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Spin path integral and quantum mechanics in the rotating frame of reference*

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Abstract: We have developed a path integral formalism of the quantum mechanics in the rotating frame of reference, and proposed a path integral description of spin degrees of freedom, which is connected to the Schwinger bosons realization of the angular momenta. We have also given several important examples for the applications in the rotating frames.

Key words: path integral, spin, rotating frame of reference, field theory

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1 Introduction

The Quantum mechanics in the rotating frame of reference has many important applications. For example, Rabi oscillation is crucial for cavity quantum electrodynamics [1] and for designing the qubit circuit of a scalable quantum computer [2–4]. And the analysis of Coriolis effect is very important in the research of unified geometric phase and spin-Hall effect in optics [5].

The main purpose of this paper is to develop a path integral description of the non-relativistic quantum mechanics in the rotating frame. We investigate a charged particle in the rotating frame with a uniform external magnetic field applied to it, and use the path integral description to explain some related experiments, e.g. the Sagnac effect [6, 7] due to the coupling between the orbital angular momentum of the particle and the rotation of the reference frame (see Ref. [8] for more specific introduction of this effect), the spin-rotation coupling analog of the Sagnac effect, and the Neutron interference [9–14] (especially in Ref. [15] for the experimental observation of the phase shift via spin-rotation coupling). We also show the application in the Rabi oscillation problem which was traditionally solved in the Hamiltonian formulation [16].

By using the spin path integral description [17] and connecting it to the Schwinger bosons realization of the algebra of angular momenta $\vec{\mathcal{J}}$, we can introduce the part of the action for the spin degrees of freedom of a charged point particle in a rotating frame of reference, and develop the corresponding spin path integral description. Combining the path integral description with coordinate variables [18], we can develop the full path integral theory of a point particle in a rotating frame. We can then derive the phase factor due to the spin-rotation coupling and its orbital angular momentum counterpart (the Sagnac effect), give a path integral solution for Rabi oscillation problem, and extend the Coriolis force from classical mechanics to quantum mechanics, by using the quantum action principle [19].

The paper is arranged as follows. In Section 2, we present the total action for the charged particle in a rotating frame of reference and formulate the path integral description for the corresponding non-relativistic quantum mechanics of the point particle. The spin-orbital coupling is also discussed. In Section 3 we give some applications on some well known quantum mechanical effects, the Sagnac effect and its spin-rotation coupling extension, the Rabi oscillation

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and the operator equation for the Coriolis force. In these examples, the rotating frame formulations are more convenient. Finally, we end with some concluding remarks.

2 The path intergal formalism of quantum mechanics in the rotating frame

2.1 Spin path integral and the Schwinger bosons

To describe the spin degrees of freedom of a non-relativistic spin $s=n/2$ particle (with mass m and electric charge e), we introduce the action [17]

$$I_{\text{Spin}}=\int \mathrm{d}t\left[\mathrm{i}\phi^{\dagger}\frac{\mathrm{d}}{\mathrm{d}t}\phi-\lambda(\phi^{\dagger}\phi-n)\right], \qquad (1)$$

where ϕ is a two components bosonic variable, $\phi^{\dagger}=(\overline{\phi^1},\overline{\phi^2})$ is its Hermitian conjugation, and λ is a Lagrange multiplier. The action (1) is invariant under a $U(2)$ transformation $\mathcal{U}:\ \phi\to\mathcal{U}\phi$, where $\mathcal{U}\in U(2)$. To realize the $SU(2)$ symmetry of spin, one can consider operations of multiplication modulo some $U(1)$ factor from $U(2)$, i.e. $\phi\to \mathrm{e}^{-\mathrm{i}\theta}\phi$, $\overline{\phi}\to \mathrm{e}^{\mathrm{i}\theta}\,\overline{\phi}$, by requiring that all the physical observables be $U(1)$ invariant. Furthermore, $N(\phi)=\phi^{\dagger}\phi$ is the conserved charge of this $U(1)$ transformation, and is also a physical observable. Other fundamental physical observables[1)] are the conservation charges $\vec{S}(\phi)=\dfrac{1}{2}\phi^{\dagger}\vec{\sigma}\phi$ of the $SU(2)$ symmetry, where $\vec{\sigma}$ are the three Pauli matrices. In the quantum theory, $\vec{S}(\phi)$ will realize $SU(2)$ symmetry algebra, which we identify with the spin. Thus ϕ is a Pauli spinor. In the path integral quantization, $\vec{S}(\phi)$ can be inserted into the path integral $\int\mathcal{D}\phi\mathcal{D}\overline{\phi}\mathcal{D}\lambda\exp(\mathrm{i}I/\hbar)$.

To see the connection between the above spin path integral and the $SU(2)$ spin algebra more clearly, we now perform the canonical quantization procedure. Firstly, $\mathrm{i}\phi^{\dagger}$ is the canonical momentum of ϕ, and the Hamiltonian for the free spin is trivial. Then ϕ is now realized as the Schwinger bosons with the commutators $\left[\phi^{\alpha},\ \overline{\phi^{\beta}}\right]=\hbar\delta^{\alpha\beta}$, where $\alpha,\ \beta=1,\ 2$ are the indices of the spinor ϕ. The constraint equation associated with λ will restrict us to the Hilbert space of n Schwinger bosons $(\phi^{\dagger}\phi-n)\left|\psi\right\rangle=0$, where $\left|\psi\right\rangle$ is an arbitrary state of the Hilbert space $\mathcal{H}_{\mathrm{s}}$ of spin wave functions, $\left|\psi\right\rangle\in\mathcal{H}_{\mathrm{s}}$. The basis of $\mathcal{H}_{\mathrm{s}}$ can be constructed by acting n creation bosons $\overline{\phi^{\alpha}}$ on the Fock vacuum[2)], $\left|\alpha_1\alpha_2\cdots\alpha_n\right\rangle=\overline{\phi^{\alpha_1}}\,\overline{\phi^{\alpha_2}}\cdots\overline{\phi^{\alpha_n}}\left|0\right\rangle$, where $\left|0\right\rangle$ is the Fock vacuum which satisfies $\phi^{\alpha}\left|0\right\rangle=0$. The spin operators $\vec{S}$ are realized as

$$\vec{S}=\frac{1}{2}\phi^{\dagger}\vec{\sigma}\phi\ . \qquad (2)$$

The action of $\vec{S}$ on $\mathcal{H}_{\mathrm{s}}$ turns out to be the $s=n/2$ representation.

2.2 Spin in the rotating frame of reference

We consider a non-inertial frame of reference $\mathcal{S}$, which rotates with angular velocity $\vec{\omega}(t)$. Firstly, we note that in terms of the variables in $\mathcal{S}$, the time derivative of the spinor $\phi(t)$ should be

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}-\frac{\mathrm{i}}{2}\vec{\omega}(t)\cdot\vec{\sigma}\phi\ , \qquad (3)$$

this result is the generalization of the similar ordinary $\mathrm{d}\vec{v}/\mathrm{d}t+\vec{\omega}(t)\times\vec{v}$, where $\vec{v}$ is a vector in $\mathcal{S}$ frame. If $\vec{\mathcal{J}}$ is the general generator of the $SU(2)$ group, in the spin-1 representation , we can rewrite $\vec{\mathcal{J}}$ as $\vec{\mathcal{L}}$: $(\mathcal{L}_k)^{l}_{\ j}:=-\mathrm{i}\epsilon_{ljk}$, here $l,\ j,\ k=1,\ 2,\ 3$. Our generalized form (3) is motivated by noticing that under an infinitesimal rotation $\left(1-\mathrm{i}\delta\vec{\theta}\cdot\vec{\mathcal{J}}\right)$, a vector $\vec{v}$ transforms as[3)]: $\vec{v}\to\vec{v}-\mathrm{i}\left(\delta\vec{\theta}\cdot\vec{\mathcal{L}}\right)\vec{v}=\vec{v}+\delta\vec{\theta}\times\vec{v}$. However, a spinor ϕ transforms as: $\phi\to\phi-\dfrac{\mathrm{i}}{2}\delta\vec{\theta}\cdot\vec{\sigma}\phi$.

Thus, in frame $\mathcal{S}$, the action of the spin is given simply by replacing the $(\mathrm{d}\phi/\mathrm{d}t)$ of Eq. (1) with $\mathrm{d}\phi/\mathrm{d}t-\dfrac{\mathrm{i}}{2}\vec{\omega}(t)\cdot\vec{\sigma}\phi$:

$$\begin{aligned}I_{\mathcal{S}}&=\int\mathrm{d}t\left[\mathrm{i}\phi^{\dagger}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\frac{\mathrm{i}}{2}\vec{\omega}\left(t\right)\cdot\vec{\sigma}\right)\phi-\lambda\left(\phi^{\dagger}\phi-n\right)\right]\\&=\int\mathrm{d}t\left[\mathrm{i}\phi^{\dagger}\frac{\mathrm{d}}{\mathrm{d}t}\phi+\vec{\omega}\left(t\right)\cdot\frac{1}{2}\phi^{\dagger}\vec{\sigma}\phi-\lambda\left(\phi^{\dagger}\phi-n\right)\right].\end{aligned} \qquad (4)$$

In terms of the path integral quantization, spin in the rotating frame is described as an insertion $\vec{S}(\phi)=\dfrac{1}{2}\phi^{\dagger}\vec{\sigma}\phi$ in $\int\mathcal{D}\phi\mathcal{D}\overline{\phi}\mathcal{D}\lambda\exp(\mathrm{i}I_{\mathcal{S}}/\hbar)$.

To compare with the Hamiltonian formalism, we then canonically quantize the action (4). The Hamiltonian $H_{\mathcal{S}}$ can be easily got[4)],

$$H_{\mathcal{S}}=-\vec{\omega}\left(t\right)\cdot\vec{S}\ . \qquad (5)$$

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1) It is easy to show that an arbitrary physical observable can be written as the function of $\vec{S}(\phi)$ and $N(\phi)$.

2) In the full Fock space, one can also construct the spin coherent state $\left|z\right\rangle$ by defining it as the eigen-states of ϕ, with $\phi^{\alpha}\left|z\right\rangle=z^{\alpha}\left|z\right\rangle$. The mean value of the spin operator $\vec{S}$ on the spin coherent states is $\left\langle z\right|\vec{S}\left|z\right\rangle=\dfrac{1}{2}z^{\dagger}\vec{\sigma}z$.

3) The transformation of components is: $v^l\to v^l-\mathrm{i}\delta\theta^k\cdot(\mathcal{L}_k)^l_{\ j}v^j$.

4) The quantization relation of the Schwinger bosons, the constraint equation and the spin Hilbert space $\mathcal{H}_{\mathrm{s}}$ are all the same as mentioned above.

This result is consistent with Refs. [6, 16].

In the static frame, one should insert $\exp\left[\mathrm{i}g\mu\int \mathrm{d}t\vec{B}\cdot\vec{S}(\phi)/\hbar\right]$ into the path integration to account for the spin-magnetism coupling, where $\mu=e/2mc$, g is the g-factor of the particle, and $\vec{B}$ is a uniform magnetic field. The total effect is to change the action from I for the free spin to I_{B} for the spin in the $\vec{B}$ field, where I_{B} is given as

$$I_{\mathrm{B}}=\int \mathrm{d}t\left[\mathrm{i}\phi^{\dagger}\frac{\mathrm{d}}{\mathrm{d}t}\phi-\lambda(\phi^{\dagger}\phi-n)+g\mu\vec{B}\cdot\vec{S}(\phi)\right]. \quad (6)$$

From action (6), one can easily get the equations of motion of the Pauli spinors ϕ,

$$\mathrm{i}\frac{\mathrm{d}\phi}{\mathrm{d}t}+\frac{1}{2}g\mu\vec{B}\cdot\vec{\sigma}\phi=0$$

and its complex conjugation. By using these equations, one can get the equation of motion of the spin in magnetic field $\vec{B}$

$$\frac{\mathrm{d}\vec{S}}{\mathrm{d}t}+g\mu\vec{B}\times\vec{S}=0. \quad (7)$$

Now we can consider the spin-magnetism coupling in $\mathcal{S}$ frame. For brevity, we assume that the angular velocity $\vec{\omega}$ is time independent. $\vec{B}$ denotes the magnetic field in the rotating frame $\mathcal{S}$. To write out the explicit form of the magnetic field $\vec{B}_{\mathrm{iner}}(t)$ in the static inertial frame, we first decompose $\vec{B}$ into the part $\vec{B}_{\parallel}$, which is parallel to $\vec{\omega}$, and the part $\vec{B}_{\perp}$, which is perpendicular to $\vec{\omega}$. The magnetic field in the static frame can then be written as $\vec{B}_{\mathrm{iner}}(t)=\vec{B}_{\parallel}+\vec{B}_{\perp}^{+}\mathrm{e}^{\mathrm{i}\omega t}+\vec{B}_{\perp}^{-}\mathrm{e}^{-\mathrm{i}\omega t}$, where the additional factors $\mathrm{e}^{\mathrm{i}\omega t}$ and $\mathrm{e}^{-\mathrm{i}\omega t}$ are due to the rotation of $\mathcal{S}$[1)]. Now, actions (4) and (6) should be jointed together into the action I_{BS},

$$I_{\mathrm{BS}}=\int \mathrm{d}t\left[\mathrm{i}\phi^{\dagger}\frac{\mathrm{d}}{\mathrm{d}t}\phi+(\vec{\omega}+g\mu\vec{B})\cdot\vec{S}(\phi)-\lambda(\phi^{\dagger}\phi-n)\right]. \quad (8)$$

In terms of the canonical quantization, the Hamiltonian in $\mathcal{S}$ is H_{BS}:

$$H_{\mathrm{BS}}=-g\mu\vec{B}\cdot\vec{S}-\vec{\omega}\cdot\vec{S}, \quad (9)$$

which is in agreement with the previous results [16].

2.3 Include the position variables

In a uniform magnetic field, the spatial part I_{x} of the total action can be written in terms of the variables of $\mathcal{S}$:

$$\begin{aligned}I_x=&\int \mathrm{d}t\left[\frac{1}{2}m\left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}+\vec{\omega}\times\vec{x}\right)^2-V(\vec{x})\right]\\&-\frac{e}{c}\int \mathrm{d}t\vec{A}(\vec{x})\cdot\left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}+\vec{\omega}\times\vec{x}\right),\end{aligned} \quad (10)$$

where $V(\vec{x})$ is the potential energy of the particle in the static inertial frame, the vector potential $\vec{A}(\vec{x})$ can be written as $\vec{A}(x)=-\frac{1}{2}\vec{B}\times\vec{x}$ for the uniform magnetic field that we are considering. Thus, I_x can be rewritten as

$$\begin{aligned}I_x=&\int \mathrm{d}t\left[\frac{1}{2}m\left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)^2+m\left(\vec{\omega}+\mu\vec{B}\right)\cdot\left(\vec{x}\times\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)\right]\\&-\int \mathrm{d}t\left[V(\vec{x})-\frac{1}{2}m(\vec{\omega}\times\vec{x})^2\right.\\&\left.-m\mu\left(\vec{B}\times\vec{x}\right)\cdot(\vec{\omega}\times\vec{x})\right].\end{aligned} \quad (11)$$

Furthermore, in the path integral quantization, one should include the factor $\int\mathcal{D}\vec{x}\exp(\mathrm{i}I_x/\hbar)$.

In the canonical quantization to (11), we take $\vec{x}$ as the canonical coordinates, the canonical momenta of $\vec{x}$ can be easily gotten:

$$\vec{p}=m\left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}+\vec{\omega}\times\vec{x}+\mu\vec{B}\times\vec{x}\right). \quad (12)$$

Finally, the Hamiltonian reads

$$H_x=\frac{\vec{p}^2}{2m}-\left(\mu\vec{B}+\vec{\omega}\right)\cdot\vec{L}+V_{\mathrm{eff}}(\vec{x}), \quad (13)$$

where $\vec{L}=\vec{x}\times\vec{p}$ is the angular momentum operator, and the effective potential $V_{\mathrm{eff}}(\vec{x})$ is given by $V_{\mathrm{eff}}(\vec{x})=V(\vec{x})+\frac{1}{2}\mu^2m\left(\vec{B}\times\vec{x}\right)\cdot\left(\vec{B}\times\vec{x}\right)$.

Thus, after including the part I_x for the position variable, the action I for the spin $s=n/2$, mass m, charge e particle in rotation frame $\mathcal{S}$ with an external magnetic field $\vec{B}$ can be written as

$$I=I_{\mathrm{BS}}+I_x. \quad (14)$$

The corresponding Hamiltonian is $H=H_{\mathrm{BS}}+H_x$.

2.4 Spin-orbital coupling

We now discuss the coupling between the spin and the spacial variables by treating it as a perturbation to I.

In the static frame, the simplest term that preserves the parity is

$$m\xi(\vec{x})\vec{S}(\phi)\cdot\left[\vec{x}\times\left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}+\mu\vec{B}\times\vec{x}\right)\right], \quad (15)$$

where the small factor $\xi(\vec{x})$ is an arbitrary function of $\vec{x}$. For the hydrogen atom,

$$\xi(\vec{x})=\frac{1}{2m}\cdot\frac{1}{|\vec{x}|}\cdot\frac{\mathrm{d}V_{\mathrm{C}}(\vec{x})}{\mathrm{d}|\vec{x}|},$$

1) To get the expressions of $\vec{B}_{\perp}^{+}$ and $\vec{B}_{\perp}^{-}$ explicitly, one can choose the coordinates to make the plane which is perpendicular to $\vec{\omega}$ being x-y plane, then $\vec{B}_{\perp}^{+}=\vec{B}_{\perp}^{x}+\mathrm{i}\vec{B}_{\perp}^{y}$ and $\vec{B}_{\perp}^{-}=\vec{B}_{\perp}^{x}-\mathrm{i}\vec{B}_{\perp}^{y}$.

where $V_C(\vec{x})$ is the Coulomb potential. Then, in $\mathcal{S}$ frame, we have the perturbation term:

$$I_{\rm SLS}=\int {\rm d}t\xi(\vec{x})m\left[\vec{x}\times\left(\frac{{\rm d}\vec{x}}{{\rm d}t}+\vec{\omega}\times\vec{x}+\mu\vec{B}\times\vec{x}\right)\right]\cdot\vec{S}(\phi). \tag{16}$$

Thus in path integral quantization, we should insert $\exp({\rm i}I_{\rm SLS}/\hbar)$ into the path integration $\int\mathcal{D}\phi\mathcal{D}\overline{\phi}\mathcal{D}\lambda\mathcal{D}\vec{x}\exp\{{\rm i}I/\hbar\}$. In the canonical quantization, the contribution of spin-orbital coupling is just the usual perturbation $V_{\rm LS}=\xi(\vec{x})\vec{L}\cdot\vec{S}$.

3 Applications

We now apply the general formalism on some well-known quantum mechanical effects or experiments concerning rotating frame of reference to give some unified interpretations of the formalism.

3.1 Neutron interference

In the neutron interference experiment, earth is the rotating frame $\mathcal{S}$. The coupling between the angular momentum and rotation of the frame will give rise to a phase shift $\Delta\varphi$. The relevant terms of the action are the kinematic term $I_{\rm k}$ plus the term $I_{\rm r}$ which contributes to the inertial force, where

$$I_{\rm k}=\int{\rm d}t\left[\frac{1}{2}m\left(\frac{{\rm d}\vec{x}}{{\rm d}t}\right)^2+{\rm i}\phi^\dagger\frac{\rm d}{{\rm d}t}\phi-\lambda(\phi^\dagger\phi-n)\right], \tag{17}$$

$$I_{\rm r}=\int{\rm d}t\left[m\left(\vec{x}\times\frac{{\rm d}\vec{x}}{{\rm d}t}\right)+\vec{S}(\phi)\right]\cdot\vec{\omega}. \tag{18}$$

In the interference experiment, the term $\exp({\rm i}I_{\rm r}/\hbar)$ in the path integration brings a phase factor

$$\exp\left({\rm i}\frac{\vec{S}}{\hbar}\cdot\oint{\rm d}t\vec{\omega}\right)\cdot\exp\left[{\rm i}\frac{2m\vec{\omega}}{\hbar}\cdot\frac{1}{2}\oint_{\rm C}(\vec{x}\times{\rm d}\vec{x})\right], \tag{19}$$

where the term $\frac{1}{2}\oint_{\rm C}(\vec{x}\times{\rm d}\vec{x})$ is just the area $\vec{A}_{\rm C}$ surrounded by the moving path of the neutron, and $\oint{\rm d}t\vec{\omega}$ equals $2T\vec{\omega}$, where T is the flying time of the neutron. Thus, the total phase shift

$$\Delta\varphi=\frac{2}{\hbar}(m\vec{A}_{\rm C}+T\vec{S})\cdot\vec{\omega}, \tag{20}$$

in which the first term is the so-called Sagnac effect [6]. In terms of the Hamiltonian formalism, the relevant terms are $H_{\rm r}=-\vec{\omega}\cdot(\vec{L}+\vec{S})$, which has been derived to explain the interference experiment done before.

3.2 Rabi oscillation

In the nuclear-magnetism resonance experiment, a magnetic momentum $g\mu$ is coupled to a controllable magnetic field $\vec{B}_\parallel$ and a transverse rotating field $\vec{B}_\perp^+{\rm e}^{{\rm i}\omega t}+\vec{B}_\perp^-{\rm e}^{-{\rm i}\omega t}$ in the static frame. The problem can be solved more easily in the frame $\mathcal{S}$ rotating along the direction of $\vec{B}_\parallel$ with the angular velocity $\vec{\omega}$. The relevant terms in the action are the kinematic terms $I_{\rm Sk}$, plus $I_{\rm Br}$:

$$I_{\rm Sk}=\int{\rm d}t\left[{\rm i}\phi^\dagger\frac{\rm d}{{\rm d}t}\phi-\lambda(\phi^\dagger\phi-n)\right], \tag{21}$$

$$I_{\rm Br}=\int{\rm d}t\left(\vec{\omega}+g\mu\vec{B}\right)\cdot\vec{S}(\phi). \tag{22}$$

At the resonant frequency $\vec{\omega}=-g\mu\vec{B}_\parallel$, the factor $\exp\left[\frac{\rm i}{\hbar}\int{\rm d}t(g\mu\vec{B}_\perp\cdot\vec{S})\right]$ will cause the state of the spin to oscillate between the up and down state with oscillating frequency

$$\omega_{\rm R}=\frac{g\mu|\vec{B}_\perp|}{2\hbar}. \tag{23}$$

In the more general case the oscillating frequency Ω is given by $\Omega^2=\left[(\vec{\omega}+g\mu\vec{B}_\parallel)/2\hbar\right]^2+\omega_{\rm R}^2$. All these results can also be derived from the Hamiltonian $H_{\rm BS}=-g\mu\vec{B}\cdot\vec{S}-\vec{\omega}\cdot\vec{S}$.

3.3 Coriolis force in quantum action principle

We now discuss the equation of motions in the rotating frame $\mathcal{S}$ derived by using Schwinger's quantum action principle to see the quantum extension of the ordinary non-inertial force in the classical theory, i.e. the Coriolis force.

The relevant term $I_{\rm C}$ of the action is

$$I_{\rm C}=\int{\rm d}t\left[\frac{1}{2}m\left(\frac{{\rm d}\vec{x}}{{\rm d}t}\right)^2+m\vec{\omega}\cdot\left(\vec{x}\times\frac{{\rm d}\vec{x}}{{\rm d}t}\right)-V(\vec{x})+\frac{1}{2}m(\vec{\omega}\times\vec{x})^2\right]. \tag{24}$$

The quantum action principle [19] tells us $\langle\psi_{\rm f}|\delta I_{\rm C}|\psi_{\rm i}\rangle=0$. In virtue of the variation of $I_{\rm C}$ we find the equations of motion:

$$\left\langle\psi_{\rm f}\left|m\frac{{\rm d}^2\vec{x}}{{\rm d}t^2}+2m\frac{{\rm d}\vec{x}}{{\rm d}t}\times\vec{\omega}+m\omega^2\vec{x}+\frac{\partial V(\vec{x})}{\partial\vec{x}}\right|\psi_{\rm i}\right\rangle=0, \tag{25}$$

which is just the quantum mechanical generalization of the ordinary Newton equation in the rotating frame. The term $\int{\rm d}t\,m\vec{\omega}\cdot\left(\vec{x}\times\frac{{\rm d}\vec{x}}{{\rm d}t}\right)$ is the origin of the Coriolis force $2m\vec{\omega}\times\frac{{\rm d}\vec{x}}{{\rm d}t}$. In terms of the Hamiltonian formalism, the Coriolis force comes from the term $\left(-\vec{\omega}\cdot\vec{L}\right)$ of the Hamiltonian $H=\vec{p}^2/2m-\vec{\omega}\cdot\vec{L}+V(\vec{x})$.

4 Concluding remarks

We have shown the path integral formalism of the non-relativistic quantum mechanics of a charged point particle in a rotating frame of reference, and give some discussions on the applications of the formalism. There are many kinds of rotating frames in both nature and technology, especially in many spin systems. To show the quantum properties of them is very helpful for us to understand the physics of nature. We mainly focus on the non-relativistic cases in this paper. It is contemplable that when the relativistic case is considered, there will be some other problems. We leave the further issues and the relativistic generalizations for future work.

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