SLIM – a formalism for linear coupled systems*

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Abstract A SLIM formalism to deal with a general, linearly coupled accelerator lattice is summarized. Its application to a wide range of accelerator calculations is emphasized.

Key words linear coupling, eigen modes, beam emittance, depolarization time, SLIM

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1 Introduction

Consider the linear optics of a circular accelerator. Following the tradition of Courant and Snyder[1], we analyze particle motion using *transport matrices*. We first form a phase space state vector

$$Z = \begin{bmatrix} x \\ x' \end{bmatrix}, \tag{1}$$

where x is the coordinate and x' is the momentum of the particle. Particle motion from position 1 to position 2 in the accelerator beamline is described by a 2×2 transport matrix $M(1\to 2)$. To calculate $M(1\to 2)$, one multiplies the transport matrices element by element from position 1 to position 2.

In this formalism, it is important to recognize that *all* beam dynamics associated with linear optics are contained in the transport matrices. In other words, these matrices should give us everything we want to know, and all physical results must be derivable from them without having to seek additional inputs. Our job is to analyze these transport matrices to extract all the physics information they contain to the fullest possible extent. Question, is how.

The way Courant-Snyder did it was to introduce a set of "auxiliary functions" $\alpha(s),\beta(s),\gamma(s),\psi(s)$, as well as the dispersion functions $\eta(s),\eta'(s)$ to deal with the off-momentum particles. The goal, of course, is to calculate all the physical quantities associated with the beam. By physical quantities, I mean quantities associated directly to the beam's measurable physical properties, such as closed orbit distortion, momentum compaction factor, betatron and synchrotron tunes, x-y coupling coefficient, rms beam sizes, bunch length, and energy spread, etc. Now note that this long list does not contain the auxiliary functions themselves. These functions play an auxiliary role helping us to calculate the physical quantities, but they themselves are not physical.

So, in the Courant-Snyder tradition, we have been doing accelerator physics in three steps:

1) find the transport matrices $M(1\to 2)$ by multiplying element matrices.

2) compute the auxiliary lattice functions α, β, γ, ψ, η, η', $\mathscr{H}(s)$, etc. using the transport matrices. (Here for those familiar with electron storage ring optics, I added another auxiliary function $\mathscr{H}(s)$.)

3) compute the beam's physical properties using the auxiliary lattice functions.

Again, step 1 contains all the physics. Step 2 contains no physics but, for a 1-D uncoupled system, the set of auxiliary functions provide sufficient flexibility to allow all physical quantities to be calculated in step 3.

Not to question its monumental importance on accelerator physics, however, the Courant-Snyder formalism is not without weakness. The main point is that it applies only to 1-D uncoupled dynamics. For example, β_x and β_y lose their meanings when x- and y-motions are coupled, and η and η' become obsolete when the synchrotron tune becomes large, or near a synchrobetatron resonance, etc. Trying to retrofit the Courant-Snyder formalism to deal with these coupled cases is often awkward if not impossible.

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To establish Step 3 above, we are accustomed to use formulae explicitly involving the auxiliary functions. Examples are easily foundavailable, as listed in Ref. [2]. But these textbook formulae work only for the 1-D uncoupled cases. In actual applications, we often have to ask what replaces these formulae when they break down? What if there is a skew quadrupole in the storage ring? What if there is a crab cavity? What happens if the tunes are near or even very close to a resonance?

Facing this weakness, one begins to wonder if there is another way to calculate the beam's physical parameters directly from the transport matrices (6×6 with general coupling) without resorting to the auxiliary functions. After all, as mentioned, these matrices contain all the information we need for the arbitrarily coupled system under study.

Indeed it is an old topic to find ways to extend the Courant-Snyder formalism, and many people have tried it. One such early effort proposed in 1979-81[3, 4] is reviewed here. There are also other efforts with a similar goal. A set of examples (nonexhaustive) are Refs. [5—11].

2 Courant-Snyder representation is not a unique choice

Let me begin by illustrating that the Courant-Snyder formalism is not unique. This is because by illustrating that, I indirectly prove that the Courant-Snyder representation, or at least a significant part of it, can only be an artifact, and if there is a good reason to, it can be replaced without loss of real contents. In other words, we want to show that the Courant-Snyder formalism is not sacred.

The basic representation that defines the Courant-Snyder formalism is

$$x(s)=\sqrt{\epsilon\beta(s)}\,\sin\psi(s),\qquad \psi(s)=\int^s\frac{ds'}{\beta(s')},\qquad (2)$$

where $\psi(s)$ is the betatron phase. The formalism is based on an eigen-analysis (normal form analysis). What one intends to do is to look for a coordinate transformation from (x,x') of Eq. (1) to (u,p_u), in such a way that the complicated dynamics of (x,x') in the accelerator lattice becomes simple uniform rotation on a circle in the (u,p_u) space. In fact, the transformation from (x,x') to (u,p_u) is well-known,

$$\begin{bmatrix}u\\ p_u\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\sqrt{\beta}} & 0\\ \dfrac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\end{bmatrix}\begin{bmatrix}x\\ x'\end{bmatrix}=\begin{bmatrix}\dfrac{x}{\sqrt{\beta}}\\ \dfrac{\alpha x+\beta x'}{\sqrt{\beta}}\end{bmatrix}.\qquad (3)$$

But there are actually an infinite number of ways to parametrize a normal form transformation; Eq. (3), chosen by Courant and Snyder, is just one of them that is particularly elegant.

Investigating further, one then finds that, even for the sake of elegance, the choice (3) is not unique. It turns out that there exists another equally elegant choice for normal form transformation:

$$\begin{bmatrix}\bar u\\ \bar p_u\end{bmatrix}=\begin{bmatrix}\sqrt{\gamma} & \dfrac{\alpha}{\sqrt{\gamma}}\\ 0 & \dfrac{1}{\sqrt{\gamma}}\end{bmatrix}\begin{bmatrix}x\\ x'\end{bmatrix}=\begin{bmatrix}\dfrac{\gamma x+\alpha x'}{\sqrt{\gamma}}\\ \dfrac{x'}{\sqrt{\gamma}}\end{bmatrix}.\qquad (4)$$

With this choice, particle motion in the $(\bar u,\bar p_u)$ space also follows a nice simple circle. In this case, instead of Eq. (2), the basic transformation we introduce looks like

$$x'(s)=\sqrt{\epsilon\gamma(s)}\,\sin\bar\psi(s),\quad \bar\psi(s)=\int^s\frac{K(s')ds'}{\gamma(s')},\qquad (5)$$

where now the betatron phase is given by $\bar\psi(s)$.

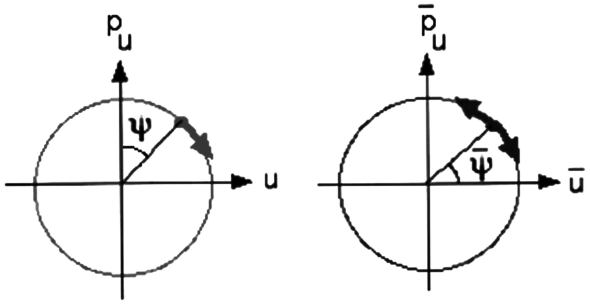


Fig. 1. Normalized coordinates in the Courant-Snyder (left) and the alternative (right) conventions.

Figure 1 shows a comparison between the normalized coordinates in the familiar Courant-Snyder and the alternative conventions. Fig. 2 shows the Courant-Snyder functions for a FODO cell, as well as the alternative betatron phase $\bar\psi(s)$. A short comparison between the Courant-Snyder and the alternative formalisms might be summarized in Table 1.

Table 1. Comparison between Courant-Snyder and alternative formalisms.

	emphasis	envelope function	betatron phase
Courant-Snyder	x	$\beta(s)$	$\psi(s)$
alternative	x'	$\gamma(s)$	$\bar\psi(s)$

From Fig. 2, one sees that the betatron phases ψ and $\bar\psi$ indeed look very different. For example, ψ advances when the particle goes through a drift space, while $\bar\psi$ stands still in a drift space and advances only when going through a quadrupole. Also, ψ always advances monotonically, while $\bar\psi$ advances

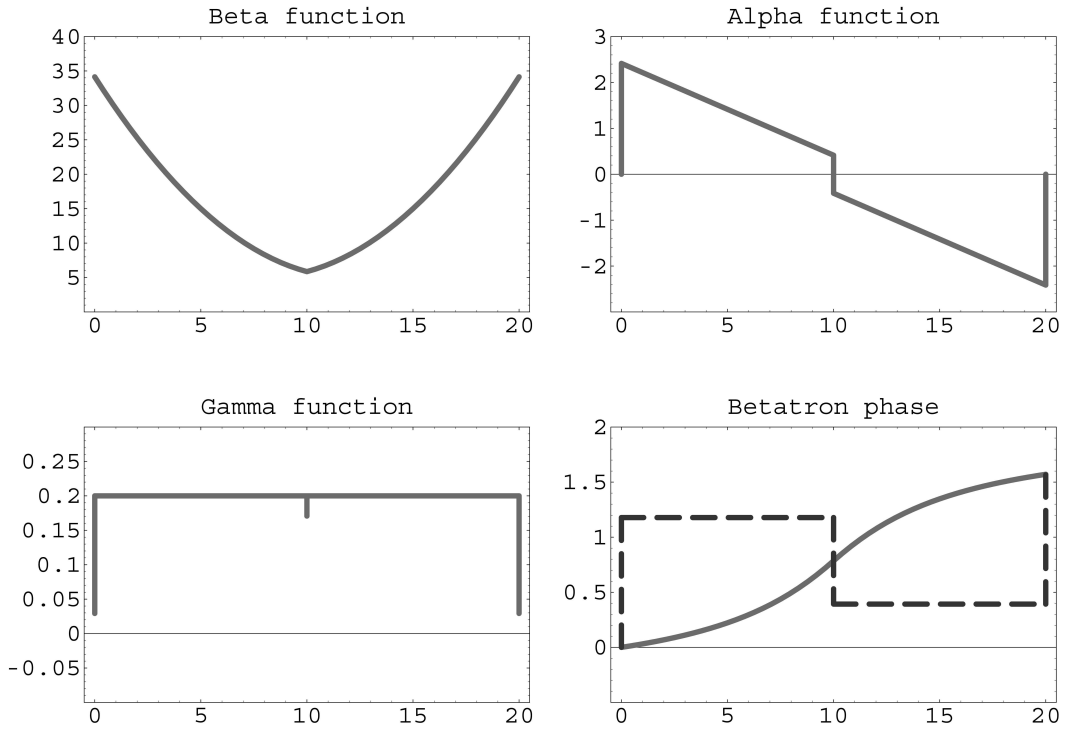


Fig. 2. The familiar Courant-Snyder functions $\beta(s), \alpha(s), \gamma(s)$ and $\psi(s)$ for a FODO cell are shown in solid line. The dashed curve gives the alternative betatron phase $\bar{\psi}(s)$.

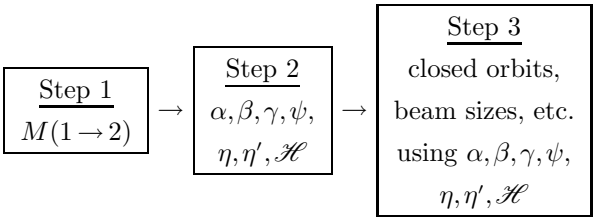
going through a focusing quadrupole and goes backwards going through a defocusing quadrupole.

Had Courant-Snyder chosen (4) and (5) for their classic paper, today we would be using a different accelerator physics language. We would not recognize many derivations in our present textbooks[2]. And yet, both representations give identical results for all physical quantities.

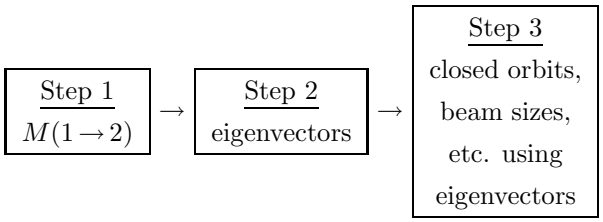
3 Replacing the auxiliary functions by eigenvectors

At this point, it is natural to contemplate the possibility of replacing all the auxiliary lattice functions by the eigenvectors of the transport matrices, like this:

Conventional scheme:



Eigenvector (SLIM) scheme:



Note that the set of all eigenvalues and eigenvectors contains all the information contained in a transport matrix (which is now 6×6 and generally coupled). There is no loss (and of course also no gain) of information by resorting to the eigen-analysis. The eigenvector scheme has the advantage that it deals readily with the weakness of Courant-Snyder formalism mentioned earlier.

Incidentally I shall call this replacement scheme the SLIM formalism, following the name of an early computer code. Using this formalism, hopefully storage ring design codes can become slimmer.

In the SLIM analysis, we aim for a single computing framework that covers a wide range of situations. No assumptions are made on the 6×6 transport matrices other than those imposed by fundamental physics such as Liouville theorem (symplecticity). RF cavities are considered longitudinal focusing elements, just like quadrupoles are in the transverse motion. A crab cavity acts as a y-z or x-z coupling element, just like a skew quadrupole as an x-y coupling element.

Synchrotron tune does not have to be small, and the three orbital tunes can be near any combination of linear resonances. Therefore, in SLIM, the following cases are treated the same way on equal footing:

1) Betatron motion and synchrotron motion
2) Coupled case and uncoupled case
3) Near resonances and away from resonances
4) Spin motion and orbital motion
5) Orbital resonances and depolarization resonances.

While applications cover a wide range, there is only one straightforward framework of actual computation. The original SLIM program, when first written, had only ∼1000 lines.

4 Calculating physical quantities using eigenvectors

I need to demonstrate how to calculate the physical quantities in SLIM. To do this, the state vector first needs to be generalized to become 6-dimensional,

$$Z=\begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{bmatrix}. \tag{6}$$

Given the linear optics of a storage ring, one first calculates the 6×6 transport matrices $M(s\to s+C)$ for one turn around position s by multiplying element matrices around the storage ring. Let me illustrate the SLIM calculation of physical quantities by a few examples.

Tunes

The six eigenvalues

$$\mathrm{e}^{\pm \mathrm{i}2\pi\nu_{\mathrm{k}}},\quad k=\mathrm{I},\mathrm{II},\mathrm{III}, \tag{7}$$

immediately give three tunes $\nu_{\mathrm{I},\mathrm{II},\mathrm{III}}$. In the nominal case, they are the horizontal, vertical, and synchrotron tunes. In case of an arbitrary coupled system, they are the tunes of the three eigenmodes.

Each eigenmode also has a pair of eigenvectors. The six eigenvectors are $E_{\mathrm{I},\mathrm{II},\mathrm{III}}$ and $E^{*}_{\mathrm{I},\mathrm{II},\mathrm{III}}$. As mentioned, all beam dynamics contained in the transport matrices are now contained in these eigenvalues and eigenvectors. Our next job is then to construct all remaining physical quantities out of these eigenvectors, without resorting to other auxiliary lattice functions. The Courant-Snyder β-function, for example, does not need to be calculated.

Closed orbits

Once all the orbital perturbations are given in the lattice, the closed orbit is calculated simply from the condition that the resulting closed-orbit vector (6) closes onto itself after one complete turn.

This result should not be taken too lightly. It should be emphasized that what one finds here is an orbit in the 6-D phase space. In the nominal uncoupled case, its 1-st and 3-rd components give the usual Δx and Δy. But it contains a lot more, such as x-y coupled closed orbit, synchrobetatron coupled orbit, orbits generated by localized RF cavities, the longitudinal closed orbit Δz, and the energy shift closed orbit $\Delta\delta$. It also contains some hidden useful information, such as the momentum compaction factor α_c, and the total synchrotron radiation energy loss U_0, both in the presence of general coupling. It is possible that some additional, more subtle effects have yet to be explored further utilizing this flexibility.

Coupling effects

Once the closed orbit is obtained, sextupoles can be included by linearizing them around the closed orbit.

Linear coupling and linear resonances, whether x-y or synchrobetatron in nature, are treated on equal footing. When we calculate the beam parameters, these coupling effects are included automatically and no approximations such as weak coupling are necessary. The calculation is exact[12].

As one example, a crab cavity, an element not readily treated by the conventional analysis, is a straightforward application here. All one has to do is to include these coupling elements in the transport matrices. Subsequent calculations of physical quantities will then include their effects automatically.

Radiation damping times

Radiation damping originates from two effects. One is that in a bending magnet, a particle with higher (lower) energy will lose more (less) energy due to synchrotron radiation. The other is that in an RF cavity, the transverse momentum will be slightly reduced because the acceleration is purely longitudinal. These effects are easily incorporated by slightly modifying the transport matrices of the bending magnets and the RF cavities. Having done so, the six eigenvalues now become

$$\mathrm{e}^{-\alpha_{\mathrm{k}}\pm \mathrm{i}2\pi\nu_{\mathrm{k}}},\quad k=\mathrm{I},\mathrm{II},\mathrm{III}, \tag{8}$$

where the additional parameters $\alpha_{\mathrm{I},\mathrm{II},\mathrm{III}}$ are the radiation damping constants for the three eigenmodes. The Orlov-Robinson-Tarasov sum rule is automatically satisfied.

In the nominal calculation using auxiliary functions, the traditional way to calculate these constants involve the calculation of the partition number $\mathcal{D}$. It then becomes a dubious venture how to extend that calculation when the system is coupled or a resonance is close by. The SLIM approach avoids the problem. Equilibrium beam sizes in an electron storage ring

When a photon of energy u is emitted, the emitting electron suffers a sudden quantum excitation on its state vector,

$$\Delta Z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -u/E_0 \end{bmatrix}. \tag{9}$$

Once the eigenvectors E_k are known, this quantum excitation can be decomposed into a linear sum of these eigen-states,

$$\begin{aligned} \Delta Z &= \sum_{k=\pm\,\mathrm{I},\mathrm{II},\mathrm{III}} e_{\mathrm{k}} E_{\mathrm{k}}, \\ e_{\mathrm{k}} &= -\mathrm{i}(u/E_0) E^*_{\mathrm{k}5}. \end{aligned} \tag{10}$$

By balancing the quantum excitations of these eigenmodes with their respective radiation damping, one then obtains the three equilibrium eigen-emittances,

$$\epsilon_{\mathrm{k}} = \frac{55}{48\sqrt{3}} \frac{r_{\mathrm{e}} \hbar \gamma^5}{m_{\mathrm{e}} c \alpha_{\mathrm{k}}} \oint \mathrm{d}s \frac{|E_{\mathrm{k}5}(s)|^2}{|\rho(s)|^3}. \tag{11}$$

The second moments of the equilibrium beam distribution are then given by

$$\langle Z_{\mathrm{i}} Z_{\mathrm{j}} \rangle = 2 \sum_{k=\mathrm{I},\mathrm{II},\mathrm{III}} \epsilon_{\mathrm{k}} \mathrm{Re}[E_{\mathrm{ki}} E^*_{\mathrm{kj}}]. \tag{12}$$

In a linear system, the beam distribution is strictly Gaussian, and the 21 quantities in (12) completely specify the equilibrium distribution, and consequently the sizes and shapes, of the beam in the 6-D phase space. Perhaps one can compare the SLIM result with the corresponding expression in a conventional 1-D uncoupled theory here. The conventional theory basically has an expression similar to Eq. (11) but with $\mathscr{H}$ replacing the eigenvector $|E_{\mathrm{k}5}|^2$, and the resulting expression gives the unperturbed horizontal beam size $\langle x^2 \rangle$ for a 1-D uncoupled beam. In contrast, Eq. (11) gives the three eigenmode emittances for a 3-D arbitrarily coupled beam, and subsequently Eq.(12) gives the 21 beam distribution moments. The eigenvector algorithm is obviously much more powerful in actual applications.

5 Adding spin dynamics

Once a generalization to 3-D dynamics is made, one may take one step further. A proton or an electron has a 4-th dimension in its dynamics. In addition to x-, y- and z-motions, it also has a dynamics involving its spin. By adding spin as its 4-th dimension, and extending the eigen-analysis from 6-D to 8-D, the SLIM formalism also calculates the spin properties of the beam.

To do so, we consider the state vector, now 8-dimensional,

$$Z = \begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \\ \alpha \\ \beta \end{bmatrix}, \tag{13}$$

where α and β are the two components describing the deviation of the spin of a particle from its nominal (usually vertical, but does not have to) direction. The transport matrices are now 8×8. In addition to the three eigentunes in Eq. (7), there is now a 4-th eigentune,

$$\nu_{\mathrm{IV}} = \text{spin precession tune}. \tag{14}$$

The reason that the SLIM formalism is particularly suitable to deal with spin dynamics is the fact that spin motion couples intimately, and in a rather complicated manner, to the orbital dynamics, and SLIM is particularly useful to deal with this spin-orbital coupling because it simply treats spin motion and orbital motions on an equal footing, and treats coupled and non-coupled motions in one single framework.

Table 2. Analogy between the mechanism for equilibrium beam emittances and mechanism for equilibrium level of spin polarization.

	diffusion	⟷	damping	beam property
orbital motion	radiation damping	⟷	quantum excitation	emittances
spin motion	radiative polarization	⟷	spin diffusion	polarization

The equilibrium polarization of an electron beam is therefore determined the same way as the equilibrium beam sizes are calculated. After an emission of a synchrotron radiation photon, the state vector of an electron suffers a sudden impact

$$\Delta Z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -u/E_0 \\ 0 \\ 0 \end{bmatrix}. \tag{15}$$

We then decompose this impact vector into a sum over eight eigenvectors. The last two components, projected to the two spin eigenvectors, represent the quantum excitation of synchrotron radiation onto the spin motion. By balancing this quantum spin diffusion against radiative polarization, one then obtains the equilibrium level of beam polarization in the same way we obtain the equilibrium beam emittances by balancing the quantum excitation again radiation damping, as illustrated in Table 2.

6 Summary

In learning accelerator physics, there is no substitute to first learn the elegant Courant-Snyder formalism for the 1-D uncoupled case. In actual accelerator applications, often one needs to deal with more complicated 2-D and 3-D coupled cases. For those applications, it is not a good idea to confine ourselves to the Courant-Snyder formalism and try, for example, to invent some "generalized β-functions". Instead, one should utilize eigen-analysis of the transport matrices. In doing so, one can deal with coupled multi-dimensional cases readily, including the 4-D cases when spin dynamics are included, in a rather straightforward manner.

References

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2 There are many such examples:

Momentum compaction factor $\alpha_c = \frac{1}{C}\oint\frac{\eta\, ds}{\rho}$

Closed orbit $\Delta x = \frac{\theta_k\sqrt{\beta_k\beta}}{2\sin\pi\nu_x}\cos[\pi\nu_x - |\psi-\psi_k|]$

Beam size $\sigma_x^2 = \sigma_{x\beta}^2 + \sigma_\delta^2\eta^2$

$$\frac{\sigma_{x\beta}^2}{\beta} = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc}\frac{\gamma^2}{1-\mathcal{D}}\frac{\oint\frac{\mathcal{H}ds}{|\rho|^3}}{\oint\frac{ds}{\rho^2}}$$

Damping partition $J_x = 1-\mathcal{D}, \quad J_y = 1, \quad J_s = 2+\mathcal{D}$

$$\mathcal{D} = \frac{\oint\frac{\eta ds}{\rho}(2K+\frac{1}{\rho^2})}{\oint\frac{ds}{\rho^2}}$$

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12 This is slightly exaggerated. The calculation as given here is valid only if the tunes stay away from the resonances by a distance of the order of the radiation damping constants. Since the radiation damping time constants are typically extremely small, the sacrifice of parameter space is negligibly small