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Critical phenomena in a disc-percolation model and their application to relativistic heavy ion collisions*

KE Hong-Wei(柯宏伟)[1] XU Ming-Mei(许明梅)[1,2] LIU Lian-Shou(刘连寿)[1,2;1)]

1 (Institute of Particle Physics, Huazhong Normal University, Wuhan 430079, China)
2 (Key Lab of Quark and Lepton Physics (Huazhong Normal University), Ministry of Education, Wuhan 430079, China)

Abstract By studying the critical phenomena in continuum-percolation of discs, we find a new approach to locate the critical point, i.e. using the inflection point of P_∞ as an evaluation of the percolation threshold. The susceptibility, defined as the derivative of P_∞, possesses a finite-size scaling property, where the scaling exponent is the reciprocal of ν, the critical exponent of the correlation length. A possible application of this approach to the study of the critical phenomena in relativistic heavy ion collisions is discussed. The critical point for deconfinement can be extracted by the inflection point of $P_{\rm QGP}$ — the probability for the event with QGP formation. The finite-size scaling of its derivative can give the critical exponent ν, which is a rare case that can provide an experimental measure of a critical exponent in heavy ion collisions.

Key words relativistic heavy ion collisions, quark deconfinement, critical phenomena, percolation

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1 Introduction

The general feature of the phase diagram of strongly interacting matter has become increasingly well established[1]. Following the region of crossover around a temperature of 170 — 190 MeV at zero baryon chemical potential $\mu_{\rm B}$, increasing $\mu_{\rm B}$ leads to a critical point, beyond which the system shows a first order transition from confined to deconfined phase.

Recently, interest in the QCD critical point progressively arose. With the plan of RHIC low energy scan and the new GSI facility, the study of relativistic heavy ion collisions is concentrating more and more on the search for the critical point and observing relevant critical phenomena, in particular measuring the critical exponents.

On the theoretical side, studying the critical phenomena in chiral symmetry restoration is available[2]. Many discussions about universality have been active[3]. The chiral condensate and Polyakov loop are proposed as order parameters for chiral restoration and color deconfinement, respectively. However, it is a pity that both of these variables can not be directly measured by experiment. What observables should we measure, how to locate the critical point and extract the corresponding critical exponents in heavy ion collisions has not been clear so far.

On the experimental side, what we observed in heavy ion collisions is the deconfined partonic degree of freedom[4]. Despite the fact that confinement is a long standing problem which has not been solved by theory, studying the critical phenomena in deconfinement phase transition is more realistic for experiments. Since chiral symmetry is hard to measure and deconfinement is easier to observe, we propose to phenomenologically study the critical phenomena from the deconfinement aspect and give some hints for experiments.

Principally speaking, it is impossible to get the critical point because of the limited system size in relativistic heavy ion collisions. In this paper, by means of studying the critical phenomena in the finite-size continuum-percolation of discs, we find that P_∞, the probability for an event for which an infinite cluster occurs, has an inflection point, which is a good approximation for the critical point. The finite-size scaling method is further used to extract the critical

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1) E-mail: liuls@iopp.ccnu.edu.cn

exponents ν from the distribution of the susceptibility. Since percolation has some resemblance to deconfinement[5], a possible application of the method used in percolation to the study of the critical phenomena in relativistic heavy ion collisions is discussed.

2 The continuum-percolation of discs

The continuum-percolation problem has various formulations among which the problem of spheres is most popular. Equally sized spheres are placed at random in a substrate. The spheres support the transport and the substrate does not. When we put enough spheres in the substrate, the overlapping spheres form an infinite cluster, and the system is able to support a long-range current. This model has been well studied and has been used to describe hopping conduction in doped semiconductors[6] and phase transitions in ferromagnetics[7]. Such a model, together with its critical behavior, also bears some resemblance to the transition from a hadron gas to a quark-gluon plasma.

In QCD, hadrons are color-singlet bound states of more basic colored objects — quarks and gluons. Hadronic matter, consisting of colorless constituents of hadronic dimensions, can turn at high temperature and/or density to a quark-gluon plasma of colored quarks and gluons as constituents in a much larger volume. This deconfinement transition leads to a color-conducting state and thus is the QCD counterpart of the insulator-conductor transition in atomic matter. Suppose hadrons have an intrinsic size. At low density we have a hadron gas. When this becomes so dense that the formation of an infinite cluster occurs, it turns into a quark-gluon plasma. In this case, the connectivity (cluster formation) determines the different states of many-body systems. The lesson learned from spin systems also indicates that cluster formation and associated critical behavior are the more general feature[5].

2.1 Model parameters

In relativistic heavy ion collisions the incoming nuclei become discs of vanishing thickness due to Lorentz contraction and the collision region is two-dimensional. So we turn to the study of two-dimensional percolation.

In a two-dimensional system spheres become discs. Discs with radius a, called cells, distribute randomly in the system of a big disc with radius R. The ratio R/a determines the system size and is denoted by s. The control variable for percolation is η — the ratio of the total area of all the cells and the area of the big disc, i.e.

$$\eta=\frac{N\pi a^2}{\pi R^2}=\frac{Na^2}{R^2}\,, \tag{1}$$

where N is the number of cells. The increase of the cell number N leads to the increase of η. The value of η at which an infinite cluster appears is the critical point and is denoted by η_c.

The definition of an infinite cluster varies. In a finite system a cluster spanning the system is called an infinite cluster. Let (r_i,θ_i) be the polar coordinates of cell i. If $r_i\in[R-a,R]$ we call the cell i a boundary cell. If cells i and j are two boundary cells belonging to the same cluster, and $|\theta_i-\theta_j|\geqslant\theta_0$ then we say the current cluster is infinite. Here θ_0 is a parameter.

Thus the control variable is η and the model parameters are s and θ_0.

2.2 Locating the critical point

When s and θ_0 take on certain values, the probability for the event for which an infinite cluster occurs is denoted by P_∞ and is an increasing function of η. For infinite system size, $P_\infty=0$ holds for all $\eta<\eta_c$. For $\eta\geqslant\eta_c$, P_∞ rises sharply and approaches unity. Systems of infinite size are unrealistic. We have investigated P_∞ for finite systems of three different sizes. The results are shown in Fig. 1.

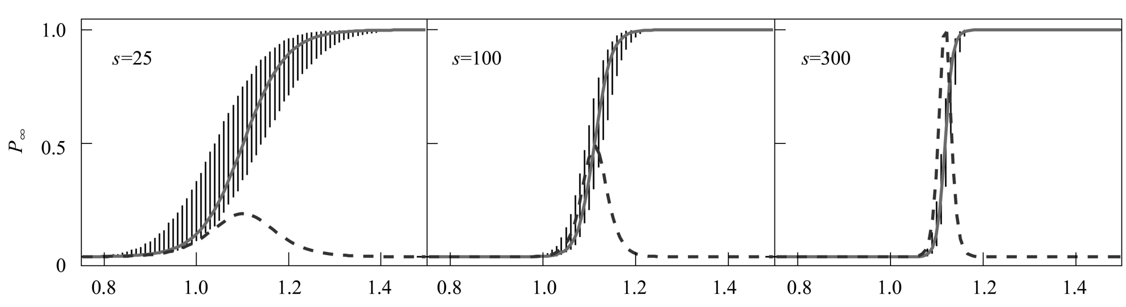

Fig. 1. P_∞ for different system sizes. From left to right the system sizes are 25, 100, 300, respectively. The error bar shown is the systematic error induced by the variation of the parameter θ_0. The statistical errors are small and not shown. The dashed curves are the susceptibility defined in Eq. (4).

Different parameters, e.g. $\theta_0 = 90°$, $135°$ and $175°$, have been tried and the induced systematic errors are shown as error-bars in the figures. The solid curves in Fig. 1 are the fitting results using the function

$$P_\infty(\eta) = \frac{1+\tanh\left[c_1(\eta-c_2)\right]}{2}\,. \qquad (2)$$

It can be seen that for all three cases the fitting function always lies within the systematic errors. The fitting function, i.e. Eq. (2), has an inflection point at $\eta = c_2$. As s increases, P_∞ tends to a step function and the inflection point c_2 tends to η_c. For finite system size, the inflection point $\eta = c_2$ can be used as an evaluation of η_c.

The behavior of c_2 versus different system sizes is shown in Fig. 2. A tendency to saturation can be seen. The saturation value of c_2 is 1.1198 ± 0.0047, i.e. the percolation threshold is $\eta_c = 1.1198 \pm 0.0047$.

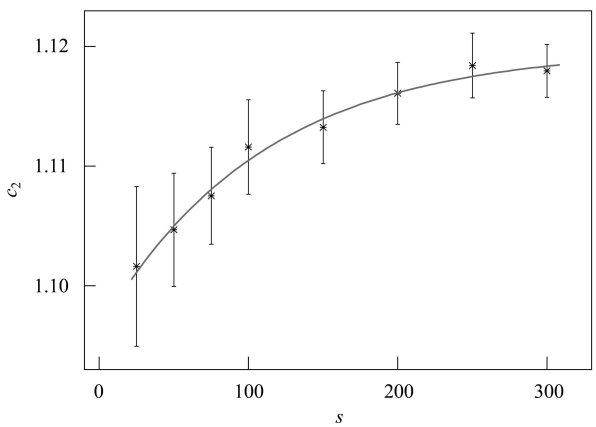

Fig. 2. The behavior of c_2 versus system size. A tendency to saturation can be seen.

For a nucleus-nucleus collision the contraction induced by the relativistic motion makes the colliding nuclei appear like a disc in which disc-like nucleons percolate. For a central Au-Au collision, the radius of the interacting region is about 7 fm and the hard core of each nucleon is about 0.1 fm, resulting in a system size of about 70. The inflection point for $s = 70$ is about 1.1054. Using this as an approximation for η_c, the error is 1.3%. So using the inflection point as an approximation for the critical point is easy to measure and exact enough.

In heavy ion collisions, when hadrons connect to form an infinite cluster, the system become color conductive and quark-gluon plasma is formed. If the formation of QGP is signaled in each collision, then P_∞ becomes the probability that the collision will produce a QGP — P_{QGP}. P_{QGP} should be measurable in experiment, provided that an available signal is found, and thus in principle the critical point can be obtained according to the inflection point of P_{QGP}.

By some authors[8] P_∞ is also referred to as the percolation cumulant, in analogy to the Binder cumulant[9], which will intersect at the critical point for different system sizes. Extracting the intersection point of P_∞ is a method to estimate the critical point. However, it only needs the collisions of one kind of ion at various energies to use the inflection-point method proposed above, while using the intersecting-point method needs collisions of more than one kind of ions at different energies. Therefore, the inflection-point method is more realistic and the value of η_c obtained for system sizes as small as Pb ($s = 71$), Au ($s = 70$) is a good estimate with an error of about 1.3%.

The inflection-point method is also applicable for another definition of P-infinity. In Ref. [10] P-infinity is defined as the probability of a cell belonging to an infinite cluster, which will be denoted by $P_\infty^{\mathrm{usual}}$ in the following. Fig. 3 shows $P_\infty^{\mathrm{usual}}$ as a function of η for $s = 1000$. In case of $s = 1000$ the system is very large and P_∞ rises like a step function, cf. Fig. 3, but $P_\infty^{\mathrm{usual}}$ has a step function sharp rise only at the bottom side ($P_\infty^{\mathrm{usual}} = 0$), while at the upper side it tends smoothly to unity.

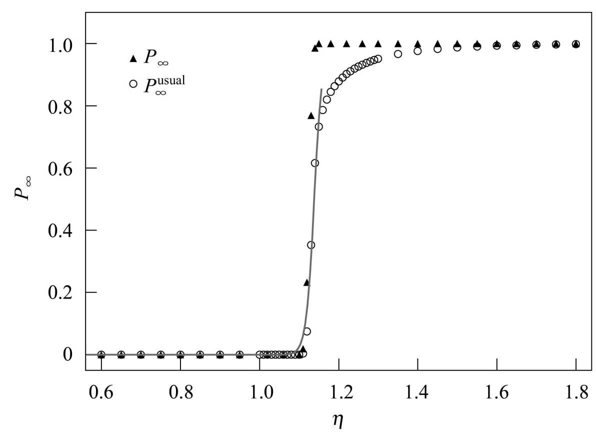

Fig. 3. $P_\infty^{\mathrm{usual}}$ (open circles) as a function of η of the system size $s = 1000$. The solid curve is a fit of the lower part with Eq. (3). For comparison P_∞ for $s = 1000$ is also shown with triangles.

In this case we use the horizontal line $P_\infty^{\mathrm{usual}} = 0.8$ to divide the function $P_\infty^{\mathrm{usual}}(\eta)$ into two parts — the lower part $P_\infty^{\mathrm{usual}}(\eta)\big|_{P_\infty^{\mathrm{usual}} \leqslant 0.8}$ and the upper part $P_\infty^{\mathrm{usual}}(\eta)\big|_{P_\infty^{\mathrm{usual}} > 0.8}$ and fit the lower part $P_\infty^{\mathrm{usual}}(\eta)\big|_{P_\infty^{\mathrm{usual}} \leqslant 0.8}$, which amounts to 80% of the height change, to Eq. (2), or alternatively, to a modified fitting function

$$P_\infty^{\mathrm{usual}}(\eta) = c_1\left[c_2 + \tanh\left(c_3(\eta - c_4)\right)\right]. \qquad (3)$$

The fitting result is shown as a solid curve in Fig. 3. The inflection point of this part-fitted curve can also be used as an approximation to the critical point.

2.3 Susceptibility and the critical exponent ν

The susceptibility is defined by the response of the system to small external forces. We define the susceptibility in percolation as the derivative of P_∞ with respect to the control variable η, i.e.

$$\kappa(\eta,s)=\frac{\partial P_\infty(\eta,s)}{\partial\eta}\,. \tag{4}$$

κ shows a peak at the inflection point of P_∞ (see the dashed line in Fig. 1). Since P_∞ becomes a step function when s tends to infinity, the peak of κ will become higher and narrower and finally diverge at $\eta_{\rm c}$.

In a realistic case the system is of finite size determined by the size of the colliding nuclei. The finite-size scaling method[11], investigating the scaling of quantities at $\eta_{\rm c}$ as a function of system size, is adopted to extract values for the critical exponents.

The finite-size scaling of P_∞ suggests[12]

$$P_\infty=\Phi\left[(\eta-\eta_{\rm c})\,s^{1/\nu}\right] \ \text{ for large } s, \quad \eta\to\eta_{\rm c}\,. \tag{5}$$

Denoting $X=(\eta-\eta_{\rm c})s^{1/\nu}$, $\Phi(X)$ is some function of X. ν is the critical exponent of the correlation length ξ, i.e.

$$\xi\propto|\eta-\eta_{\rm c}|^{-\nu} \ \text{ for } \eta\to\eta_{\rm c}\,. \tag{6}$$

Then, the susceptibility of P_∞ is given by

$$\kappa(\eta,s)=\frac{\partial P_\infty(\eta,s)}{\partial\eta}=s^{1/\nu}\frac{{\rm d}\Phi}{{\rm d}X}\,. \tag{7}$$

For $\eta=\eta_{\rm c}$, $\left.\frac{{\rm d}\Phi}{{\rm d}X}\right|_{X=0}$ is a constant and

$$\kappa(\eta_{\rm c},s)\propto s^{1/\nu} \ \text{ for large } s\,. \tag{8}$$

Thus the divergent behavior of κ near $\eta_{\rm c}$ is related to the critical exponent of the correlation length.

Using the critical point $\eta_{\rm c}$ extracted from P_∞, we evaluate $\kappa(\eta_{\rm c},s)$ as a function of s, shown in Fig. 4. Fitting it to Eq. (8) we obtain $1/\nu=0.739\pm0.041$, $\nu=1.353\pm0.075$. The exponent ν obtained agrees within 1.5% with the result from other calculations of two-dimensional percolation[10].

The critical exponent of the correlation length plays a special role in the theory of critical phenomena, because the scaling behavior of other quantities depends on the relative magnitude of the correlation length and system size. Hence, the critical exponent of the correlation length appears in the scaling relation of various quantities. By definition the correlation length is the distance at which the correlation function reduces to $1/e$[10]. However, the correlation function usually has non-monotonic behavior, e.g. in case of a liquid, the correlation function shows damped oscillations[13], which makes the correlation length hard to measure. Now we see that using the scaling behavior of the susceptibility, the critical exponent of the correlation length can easily be obtained[12].

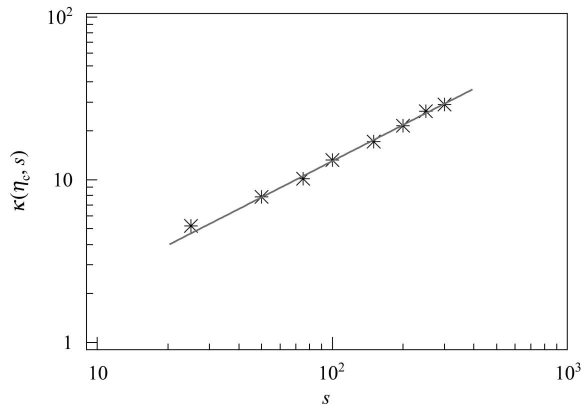

Fig. 4. The distribution of the susceptibility κ at the critical point $\eta_{\rm c}$ as a function of the system size s. The solid line is a fit with Eq. (8).

3 Application to relativistic heavy ion collisions

Recently, some kind of scaled third order moment of the transverse momentum[14]

$$D_3=\frac{\langle p_{\rm t}\rangle^3}{\langle p_{\rm t}^3\rangle} \tag{9}$$

has been proposed as a possible signal of the critical point (CP) in the QCD phase diagram. By imposing a temperature gradient to the two-dimensional continuum percolation of discs and assuming the transverse momentum for each cell takes on the value of thermal momentum determined by the temperature, $D_3(\eta)$ behaves like a step function, similar to that of $P_\infty^{\rm usual}(\eta)$ described above, cf. Fig. 5 and Fig. 3. This similarity makes the methods developed in Sect. 2 applicable to D_3, i.e. we can locate the CP and determine the critical exponent of a susceptibility related to D_3.

A typical result for D_3 and the fitting result of the lower part (80% height change) with Eq. (3) is shown in Fig. 5. The inflection point c_4 is regarded as the approximation to the critical point $\eta_{\rm c}$. With the increasing of system size, the inflection point c_4 has an asymptotic behavior which is shown in Fig. 6. The saturation value indicates $\eta_{\rm c}=1.124$.

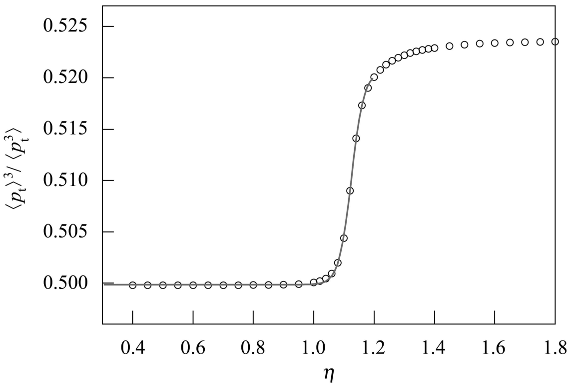

Fig. 5. D_3 as a function of η with system size $s=500$. The red line is a fit with Eq. (3).

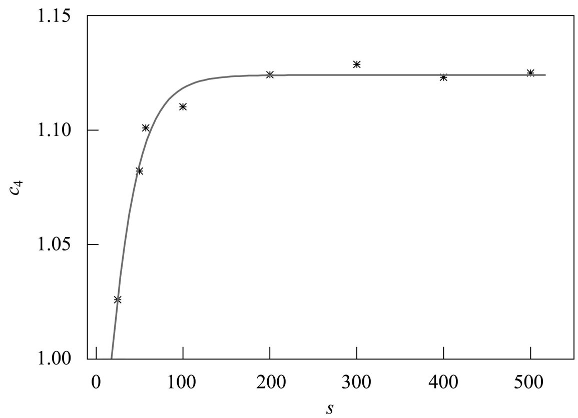

Fig. 6. The behavior of c_4 versus different system size showing saturation.

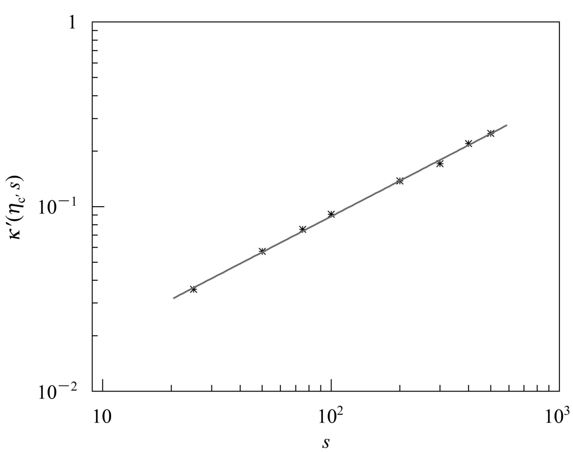

Fig. 7. The distribution of the susceptibility κ' at the critical point η_c as a function of the system size s. The solid line is a fit with Eq. (11).

Similar to Eq. (4) we define the susceptibility related to D_3 as

$$\kappa'(\eta,s)=\frac{\partial D_3(\eta,s)}{\partial\eta}, \tag{10}$$

and a critical exponent ν' for this thermal system as

$$\kappa'(\eta_c,s)\propto s^{1/\nu'}. \tag{11}$$

Investigating the finite-size scaling of κ', $1/\nu'=0.642$ is obtained, cf. Fig. 7.

4 Conclusion

What we are interested in is the phenomena at the vicinity of the critical point, in particular the process occurring while the system produced in relativistic heavy ion collisions evolves passing through the critical point. This is a second-order phase transition process[15] and can be well described by the continuum-percolation model[5].

In general, when a system evolves passing through a critical point some characteristic quantities of the system will have a sudden change in value. For an infinite system such a sudden change is of Heaviside step function form. The place of this discontinuous change is just the critical point of the system in consideration. In the case where the system is of finite size the discontinuity will be smoothed to a continuous abrupt change of the variable.

In this paper we study critical phenomena in continuum percolation of discs. For an infinite system P_∞ or P_∞^{usual} versus the control parameter has a step function discontinuity, which is referred to as the percolation threshold. In a finite-size system this discontinuity will be smoothed to a continuous abrupt change of the value of P_∞ or P_∞^{usual}. It is shown that the continuous curve of P_∞ or P_∞^{usual} versus the control parameter can be fitted well with a hypertangent-type function as shown in Eq. (2) or Eq. (3). It is shown that using the inflection point of this function as an estimate for the percolation threshold is a good approximation with the error being less than 2%. The critical exponent ν for the correlation length is extracted from the distribution of the susceptibility by using the finite-size scaling method.

For heavy ion collisions, different colliding nuclei have been tried, e.g. Pb, Au, Cu, S, C etc., which produce systems of different sizes. The colliding energy plays the role of a control variable. P_∞ in percolation corresponds to the probability of events with QGP in heavy ion collision experiments, denoted as P_{QGP}. Studying the inflection point of P_{QGP} as a function of $\sqrt{s_{\text{NN}}}$, the critical point can be extracted. A susceptibility defined as the derivative of P_{QGP} determines the critical exponent ν which is a rare case in which one can experimentally measure the critical exponent in heavy ion collisions. This approach is worth trying in future studes of the critical phenomena in real relativistic heavy ion collisions.

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