Stable solutions for q$\bar{q}$ bound states with a regularized Breit potential*

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Abstract The Breit interaction contains singular terms which may lead to an instability in quark-antiquark bound state calculations. We regularize the Breit interaction by multiplying the singular terms in momentum space by the form factor $\mu^2/(q^2+\mu^2)$ such that the interaction is not singular at the origin and the intermediate- and long-range parts of the interaction remain unchanged. The singular terms in the Breit potential find their stable contributions in the calculations after being multiplied by the form factor with different powers. Such a regularized Breit potential with a linear and a relativistically corrected confining potential are applied to the study of q$\bar{q}$ bound states. The spectra for most familiar mesons are consistently obtained and agree well with the experimental data.

Key words quark model, regularized Breit potential, meson spectra, relativistically corrected confining potential

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1 Introduction

In the description of hadronic phenomena, non-relativistic and relativistically corrected potential quark models have been successfully applied to describe many properties of low-lying hadronic states[1—11]. Progress has been made in studying the hadron bound states and the fine structure of the mass spectra[1—6]. The success of the potential model also promotes its applications to scattering problems[1, 6—9, 12, 13]. However, it is still a challenging problem to explain all the meson spectra (from light to heavy) within an appropriate quark-potential model.

The earliest version of a complete one-gluon exchange potential up to the second order in the relative velocity v is the Fermi-Breit potential (or just the Breit potential)[14]. Because the Breit potential contains terms which become attractively singular faster than $-1/r^2$ when r approaches zero, the direct use of the Breit potential in the Schrödinger equation for the solution of the q$\bar{q}$ bound states may lead to an instability of the solution[12, 14, 15].

In our previous work[16], the meson mass spectra were calculated using the Breit potential. The wave functions were expanded into a set of Gaussian basis functions with different widths. We found that the results are dependent on the width parameter β and the number N of the basis functions. They changed appreciably when β and N were changed. This is the so-called instability of the solution because a stable result should be independent of the basis functions and only depend on the potential parameters and the wave function used (the expansion coefficients of the wave function in the chosen basis).

In order to remove the instability we will adopt the following approach to utilize the Breit interaction in the present work. We regularize the singular terms in the Breit potential by multiplying them by the form factor $\mu^2/(q^2+\mu^2)$ in momentum space, where q is the momentum transfer and μ is a momentum cut-off parameter. This regularization pre-

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serves the intermediate-scale behavior of the potential terms meanwhile it removes their singular short distance property. We find that the calculated mass spectra are completely stable and produce the experimental data with high accuracy if we apply the form factor to the spin-spin and spin-orbit coupling terms once and to the orbit-orbit coupling term twice.

On the other hand, we used in our previous work[16] for simplicity a linear confining potential. In the present work we will take the linear confining potential as well as a confining potential with a relativistic correction up to second order in the velocity v. We find that the results for both of the two confining potentials are consistent.

The paper is arranged as follows. In section 2 we discuss the potential we use and the regularization of the Breit potential. In section 3 the matrix equation for the bound states of q$\bar{\text{q}}$ and the meson wave functions are introduced. In section 4, the matrix elements of the regularized Breit potential and the confining potential are presented. Finally, we discuss our results in section 5 and give a summary of the paper in section 6.

2 Regularization of the Breit potential

Mesons are quark-antiquark bound states. The complete one-gluon exchange potential up to $O(1/c^2)$ for these states, the Fermi-Breit potential, can be expressed in the center-of-mass frame as[17]

$$V^{\mathrm{B}}(\boldsymbol{r}) = \boldsymbol{C}_{\mathrm{ij}}\,\alpha_{\mathrm{s}}\left\{\frac{1}{r}-\frac{\pi}{2}\,\delta(\boldsymbol{r})\frac{(m_{\mathrm{i}}^2+m_{\mathrm{j}}^2)}{m_{\mathrm{i}}^2 m_{\mathrm{j}}^2}+\frac{1}{2m_{\mathrm{i}}m_{\mathrm{j}}}\left(\frac{\boldsymbol{p}^2}{r}+\frac{\boldsymbol{r}\cdot(\boldsymbol{r}\cdot\boldsymbol{p})\boldsymbol{p}}{r^3}\right)-\frac{2\pi}{3m_{\mathrm{i}}m_{\mathrm{j}}}(\boldsymbol{\sigma}_{\mathrm{i}}\cdot\boldsymbol{\sigma}_{\mathrm{j}})\,\delta(\boldsymbol{r})-\frac{1}{4m_{\mathrm{i}}m_{\mathrm{j}}r^3}(\boldsymbol{r}\times\boldsymbol{p})\times\left[\left(2+\frac{m_{\mathrm{j}}}{m_{\mathrm{i}}}\right)\boldsymbol{\sigma}_{\mathrm{i}}+\left(2+\frac{m_{\mathrm{i}}}{m_{\mathrm{j}}}\right)\boldsymbol{\sigma}_{\mathrm{j}}\right]-\frac{3}{4m_{\mathrm{i}}m_{\mathrm{j}}r^3}S_{\mathrm{ij}}^r\right\}+\boldsymbol{C}_{\mathrm{ij}}(-V_0)\,, \tag{1}$$

where $\boldsymbol{C}_{\mathrm{ij}}$ is the color matrix, α_{s} is the QCD coupling constant, $r=|\boldsymbol{r}|=|\boldsymbol{r}_1-\boldsymbol{r}_2|$ is the distance between quark i and quark j, m_{i} and m_{j} are the masses of the constituent quarks, $\boldsymbol{p}\equiv\boldsymbol{p}_{\mathrm{i}}=-\boldsymbol{p}_{\mathrm{j}}$ is the quark momentum, $\boldsymbol{s}_{\mathrm{i}}=\frac{1}{2}\boldsymbol{\sigma}_{\mathrm{i}}$ is the spin of the quark i. In Eq. (1), the last term is a constant potential used to adjust the meson mass in solving the Schrödinger equation, and

$$S_{\mathrm{ij}}^r=\frac{(\boldsymbol{r}\cdot\boldsymbol{\sigma}_{\mathrm{i}})(\boldsymbol{r}\cdot\boldsymbol{\sigma}_{\mathrm{j}})}{r^2}-\frac{1}{3}(\boldsymbol{\sigma}_{\mathrm{i}}\cdot\boldsymbol{\sigma}_{\mathrm{j}}) \tag{2}$$

is the tensor-force operator. The average of S_{ij}^r over $\boldsymbol{r}$ is zero. In calculations we find that the effect of the tensor-force term is small after regularization and can be neglected.

From Eq. (1) one can see that the Breit interaction contains terms becoming singular faster than $1/r^2$ when r approaches zero (notice that $\delta(\boldsymbol{r})\sim r^{-3}$ and $\boldsymbol{p}\sim r^{-1}$). This singularity may lead to an instability of the solution for q$\bar{\text{q}}$ bound states[12, 14, 15]. In order to obtain stable solutions for the bound states we regularize the Breit potential by multiplying these singular terms by the form factor $\mu^2/(\boldsymbol{q}^2+\mu^2)$ in momentum space. Here $\boldsymbol{q}$ is momentum transfer and μ is a momentum cut-off parameter. With this approach the singular terms are regularized at short relative separations (large momentum transfer) and they nonetheless retain the same form at large relative separations (small momentum transfer).

In momentum space the Breit potential can be expressed as[7, 18—20]

$$V^{\mathrm{B}}(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{C}_{\mathrm{ij}}(4\pi\alpha_{\mathrm{s}})\left\{\frac{1}{\boldsymbol{q}^2}-\frac{(m_{\mathrm{i}}^2+m_{\mathrm{j}}^2)}{8m_{\mathrm{i}}^2m_{\mathrm{j}}^2}+\frac{1}{m_{\mathrm{i}}m_{\mathrm{j}}\,\boldsymbol{q}^2}\left[\boldsymbol{p}^2-\frac{(\boldsymbol{p}\cdot\boldsymbol{q})^2}{\boldsymbol{q}^2}\right]-\frac{2}{3}\frac{(\boldsymbol{\sigma}_{\mathrm{i}}\cdot\boldsymbol{\sigma}_{\mathrm{j}})}{4m_{\mathrm{i}}m_{\mathrm{j}}}+\frac{1}{4m_{\mathrm{i}}m_{\mathrm{j}}}\frac{(\mathrm{i}\boldsymbol{q}\times\boldsymbol{p})}{\boldsymbol{q}^2}\cdot\left[(2+\frac{m_{\mathrm{j}}}{m_{\mathrm{i}}})\boldsymbol{\sigma}_{\mathrm{i}}+(2+\frac{m_{\mathrm{i}}}{m_{\mathrm{j}}})\boldsymbol{\sigma}_{\mathrm{j}}\right]\right\}+\boldsymbol{C}_{\mathrm{ij}}(2\pi)^3(-V_0)\delta(\boldsymbol{q})\,, \tag{3}$$

where $\boldsymbol{p}=\frac{1}{2}(\boldsymbol{\kappa}'+\boldsymbol{\kappa})$, $\boldsymbol{q}=\boldsymbol{\kappa}'-\boldsymbol{\kappa}$, $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$ are the relative momenta of the two quarks in the initial and final state respectively[8]. We next discuss the regularization of the terms in Eq. (3) one by one, and then transform the regularized interactions into coordinate space.

The first term in Eq. (3), the color-Coulomb potential, needs not to be regularized. Denoting the ith term in Eq. (1) with $V_{\mathrm{i}}^{\mathrm{B}}$ and using $U_{\mathrm{i}}^{\mathrm{B}}$ for the corresponding regularized potential, we get

$$U_1^{\mathrm{B}}(\boldsymbol{r})=V_1^{\mathrm{B}}(\boldsymbol{r})=\boldsymbol{C}_{\mathrm{ij}}\,\alpha_{\mathrm{s}}\frac{1}{r}\,. \tag{4}$$

For the second term in Eq. (3), the regularized form in momentum space is

$$-\boldsymbol{C}_{\mathrm{ij}}(4\pi\alpha_{\mathrm{s}})\frac{(m_{\mathrm{i}}^2+m_{\mathrm{j}}^2)}{8m_{\mathrm{i}}^2m_{\mathrm{j}}^2}\frac{\mu^2}{\boldsymbol{q}^2+\mu^2}\,.$$

Transforming it into coordinate space, we have

$$U_2^{\rm B}(\boldsymbol{r})=-\boldsymbol{C}_{\rm ij}\,\alpha_{\rm s}\frac{(m_{\rm i}^2+m_{\rm j}^2)}{8m_{\rm i}^2m_{\rm j}^2}\left(\frac{\mu^2{\rm e}^{-\mu r}}{r}\right). \tag{5}$$

The third term in Eq. (3) is the orbit-orbit coupling interaction. We find that it is a main source of the instability in the calculations and this instability can not be removed if we apply the form factor only once to this term. So in our regularization we multiply the term twice by the form factor. The corresponding regularized form of the interaction in coordinate space is then

$$U_3^{\rm B}(\boldsymbol{r})=2V_3'(\boldsymbol{r})-(1-{\rm e}^{-\mu r})V_3^{\rm B}(\boldsymbol{r})-\frac{\boldsymbol{C}_{\rm ij}\,\alpha_{\rm s}}{2m_{\rm i}m_{\rm j}}(\mu r e^{-\mu r})\left\{\frac{\boldsymbol{p}^2}{r}-\frac{\boldsymbol{r}\cdot(\boldsymbol{r}\cdot\boldsymbol{p})\boldsymbol{p}}{r^3}\right\}, \tag{6}$$

where

$$V_3'(\boldsymbol{r})=V_3^{\rm B}(\boldsymbol{r})+\frac{\boldsymbol{C}_{\rm ij}\,\alpha_{\rm s}}{m_{\rm i}m_{\rm j}}\left\{-{\rm e}^{-\mu r}\left[\frac{\boldsymbol{p}^2}{r}-\frac{\boldsymbol{r}\cdot(\boldsymbol{r}\cdot\boldsymbol{p})\boldsymbol{p}}{r^3}\right]+\mu^{-2}r^{-2}\left[\frac{\boldsymbol{p}^2}{r}-3\frac{\boldsymbol{r}\cdot(\boldsymbol{r}\cdot\boldsymbol{p})\boldsymbol{p}}{r^3}\right]-\mu^{-2}(\mu+r^{-1})\frac{{\rm e}^{-\mu r}}{r}\left[\frac{\boldsymbol{p}^2}{r}-3\frac{\boldsymbol{r}\cdot(\boldsymbol{r}\cdot\boldsymbol{p})\boldsymbol{p}}{r^3}\right]\right\}. \tag{7}$$

The fourth and fifth terms in Eq. (3) are the spin-spin and spin-orbit interactions. We regularize them by multiplying with the form factor in momentum space. Their regularized expressions in coordinate space are

$$U_4^{\rm B}(\boldsymbol{r})=-\frac{\boldsymbol{C}_{\rm ij}\,\alpha_{\rm s}}{3m_{\rm i}m_{\rm j}}(\boldsymbol{\sigma}_{\rm i}\cdot\boldsymbol{\sigma}_{\rm j})\left(\frac{\mu^2{\rm e}^{-\mu r}}{2\,r}\right), \tag{8}$$

$$U_5^{\rm B}(\boldsymbol{r})=-\frac{\boldsymbol{C}_{\rm ij}\,\alpha_{\rm s}}{4m_{\rm i}m_{\rm j}}\frac{(\boldsymbol{L}\cdot\boldsymbol{\sigma})}{r^3}\left[1-(1+\mu r)\,{\rm e}^{-\mu r}\right], \tag{9}$$

where $\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}$ and $\boldsymbol{\sigma}=\left(2+\frac{m_{\rm j}}{m_{\rm i}}\right)\boldsymbol{\sigma}_{\rm i}+\left(2+\frac{m_{\rm i}}{m_{\rm j}}\right)\boldsymbol{\sigma}_{\rm j}$.

The last constant term in Eq. (3) does not need to be regularized,

$$U_6^{\rm B}(\boldsymbol{r})=V_6^{\rm B}(\boldsymbol{r})=\boldsymbol{C}_{\rm ij}(-V_0). \tag{10}$$

In addition to the Breit potential, the interaction between the quark and antiquark also includes the confining potential, which is usually taken to be proportional to their separation r[8, 19]:

$$V^{\rm c}(\boldsymbol{r})=-\boldsymbol{C}_{\rm ij}\left(\frac{3}{4}b\right)r\,, \tag{11}$$

where $\boldsymbol{C}_{\rm ij}$ is the same color matrix as in the Breit potential and b is a string tension coefficient. Using the standard QED scattering amplitude technique[18], one can get the confining potential up to $O(1/c^2)$ as

$$V^{\rm c}(\boldsymbol{p},\boldsymbol{q})\;=\;\boldsymbol{C}_{\rm ij}\left\{\frac{b^{\rm s}+b^{\rm v}}{\boldsymbol{q}^4}-\frac{(m_{\rm i}^2+m_{\rm j}^2)}{8m_{\rm i}^2m_{\rm j}^2}\frac{b^{\rm s}+b^{\rm v}}{\boldsymbol{q}^2}-\frac{(m_{\rm i}^2+m_{\rm j}^2)}{2m_{\rm i}^2m_{\rm j}^2}b^{\rm s}\frac{\boldsymbol{p}^2+\boldsymbol{p}\cdot\boldsymbol{q}}{\boldsymbol{q}^4}+\frac{b^{\rm v}-b^{\rm s}}{4m_{\rm i}m_{\rm j}}\left(\frac{m_{\rm j}}{m_{\rm i}}\boldsymbol{\sigma}_{\rm i}+\frac{m_{\rm i}}{m_{\rm j}}\boldsymbol{\sigma}_{\rm j}\right)\cdot\frac{{\rm i}\boldsymbol{q}\times\boldsymbol{p}}{\boldsymbol{q}^4}\right\}, \tag{12}$$

where $b^{\rm s}$ and $b^{\rm v}$ are the string tension coefficients for the scalar and vectorial interactions. In coordinate space the confining potential is given by

$$V^{\rm c}(\boldsymbol{r})\;=\;\boldsymbol{C}_{\rm ij}\left\{-\left(\frac{3}{4}b\right)r-\left(\frac{3}{2}b\right)\frac{(m_{\rm i}^2+m_{\rm j}^2)}{8m_{\rm i}^2m_{\rm j}^2}\frac{1}{r}-\left(\frac{3}{2}a\right)\frac{(m_{\rm i}^2+m_{\rm j}^2)}{8m_{\rm i}^2m_{\rm j}^2}\left[-r\boldsymbol{p}^2+\frac{1}{r}({\rm i}\boldsymbol{r}\cdot\boldsymbol{p})\right]-\frac{3(b-a)}{16m_{\rm i}m_{\rm j}r}\boldsymbol{L}\cdot\left(\frac{m_{\rm j}}{m_{\rm i}}\boldsymbol{\sigma}_{\rm i}+\frac{m_{\rm i}}{m_{\rm j}}\boldsymbol{\sigma}_{\rm j}\right)\right\}, \tag{13}$$

where $b=(b^{\rm s}+b^{\rm v})/(6\pi)$ and $a=b^{\rm s}/(3\pi)$. They are two adjustable parameters related the the confining potential. One can see that the first term in Eq. (13) is the linear confining potential usually used, Eq. (11). It can also be seen that the powers of r for the terms in Eq. (13) are greater than -2. So there is no need for regularizing the confining potential.

The quark-antiquark regularized potential is then

$$U(\boldsymbol{r})\;=\;U_1^{\rm B}(\boldsymbol{r})+U_2^{\rm B}(\boldsymbol{r})+U_3^{\rm B}(\boldsymbol{r})+U_4^{\rm B}(\boldsymbol{r})+U_5^{\rm B}(\boldsymbol{r})+U_6^{\rm B}(\boldsymbol{r})+V^{\rm c}(\boldsymbol{r}). \tag{14}$$

3 Meson bound-state matrix equation and wave function

The Schrödinger equation for meson bound states in coordinate space is

$$\frac{\boldsymbol{p}^2}{2\mu_r}\varPhi(\boldsymbol{r})+U(\boldsymbol{r})\varPhi(\boldsymbol{r})=E\varPhi(\boldsymbol{r}), \tag{15}$$

where $\mu_r=m_{\rm i}m_{\rm j}/(m_{\rm i}+m_{\rm j})$, $\boldsymbol{p}$, and $\boldsymbol{r}$ are the reduced mass, center-of-mass momentum, and relative coordinate between the quark and antiquark respectively, $\varPhi(\boldsymbol{r})$ is the meson wave function. The energy E and the total mass M of the meson satisfy $E=M-m_{\rm i}-m_{\rm j}$.

We expand the meson wave function $\varPhi(\boldsymbol{r})$ into a set of basis functions $\phi_{nl}(\boldsymbol{r})$:

$$\varPhi(\boldsymbol{r})=\sum_n a_n\phi_{nl}(\boldsymbol{r}). \tag{16}$$

Inserting this expansion into Eq. (15) and multiplying each term by $\phi^{\dagger}_{ml'}(\boldsymbol{r})$ from the left, we get

$$\sum_{n,\,l} a_n\left[\phi^{\dagger}_{ml'}(\boldsymbol{r})\frac{\boldsymbol{p}^2}{2\mu_r}\phi_{nl}(\boldsymbol{r})+\phi^{\dagger}_{ml'}(\boldsymbol{r})U(\boldsymbol{r})\phi_{nl}(\boldsymbol{r})\right]= E\sum_{n,\,l} a_n\phi^{\dagger}_{ml'}(\boldsymbol{r})\phi_{nl}(\boldsymbol{r})\,. \tag{17}$$

By integrating this equation over the whole coordinate space, we obtain

$$\sum_{n,\,l} a_n\left[T_{mn}+U_{mn}\right]=E\sum_{n,\,l} a_n B_{mn}\,, \tag{18}$$

where

$$T_{mn}=\langle ml'|T|nl\rangle=(2\pi)^3\int \mathrm{d}\boldsymbol{r}\,\phi^{\dagger}_{ml'}(\boldsymbol{r})\,\frac{\boldsymbol{p}^2}{2\mu_r}\phi_{nl}(\boldsymbol{r})\,, \tag{19}$$

$$U_{mn}=\langle ml'|U|nl\rangle=(2\pi)^3\int \mathrm{d}\boldsymbol{r}\phi^{\dagger}_{ml'}(\boldsymbol{r})U(\boldsymbol{r})\phi_{nl}(\boldsymbol{r})\,, \tag{20}$$

$$B_{mn}=\langle ml'|nl\rangle=(2\pi)^3\int \mathrm{d}\boldsymbol{r}\,\phi^{\dagger}_{ml'}(\boldsymbol{r})\phi_{nl}(\boldsymbol{r})\,. \tag{21}$$

In the present paper we use the same basis functions as in Ref. [8]. In coordinate space they are given by

$$\phi_{nl}(\boldsymbol{r})=R_{nl}(r)\,Y_{lm_l}(\hat{\boldsymbol{r}})=N_{nl}\,r^l\exp\Big(-\frac{n\beta^2}{2}r^2\Big)Y_{lm_l}(\hat{\boldsymbol{r}})\,, \tag{22}$$

where

$$N_{nl}=\frac{(\sqrt{2}\,\mathrm{i})^l}{4\pi}\sqrt{\frac{(2/\sqrt{\pi})^3}{(2l+1)!!}}\,(n\beta^2)^{\frac{1}{2}(l+\frac{3}{2})}\,, \tag{23}$$

and β is the width parameter of the basis. In our calculations we take the same β values for the mesons as in Ref. [8]. The radial basis wave functions $R_{nl}(r)$ are products of r^l with Gaussian functions of different widths. The advantage of using Gaussian bases is that one can get analytical expressions for the matrix elements. Then, the expansion coefficients can be determined by solving the Schrödinger equation numerically. In the calculations, we take six Gaussian basis functions ($n=1,2,3,4,5,6$) as in Refs. [8, 22].

4 The matrix elements of the regularized potential

In this section, we calculate the matrix elements of the regularized potential. After knowing the matrix elements used in Eq. (18) one can obtain the eigen values of E by solving Eq. (18) numerically and finally obtain the meson masses.

In general the matrix elements of the interaction potential can be expressed as

$$U_{mn} = \sum_{m_l m_s}\sum_{m'_l m'_s}\langle lm_l sm_s|Jm_J\rangle\langle l'm'_l s'm'_s|Jm'_J\rangle\times (2\pi)^3\int \mathrm{d}\boldsymbol{r}\phi^{\dagger}_{ml'}(\boldsymbol{r})\chi^{\dagger}_{sm'_s}c^{\dagger}(\mathrm{ij})U(\boldsymbol{r})\phi_{nl}(\boldsymbol{r})\times \chi_{sm_s}c(\mathrm{ij}), \tag{24}$$

where l, S, and J are the quantum numbers of the orbital angular momentum, the spin, and total angular momentum of the meson. m_l, m_s and m_J are the corresponding magnetic quantum numbers, χ_{sm_s} is the spin wave function, and $c(\mathrm{ij})$ is the color wave function of the meson.

Below we give the matrix elements of the regularized potential Eq. (14) for the states with $l=0$ and 1 which we consider in this paper. Because deriving the final results is a rather cumbersome procedure, we give here only the results. For the first term in Eq. (14) the matrix element is given by

$$(U^{\mathrm{B}}_1)_{mn}=\frac{C_f\alpha_s}{(2\pi)^{1/2}}\beta\frac{2^{(l+1)}l!}{(2l+1)!!}\sqrt{m+n}B_{mn}\,, \tag{25}$$

where $C_f=-\dfrac{4}{3}$,

$$B_{mn} = (2\pi)^3\int \mathrm{d}\boldsymbol{r}\phi^{\dagger}_{ml'}(\boldsymbol{r})\phi_{nl}(\boldsymbol{r})= \Big(\frac{2\sqrt{mn}}{m+n}\Big)^{l+3/2}\delta_{l'l}\,. \tag{26}$$

For the second term in Eq. (14) the matrix element is given by

$$(U^{\mathrm{B}}_2)_{mn} = -C_f\alpha_s\frac{(m_{\mathrm{i}}^2+m_{\mathrm{j}}^2)}{8m_{\mathrm{i}}^2m_{\mathrm{j}}^2}A_l\mu^2\times \Big[(1-l)w_1+lw_3\Big], \tag{27}$$

where

$$A_l=\frac{2^{(l+2)}}{(2l+1)!!\sqrt{\pi}}\beta^{(2l+3)}(mn)^{\frac{1}{4}(2l+3)}, \tag{28}$$

$$w_n=\int_0^{\infty}\mathrm{d}x\cdot x^n\mathrm{e}^{(-\nu x^2-\mu x)},\qquad (n=0,1,2,\cdots), \tag{29}$$

and $\nu=\dfrac{1}{2}\beta^2(m+n)$. The explicit expressions of w_n can be found in Appendix A. For the third term in Eq. (14) the matrix element is given by

$$(U_3^{\rm B})_{mn} = (V_3)_{mn} - \frac{C_f\alpha_s}{m_{\rm i}m_{\rm j}}A_l(n\beta^2)\Big\{\frac{l!}{\nu^l}\frac{4l}{\mu^2} + \big[(1-l)\mu^2-4l\big]\frac{w_1}{\mu^2} + \big[(1-l)\mu^2-8l\big]\frac{w_2}{\mu} - l\mu w_4\Big\} + \frac{C_f\alpha_s}{m_{\rm i}m_{\rm j}}A_l(n^2\beta^4)\Big\{\frac{l!}{\nu^{(l+1)}}\frac{2}{\mu^2} - 4(1-l)\frac{w_1+w_2}{\mu^2} - \big[4l+(l-1)\mu^2\big]\frac{w_3}{\mu^2} - 4l\frac{w_4}{\mu} - lw_5\Big\}, \qquad (30)$$

where

$$(V_3)_{mn} = \frac{C_f\alpha_s}{m_{\rm i}m_{\rm j}}\beta^3\frac{4mn}{\sqrt{m+n}}\frac{B_{mn}}{(2\pi)^{1/2}}, \qquad (l=0), \qquad (31)$$

$$(V_3)_{mn} = \frac{C_f\alpha_s}{m_{\rm i}m_{\rm j}}\beta^3\frac{B_{mn}}{(2\pi)^{1/2}}\frac{2^l l(l-1)!}{(2l+1)!!}\Big\{(l+1)\frac{4mn}{\sqrt{m+n}} - (l-1)\frac{(m+n)^{3/2}}{2}\Big\}, \ (l>0). \qquad (32)$$

For the fourth term in Eq. (14) the matrix element is given by

$$(U_4^{\rm B})_{mn} = -\frac{C_f\alpha_s}{3m_{\rm i}m_{\rm j}}A_l\big[s(s+1)-3/2\big]\mu^2\big[(1-l)w_1+lw_3\big]. \qquad (33)$$

For the fifth term in Eq. (14) the matrix element is zero for $l=0$. For $l>0$ it is given by

$$(U_5^{\rm B})_{mn} = C_f\frac{\sqrt{6}\alpha_s}{4m_{\rm i}m_{\rm j}}\Big(4+\frac{m_{\rm i}}{m_{\rm j}}+\frac{m_{\rm j}}{m_{\rm i}}\Big)\times (\hat{s})^2\hat{l}\sqrt{l(l+1)}(-1)^l\times \Big[\frac{\beta^3}{(2\pi)^{1/2}}\frac{2^l(l-1)!}{(2l+1)!!}(m+n)^{3/2}B_{mn} - A_l(w_1+\mu w_2)\Big]\times (-1)^{1+J}\begin{Bmatrix} s & s & 1\\ l & l & J\end{Bmatrix}\begin{Bmatrix} s & s & 1\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}, \qquad (l>0), \qquad (34)$$

where the symbol $\hat{A}$ denotes $\sqrt{2A+1}$. For the constant term in Eq. (14) the matrix element is

$$(U_6^{\rm B})_{mn} = C_f(-V_0)B_{mn}. \qquad (35)$$

The matrix elements for the terms of the confining potential Eq. (13) are given by

$$(V_1^{\rm c})_{mn} = -C_f\frac{3b}{(2\pi)^{1/2}\beta}\frac{2^l(l+1)!}{(2l+1)!!}\frac{B_{mn}}{\sqrt{m+n}}, \qquad (36)$$

$$(V_2^{\rm c})_{mn} = -\Big(\frac{3}{2}b\Big)\frac{(m_{\rm i}^2+m_{\rm j}^2)}{8m_{\rm i}^2m_{\rm j}^2}\frac{(U_1^{\rm B})_{mn}}{\alpha_s}, \qquad (37)$$

$$(V_3^{\rm c})_{mn} = -C_f\Big(\frac{3}{2}a\Big)\frac{(m_{\rm i}^2+m_{\rm j}^2)}{8m_{\rm i}^2m_{\rm j}^2}A_l\Big\{\frac{(l+2)!}{2\nu^{(l+3)}}n^2\beta^4 - \frac{(l+2)!}{\nu^{(l+2)}}n\beta^2 + \frac{ll!}{2\nu^{(l+1)}}\Big\}, \qquad (38)$$

$$(V_4^{\rm c})_{mn} = C_f\frac{3(b-a)}{4}\frac{\sqrt{6}}{4m_{\rm i}m_{\rm j}}\Big(\frac{m_{\rm i}}{m_{\rm j}}+\frac{m_{\rm j}}{m_{\rm i}}\Big)\times \hat{s}^2\hat{l}\sqrt{l(l+1)}A_l\frac{(-1)^l l!}{2\nu^{(l+1)}}\times (-1)^{1+J}\begin{Bmatrix} s & s & 1\\ l & l & J\end{Bmatrix}\begin{Bmatrix} s & s & 1\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}. \qquad (39)$$

Additionally, the kinetic energy matrix element is given by

$$T_{mn} = (2l+3)\frac{mn}{m+n}B_{mn}\frac{\beta^2}{2\mu_r}. \qquad (40)$$

5 Results

In Table 1 we list the experimental masses of 28 mesons, our calculated results with the regularized Breit potential, the results from previous literature[8, 9], and the expanding coefficients of the meson wave functions for the linear confining potential. For comparison we list in the brackets in the third column the results calculated with the relativistically corrected confining potential. It can been seen that the results for both confining potentials are almost consistent. In our model, the adjustable parameters are the confining potential parameters a and b, the constant potential term V_0, five quark masses $m_{\rm u}=m_{\rm d}$, m_s, $m_{\rm c}$, $m_{\rm b}$, the momentum cut-off parameter μ in the form factor, and the expanding coefficients of the wave functions into the Gaussian basis[8]. In the calculations we employ the running coupling constant, $\alpha_s(Q^2) = \dfrac{12\pi}{(33-2n_f)\ln(A+Q^2/B^2)}$, as in Ref. [8]. Here Q^2 is the square of the experimental mass of the meson, n_f is the flavor number of the meson, and A and B are taken to be 10 and 0.31 GeV as in Ref. [8]. Our input quantities are the 28 experimental meson masses. The values of the adjustable parameters are determined by minimizing the χ^2 between the 28 experimental meson masses and the corresponding calculated meson masses. For the results corresponding to the linear confining potential Eq. (11), the values of the adjustable potential parameters are: $b = 0.197$ GeV^2, $V_0 = -0.597$ GeV,

Table 1. Results of meson masses and the expanding coefficients $\{a_n\}$ of the wave functions. The masses are quoted in GeV.

meson	$M^{\rm exp}$	$M^{\rm th}$	$M^{[9]}$	$M^{[8]}$	a_1	a_2	a_3	a_4	a_5	a_6
$\pi(1^1S_0)$	0.140	0.140(0.140)	0.143	0.140	−0.030	−3.367	14.570	−32.115	32.589	−12.765
$K(1^1S_0)$	0.494	0.498(0.507)	0.494	0.495	0.796	−0.069	−1.823	7.019	−8.560	3.736
$K^*(1^3S_1)$	0.892	0.881(0.888)	0.907	0.904	2.078	−5.338	13.623	−19.485	14.374	−4.249
$\rho(1^3S_1)$	0.770	0.772(0.772)	0.788	0.774	1.690	−3.801	10.417	−15.690	12.200	−3.813
$\phi(1^3S_1)$	1.020	0.970(0.977)	1.031	0.992	1.016	−1.089	3.367	−4.735	3.467	−0.992
$b_1(1^1P_1)$	1.235	1.253(1.169)	1.397	1.330	1.657	−3.127	7.517	−10.248	7.298	−2.066
$a_1(1^3P_1)$	1.260	1.243(1.181)	1.573	1.353	1.727	−3.474	8.797	−12.819	9.800	−3.029
$\phi(2^3S_1)$	1.686	1.811(1.598)	1.852	1.870	5.222	−20.383	46.688	−63.877	45.176	−12.825
$D(1^1S_0)$	1.869	1.983(2.019)	1.865	1.913	1.646	−3.357	8.044	−10.453	6.941	−1.788
$D^*(1^3S_1)$	2.010	2.043(2.072)	1.998	1.998	1.884	−4.319	10.649	−14.677	10.436	−2.958
$D_s(1^1S_0)$	1.969	2.028(2.039)	1.976	2.000	0.989	−0.781	1.620	−0.810	−0.519	0.554
$D_s^*(1^3S_1)$	2.112	2.084(2.097)	2.121	2.072	1.184	−1.514	3.808	−4.690	2.911	−0.661
$D_1(1^1P_1)$	2.422	2.522(2.370)	2.408	2.506	1.970	−4.223	9.833	−13.422	9.591	−2.754
$D_2(1^3P_2)$	2.460	2.507(2.486)	2.381	2.514	1.957	−4.199	9.901	−13.695	9.929	−2.901
$\eta_c(1^1S_0)$	2.979	3.022(3.006)	2.978	3.033	0.809	−0.111	−0.329	2.576	−3.466	1.585
$J/\psi(1^3S_1)$	3.097	3.051(3.038)	3.128	3.069	0.908	−0.479	0.812	0.497	−1.596	0.913
$h_c(1^1P_1)$	3.570	3.461(3.438)	3.520	3.462	1.475	−2.259	5.387	−7.261	5.124	−1.441
$\chi_c(1^3P_1)$	3.525	3.463(3.455)	3.507	3.466	1.504	−2.383	5.776	−7.986	5.796	−1.689
$\psi'(2^3S_1)$	3.686	3.687(3.662)	3.689	3.693	5.279	−20.781	48.727	−68.050	49.253	−14.387
$B(1^1S_0)$	5.279	5.385(5.416)	5.272	5.322	2.398	−6.566	15.883	−21.705	15.278	−4.286
$B^*(1^3S_1)$	5.324	5.399(5.427)	5.319	5.342	2.457	−6.820	16.541	−22.716	16.077	−4.542
$B_s(1^1S_0)$	5.369	5.411(5.431)	5.368	5.379	1.650	−3.307	7.899	−10.296	6.872	−1.787
$B_s^*(1^3S_1)$	5.416	5.424(5.444)	5.426	5.396	1.706	−3.532	8.508	−11.287	7.693	−2.063
$\Upsilon(1^3S_1)$	9.460	9.463(9.454)	9.453	9.495	−0.087	−2.120	7.000	−14.748	14.427	−5.545
$\chi_b(1^3P_1)$	9.899	9.814(9.814)	9.889	9.830	0.635	0.410	−0.217	0.483	−0.477	0.218
$\Upsilon(2^3S_1)$	10.020	9.933(9.937)	10.023	9.944	3.337	−9.080	21.357	−33.533	27.020	−8.906
$\chi_b(2^3P_1)$	10.260	10.163(10.171)	10.257	10.166	3.112	−9.030	18.001	−24.326	17.376	−5.084
$\Upsilon(3^3S_1)$	10.350	10.331(10.332)	10.359	10.340	−7.363	39.761	−93.973	120.174	−79.911	20.876

$m_{\rm u}=m_{\rm d}=0.358$ GeV, $m_s=0.541$ GeV, $m_{\rm c}=1.739$ GeV, $m_{\rm b}=5.061$ GeV, and $\mu=0.918$ GeV. For the results corresponding to the relativistically corrected confining potential Eq. (13), the values of the adjustable potential parameters are: $a=0.097$ GeV^2, $b=0.211$ GeV^2, $V_0=-0.601$ GeV, $m_{\rm u}=m_{\rm d}=0.351$ GeV, $m_s=0.505$ GeV, $m_{\rm c}=1.718$ GeV, $m_{\rm b}=5.051$ GeV, and $\mu=1.010$ GeV. These values are in a reasonable range.

As is well-known, an attractive potential that becomes singular faster than $1/r^2$ as r approaches zero rises to the well-known "Landau fall"[12, 14, 15], in which the wave function collapses and oscillates without limit as r approaches zero. From Eq. (1) one can see that the terms corresponding to the $\delta(\boldsymbol{r})$ function (behaving like r^{-3} for $r\sim 0$),

$$-\frac{\boldsymbol{C}_{\rm ij}\alpha_s\pi}{m_{\rm i}m_{\rm j}}\left(\frac{m_{\rm i}}{2m_{\rm j}}+\frac{m_{\rm j}}{2m_{\rm i}}+\frac{2\boldsymbol{\sigma}_{\rm i}\cdot\boldsymbol{\sigma}_{\rm j}}{3}\right)\delta(\boldsymbol{r}),$$

and the spin-orbit interaction term,

$$-\frac{\boldsymbol{C}_{\rm ij}\alpha_s}{4m_{\rm i}m_{\rm j}r^3}(\boldsymbol{r}\times\boldsymbol{p})\cdot\left[\left(2+\frac{m_{\rm j}}{m_{\rm i}}\right)\boldsymbol{\sigma}_{\rm i}+\left(2+\frac{m_{\rm i}}{m_{\rm j}}\right)\boldsymbol{\sigma}_{\rm j}\right],$$

has alternating signs, whereas the orbit-orbit interaction term, $\frac{\boldsymbol{C}_{\rm ij}\alpha_s}{2m_{\rm i}m_{\rm j}}\left(\frac{\boldsymbol{p}^2}{r}+\frac{\boldsymbol{r}\cdot(\boldsymbol{r}\cdot\boldsymbol{p})\boldsymbol{p}}{r^3}\right)$, is always negative. Correspondingly, the orbit-orbit potential is always attractive. Our calculations showed that it is difficult to obtain stable solutions for the mesons by applying the regularization only once to the orbit-orbit potential. However, stable meson spectra can be obtained by applying the regularization twice to the orbit-orbit coupling term and once to the other singular terms. The stable results are independent of the basis functions used in the calculations.

6 Summary

In summary, we regularized the Breit potential by multiplying the singular terms by the form factor $\mu^2/(q^2+\mu^2)$ in momentum space. Using the regularized Breit potential and the confining potentials in the linear and relativistically corrected form, we calculated the low-lying mesons spectra. We found that the results for both of the two confining potentials are in good agreement with the experimental data. Our calculations indicate that the results of the the $q\bar{q}$ bound states are stable for the regularized potential. Hopefully this regularized potential will be used in future investigations for the bound states of baryons as well as some relevant hadron scattering problems.

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Appendix A:

Useful integration results from Eq. (29)

$$w_0 = \frac{1}{2}\sqrt{\frac{\pi}{\nu}}\mathrm{e}^{\frac{\mu^2}{4\nu}}\left[1-\mathrm{erf}\left(\frac{\mu}{2\sqrt{\nu}}\right)\right], \tag{A1}$$

$$w_1 = \frac{1}{2\nu}(1-\mu w_0), \tag{A2}$$

$$w_2 = \frac{1}{2\nu^2}\left[-\frac{\mu}{2}+\left(\nu+\frac{\mu^2}{2}\right)w_0\right], \tag{A3}$$

$$w_3 = \frac{1}{2\nu^3}\left[\nu+\frac{\nu^2}{4}-\left(3\nu\frac{\mu}{2}+\frac{\mu^3}{4}\right)w_0\right], \tag{A4}$$

$$w_4 = \frac{1}{4\nu^4}\left[-5\nu\frac{\mu}{2}-\frac{\mu^3}{4}+\left(3\nu^2+3\nu\mu^2+\frac{\mu^4}{4}\right)w_0\right], \tag{A5}$$

$$w_5 = \frac{1}{4\nu^5}\left[4\nu^2+9\nu\frac{\mu^2}{4}+\frac{\mu^4}{8}-\left(15\nu^2\frac{\mu}{2}+5\nu\frac{\mu^3}{2}+\frac{\mu^5}{8}\right)w_0\right], \tag{A6}$$

where $\mathrm{erf}(\boldsymbol{x})$ is error function.