y^2}(X^2-Z^2\gamma_0^2)(Y^2-Z^2\gamma_0^2)} -\right.$$

$$\left.\frac{\mathrm{i}XZ\gamma_0(Y^2-Z^2\gamma_0^2)F_{\text{elliptic}}\left(\arcsin\left(\frac{\sqrt{X^2-Y^2}}{X}\right), \frac{X^2-Z^2\gamma_0^2}{X^2-Y^2}\right)}{4\sqrt{2}\pi^{\frac{3}{2}}\varepsilon_0 XZ\sqrt{-X^2+Y^2}(X^2-Z^2\gamma_0^2)(Y^2-Z^2\gamma_0^2)}\right).$$

3 Lie map and factorization

The mapping is given by the expression

$$M = \exp\left(:-\int_{z_0}^{z} H\mathrm{d}z:\right). \quad (24)$$

Inserting the expansion into the expression (24) and imagining that the result is written in factored product form, the map can be written as

$$M = \exp\left(-\int_0^z :H_2:+:H_3:+\cdots\mathrm{d}z\right) = \cdots\exp(:f_3:)\exp(:f_2:). \quad (25)$$

Here f_2 and f_3 can be given by the expression[4]

$$f_2 = -\int_{z_0}^{z} H_2\mathrm{d}z, \quad f_3 = -\int_{z_0}^{z} H_3^{\text{int}}\mathrm{d}z. \quad (26)$$

Here H_3^{int} is defined as

$$H_3^{\text{int}}(z) = H_3(M_2 z). \quad (27)$$

One can obtain linear and the second order approximation of the final coordinates by

$$\xi_1 = \exp(:f_2:)\xi, \quad \xi_2 =:f_3:\xi_1. \quad (28)$$

4 Particle trajectory

Let the particle transport through a small segment of l in the z direction. The first factor acts on the variables and gives the linear matrix approximation. If $k_x^2 > 0$, the linear matrix transforms the variables according to the rule

$$\begin{bmatrix} x_1 \\ p_{x_1} \\ y_1 \\ p_{y_1} \\ \tau_1 \\ p_{\tau_1} \end{bmatrix} = \begin{bmatrix} \cos(k_x l) & \frac{\sin(k_x l)}{p_0 k_x} & 0 & 0 & 0 & 0 \\ -p_0 k_x \sin(k_x l) & \cos(k_x l) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(k_y l) & \frac{\sinh(k_y l)}{p_0 k_y} & 0 & 0 \\ 0 & 0 & p_0 k_y \sinh(k_y l) & \cosh(k_y l) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cosh(k_\tau l) & \frac{\sinh(k_\tau l)}{\gamma_0^2\beta_0^2 c^2 p_0 k_\tau} \\ 0 & 0 & 0 & 0 & \gamma_0^2\beta_0^2 c^2 p_0 k_\tau \sinh(k_\tau l) & \cosh(k_\tau l) \end{bmatrix} \begin{bmatrix} x \\ p_x \\ y \\ p_y \\ \tau \\ p_\tau \end{bmatrix}. \quad (29)$$

Physically, they describe horizontal focusing, vertical defocusing, and longitudinal separating action of intense beam through quadrupole in linear matrix approximation.

The second order correction can be obtained according to Eq. (21—23) and the formulas are

$$\begin{aligned}
x_2 = & \; x\tau\Bigg\{\beta_0 c k_x\Bigg\{\frac{[k_\tau^2+2k_x^2(1+\gamma_0^2)]\sin(k_x l)-[k_\tau^2\gamma_0^2+2k_x^2(1+\gamma_0^2)]\sin(k_x l)\cosh(k_\tau l)}{4k_x^2+k_\tau^2}+ \\
& \frac{k_x k_\tau(-1+\gamma_0^2)\cos(k_x l)\sinh(k_\tau l)}{4k_x^2+k_\tau^2}\Bigg\}\Bigg\}- \\
& x p_\tau\Bigg\{k_x\frac{[k_\tau^2\gamma_0^2+2k_x^2(1+\gamma_0^2)]\sin(k_x l)\sinh(k_\tau l)-k_x k_\tau(-1+\gamma_0^2)\cos(k_x l)[\cosh(k_\tau l)-1]}{k_\tau(4k_x^2+k_\tau^2)p_0\beta_0 c\gamma_0^2}\Bigg\}+ \\
& p_x\tau\Bigg\{\beta_0 c\frac{[k_\tau^2\gamma_0^2+2k_x^2(1+\gamma_0^2)]\cos(k_x l)[\cosh(k_\tau l)-1]+k_x k_\tau(-1+\gamma_0^2)\sin(k_x l)\sinh(k_\tau l)}{(4k_x^2+k_\tau^2)p_0}\Bigg\}+ \\
& p_x p_\tau\Bigg\{\frac{[k_\tau^2\gamma_0^2+2k_x^2(1+\gamma_0^2)]\cos(k_x l)\sinh(k_\tau l)+k_x k_\tau(-1+\gamma_0^2)\sin(k_x l)[\cosh(k_\tau l)+1]}{k_\tau(4k_x^2+k_\tau^2)p_0^2\beta_0 c\gamma_0^2}\Bigg\},
\end{aligned} \tag{30}$$

$$\begin{aligned}
p_{x_2} = & -x\tau\Bigg\{p_0\beta_0 c k_x^2\frac{[k_\tau^2+2k_x^2(1+\gamma_0^2)]\cos(k_x l)[\cosh(k_\tau l)-1]-k_x k_\tau(-1+\gamma_0^2)\sin(k_x l)\sinh(k_\tau l)}{4k_x^2+k_\tau^2}\Bigg\}- \\
& x p_\tau\Bigg\{k_x^2\frac{[k_\tau^2+2k_x^2(1+\gamma_0^2)]\cos(k_x l)\sinh(k_\tau l)-k_x k_\tau(-1+\gamma_0^2)\sin(k_x l)[\cosh(k_\tau l)+1]}{(4k_x^2+k_\tau^2)k_\tau\beta_0 c\gamma_0^2}\Bigg\}- \\
& p_x\tau\Bigg\{\beta_0 c k_x\Bigg\{\frac{-[k_\tau^2\gamma_0^2+2k_x^2(1+\gamma_0^2)]\sin(k_x l)+[k_\tau^2+2k_x^2(1+\gamma_0^2)]\sin(k_x l)\cosh(k_\tau l)}{4k_x^2+k_\tau^2}+ \\
& \frac{k_x k_\tau(-1+\gamma_0^2)\cos(k_x l)\sinh(k_\tau l)}{4k_x^2+k_\tau^2}\Bigg\}\Bigg\}- \\
& p_x p_\tau\Bigg\{k_x\frac{[k_\tau^2+2k_x^2(1+\gamma_0^2)]\sin(k_x l)\sinh(k_\tau l)+k_x k_\tau(-1+\gamma_0^2)\cos(k_x l)[\cosh(k_\tau l)-1]}{k_\tau(4k_x^2+k_\tau^2)p_0\beta_0 c\gamma_0^2}\Bigg\},
\end{aligned} \tag{31}$$

$$\begin{aligned}
y_2 = & \; y\tau\Bigg\{\beta_0 c k_y\Bigg\{\frac{[k_\tau^2-2k_y^2(1+\gamma_0^2)]\sinh(k_y l)+[-k_\tau^2\gamma_0^2+2k_y^2(1+\gamma_0^2)]\sinh(k_y l)\cosh(k_\tau l)}{4k_y^2-k_\tau^2}+ \\
& \frac{k_y k_\tau(-1+\gamma_0^2)\cosh(k_y l)\sinh(k_\tau l)}{4k_y^2-k_\tau^2}\Bigg\}\Bigg\}+ \\
& y p_\tau\Bigg\{k_y\frac{[-k_\tau^2\gamma_0^2+2k_y^2(1+\gamma_0^2)]\sinh(k_y l)\sinh(k_\tau l)+k_y k_\tau(-1+\gamma_0^2)\cosh(k_y l)[\cosh(k_\tau l)-1]}{k_\tau(4k_y^2-k_\tau^2)p_0\beta_0 c\gamma_0^2}\Bigg\}+ \\
& p_y\tau\Bigg\{\beta_0 c\frac{[-k_\tau^2\gamma_0^2+2k_y^2(1+\gamma_0^2)]\cosh(k_y l)[\cosh(k_\tau l)-1]+k_y k_\tau(-1+\gamma_0^2)\sinh(k_y l)\sinh(k_\tau l)}{(4k_y^2-k_\tau^2)p_0}\Bigg\}+ \\
& p_y p_\tau\Bigg\{\frac{[-k_\tau^2\gamma_0^2+2k_y^2(1+\gamma_0^2)]\cosh(k_y l)\sinh(k_\tau l)+k_y k_\tau(-1+\gamma_0^2)\sinh(k_y l)[\cosh(k_\tau l)+1]}{k_\tau(4k_y^2-k_\tau^2)p_0^2\beta_0 c\gamma_0^2}\Bigg\},
\end{aligned} \tag{32}$$

$$p_{y_2} = y\tau\left\{p_0\beta_0 ck_y^2\frac{[-k_\tau^2+2k_y^2(1+\gamma_0^2)]\cosh(k_yl)[\cosh(k_\tau l)-1]-k_yk_\tau(-1+\gamma_0^2)\sinh(k_yl)\sinh(k_\tau l)}{4k_y^2-k_\tau^2}\right\}+$$
$$yp_\tau\left\{k_y^2\frac{[-k_\tau^2+2k_y^2(1+\gamma_0^2)]\cosh(k_yl)\sinh(k_\tau l)-k_yk_\tau(-1+\gamma_0^2)\sinh(k_yl)[\cosh(k_\tau l)+1]}{(4k_y^2-k_\tau^2)k_\tau\beta_0 c\gamma_0^2}\right\}+$$
$$p_y\tau\left\{\beta_0 ck_y\left\{\frac{[k_\tau^2\gamma_0^2-2k_y^2(1+\gamma_0^2)]\sinh(k_yl)+[-k_\tau^2+2k_y^2(1+\gamma_0^2)]\sinh(k_yl)\cosh(k_\tau l)}{4k_y^2-k_\tau^2}-\right.\right.$$
$$\left.\left.\frac{k_yk_\tau(-1+\gamma_0^2)\cosh(k_yl)\sinh(k_\tau l)}{4k_y^2-k_\tau^2}\right\}\right\}+$$
$$p_yp_\tau\left\{k_y\frac{[-k_\tau^2+2k_y^2(1+\gamma_0^2)]\sinh(k_yl)\sinh(k_\tau l)-k_yk_\tau(-1+\gamma_0^2)\cosh(k_yl)[\cosh(k_\tau l)-1]}{k_\tau(4k_y^2-k_\tau^2)p_0\beta_0 c\gamma_0^2}\right\}, \tag{33}$$

$$\tau_2 = x^2\left\{k_x^2\frac{-k_xk_\tau(-1+\gamma_0^2)\sin(2k_xl)+(k_\tau^2+2k_x^2(1+\gamma_0^2))\sinh(k_\tau l)}{2k_\tau(4k_x^2+k_\tau^2)\beta_0 c\gamma_0^2}\right\}+$$
$$p_x^2\left\{\frac{k_xk_\tau(-1+\gamma_0^2)\sin(2k_xl)+(k_\tau^2\gamma_0^2+2k_x^2(1+\gamma_0^2))\sinh(k_\tau l)}{2k_\tau(4k_x^2+k_\tau^2)p_0^2\beta_0 c\gamma_0^2}\right\}+$$
$$y^2\left\{-k_y^2\frac{-k_yk_\tau(-1+\gamma_0^2)\sinh(2k_yl)+[-k_\tau^2+2k_y^2(1+\gamma_0^2)\sinh(k_\tau l)]}{2k_\tau(4k_y^2-k_\tau^2)\beta_0 c\gamma_0^2}\right\}+$$
$$p_y^2\left\{\frac{k_yk_\tau(-1+\gamma_0^2)\sinh(2k_yl)+(-k_\tau^2\gamma_0^2+2k_y^2(1+\gamma_0^2))\sinh(k_\tau l)}{2k_\tau(4k_y^2-k_\tau^2)p_0^2\beta_0 c\gamma_0^2}\right\}+$$
$$xp_x\left\{k_x^2\frac{(-1+\gamma_0^2)[\cos(2k_xl)-\cosh(k_\tau l)]}{(4k_x^2+k_\tau^2)p_0\beta_0 c\gamma_0^2}\right\}+yp_y\left\{k_y^2\frac{(-1+\gamma_0^2)[\cosh(2k_yl)-\cosh(k_\tau l)]}{(4k_y^2-k_\tau^2)p_0\beta_0 c\gamma_0^2}\right\}+$$
$$p_\tau^2\left\{\frac{\sinh(k_\tau l)+\sinh(2k_\tau l)}{2k_\tau p_0^2\beta_0^3c^3\gamma_0^2}\right\}+\tau^2\left\{k_\tau\beta_0 c\frac{\gamma_0^2[-1-2\gamma_0^2+2\gamma_0^2\cosh(k_\tau l)]\sinh(k_\tau l)}{2}\right\}+$$
$$p_\tau\tau\left\{\frac{[-1+\cosh(k_\tau l)][1+2\cosh(k_\tau l)]}{p_0\beta_0 c}\right\}, \tag{34}$$

$$p_{\tau_2} = x^2\left\{k_x^2p_0\beta_0 c\frac{-(4k_x^2+k_\tau^2)(1+\gamma_0^2)+k_\tau^2(-1+\gamma_0^2)\cos(2k_xl)+2[k_\tau^2+2k_x^2(1+\gamma_0^2)]\cosh(k_\tau l)}{4(4k_x^2+k_\tau^2)}\right\}+$$
$$p_x^2\left\{\beta_0 c\frac{-(4k_x^2+k_\tau^2)(1+\gamma_0^2)-k_\tau^2(-1+\gamma_0^2)\cos(2k_xl)+2[k_\tau^2\gamma_0^2+2k_x^2(1+\gamma_0^2)]\cosh(k_\tau l)}{4(4k_x^2+k_\tau^2)p_0}\right\}+$$
$$y^2\left\{-k_y^2p_0\beta_0 c\frac{-(4k_y^2-k_\tau^2)(1+\gamma_0^2)-k_\tau^2(-1+\gamma_0^2)\cosh(2k_yl)+2[-k_\tau^2+2k_y^2(1+\gamma_0^2)]\cosh(k_\tau l)}{4(4k_y^2-k_\tau^2)}\right\}+$$
$$p_y^2\left\{\beta_0 c\frac{-(4k_y^2-k_\tau^2)(1+\gamma_0^2)+k_\tau^2(-1+\gamma_0^2)\cosh(2k_yl)+2[-k_\tau^2\gamma_0^2+2k_y^2(1+\gamma^2)]\cosh(k_\tau l)}{4(4k_y^2-k_\tau^2)p_0}\right\}+$$
$$p_xx\left\{-k_xk_\tau\beta_0 c\frac{(-1+\gamma_0^2)[-k_\tau\sin(2k_xl)+2k_x\sinh(k_\tau l)]}{2(4k_x^2+k_\tau^2)}\right\}+$$
$$p_yy\left\{-k_yk_\tau\beta_0 c\frac{(-1+\gamma_0^2)[-k_\tau\sinh(2k_yl)+2k_y\sinh(k_\tau l)]}{2(4k_y^2-k_\tau^2)}\right\}+$$
$$\tau^2\left\{k_\tau^2p_0\beta_0^3c^3\gamma_0^2\frac{[-\gamma_0^2+(1+\gamma_0^2)\cosh(k_\tau l)][-1+2\cosh(k_\tau l)]}{2}\right\}+$$
$$p_\tau^2\left\{\frac{[-1+\cosh(k_\tau l)][1+2\gamma_0^2+(1+\gamma_0^2)\cosh(k_\tau l)]}{2p_0\beta_0 c\gamma_0^2}\right\}+p_\tau\tau\left\{k_\tau\beta_0 c[-\gamma_0^2+(1+\gamma_0^2)\cosh(k_\tau l)]\sinh(k_\tau l)\right\}. \tag{35}$$

The third and higher order correction can be calculated similarly.

5 Discussion

If $k_x^2 < 0$, to get M_2, just substitute k_x with $\mathrm{i}k_x$. In this case, the particle coordinates related to the reference particle will have an exponential increase with the growth of transport distance.

Just as expected, when the beam current is low, the effects of the quadrupole are prominent and the total effects are focusing in the x direction and defocusing in the y direction. With the growth of beam current, the space charge effects and thus the defocusing effects will increase. When the beam current is high enough to exceed the confinement of the quadrupole, the total effects are defocusing both horizontally and vertically. In either case the effects of space charge effects in the longitudinal are to increase the separating distance.

6 Conclusion

According to the calculation, the polynomials approximations of particle trajectory with beam having K-V distribution and Gaussian distribution have the same form but different coefficients. It has also been shown that the effect of space charge effects is defocusing. When the space charge effects exceed the confinement of the quadrupole, the bench will grow up rapidly.

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