Entanglement property in matrix product spin systems*

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Abstract: We study the entanglement property in matrix product spin-ring systems systemically by von Neumann entropy. We find that: (i) the Hilbert space dimension of one spin determines the upper limit of the maximal value of the entanglement entropy of one spin, while for multiparticle entanglement entropy, the upper limit of the maximal value depends on the dimension of the representation matrices. Based on the theory, we can realize the maximum of the entanglement entropy of any spin block by choosing the appropriate control parameter values. (ii) When the entanglement entropy of one spin takes its maximal value, the entanglement entropy of an asymptotically large spin block, i.e. the renormalization group fixed point, is not likely to take its maximal value, and so only the entanglement entropy S_n of a spin block that varies with size n can fully characterize the spin-ring entanglement feature. Finally, we give the entanglement dynamics, i.e. the Hamiltonian of the matrix product system.

Key words: matrix product state (MPS), entanglement, von Neumann entropy

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1 Introduction

The study of the matrix product state (MPS) has recently become a much more intensive research subject, and mainly includes two aspects. One is the ability of the matrix product state to characterize quantum many-body systems [1–4], and the other is the quantum phase transitions in matrix product systems [5, 6]. Related articles [1–4] show that for one-dimensional spin lattice models, every many-body state, in particular every ground state (GS) of a finite many-body system dictated and characterized by a local Hamiltonian, can be represented as a matrix product state. The power of this MPS representation stems from the fact that in many cases an MPS with a set of low-dimensional representative matrices already yields a very good approximation to the ground state of the many-body system [7], such as the Greenberger-Horne-Zeilinger state of the form $A_1 = |+\cdots+\rangle + |-\cdots-\rangle$ [8] with $A_1 = |0\rangle\langle 0|$ and $A_2 = |1\rangle\langle 1|$, the cluster state [9], which is a unique ground state of the three-body interactions $\sum_i \sigma_i^z \sigma_{i+1}^x \sigma_{i+2}^z$ and is represented by the matrices

$$\left\{ A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},\ \ A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right\},$$

and the exact matrix product ground state of the Affleck-Kennedy-Lieb-Tasaki model [10] specified by $\{A_i\} = \{\sigma_z, \sqrt{2}\sigma_+, -\sqrt{2}\sigma_-\}$. MPSs are therefore undoubtedly a new powerful and convenient playground for studying one-dimensional spin lattice model theory. In particular the entanglement theory of one-dimensional spin lattice models, apart from quantum phase transitions, by use of the quantum information theory which deals primarily with the quantum ground state, and the corresponding parent Hamiltonian that may be constructed such that the MPS is exactly the GS [1, 10, 11].

In this paper, we investigate the entanglement property of matrix product states systemically by using von Neumann entropy. For a given MPS, there is a set of representation matrices, and we concretely study how the representation matrices, includ-

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ing their number and dimension, influence MPS entanglement.

2 Model and method

Let us begin with the one-dimensional translation invariant MPS:

$$|\Psi\rangle = \frac{1}{\sqrt{\mathcal{N}}}\sum_{i_1,\cdots,i_N=1}^{d}\mathrm{Tr}(A_{i_1}\cdots A_{i_N})|i_1,\cdots,i_N\rangle, \quad (1)$$

where d is the Hilbert space dimension of one site in the spin-ring, and a set of $D\times D$ matrices $\{A_i, i=1,\cdots,d\}$ called representation matrices parameterize the correlations of the N-spin state with the dimension $D\leqslant d^{N/2}$ [2].

$$E=\sum_{i=1}^{d}\bar{A}_i\otimes A_i$$

contained in the normalization factor $\mathcal{N}=\mathrm{Tr}E^N$ is the so-called transfer matrix, and the bar symbol denotes the complex conjugation. Given a matrix product state, the reduced density matrix of k adjacent spins is given by

$$\rho_{i_1\cdots i_k,j_1\cdots j_k}=\frac{\mathrm{Tr}((\bar{A}_{i_1}\cdots\bar{A}_{i_k}\otimes A_{j_1}\cdots A_{j_k})E^{N-k})}{\mathrm{Tr}(E^N)}, \quad (2)$$

in the thermodynamic limit, which gives

$$\rho_{i_1\cdots i_k,j_1\cdots j_k}=\frac{\langle\lambda^{\mathrm{L}}_{\max}|\bar{A}_{i_1}\cdots\bar{A}_{i_k}\otimes A_{j_1}\cdots A_{j_k}|\lambda^{\mathrm{R}}_{\max}\rangle}{\lambda^k_{\max}}, \quad (3)$$

where $|\lambda^{\mathrm{R}}_{\max}\rangle$ and $|\lambda^{\mathrm{L}}_{\max}\rangle$ are the normalized right and left eigenvectors corresponding to the largest absolute eigenvalue $\lambda_{\max}$ of the transfer matrix E.

2.1 The entanglement property of MPSs

Here we consider the MPS $|\Psi\rangle$ with representation matrices

$$A_1=\begin{bmatrix}1&0\\0&g\end{bmatrix},\quad A_2=\begin{bmatrix}0&a\\0&0\end{bmatrix},\quad A_3=\begin{bmatrix}0&0\\g&0\end{bmatrix}, \quad (4)$$

where $g>0$ and $a>0$. Now let us study the entanglement property of the MPS in detail. Considering our system, we shall adopt the von Neumann entropy, which [12–17] according to bipartition parameterization by the adjacent spin number n of a $\mathcal{B}_n$ spin block is,

$$S_n=-\mathrm{Tr}(\rho_n\log_2\rho_n), \quad (5)$$

where $\rho_n=\mathrm{Tr}_{\bar{\mathcal{B}}_n}\rho$ is the reduced density matrix for the $\mathcal{B}_n$ block of n adjacent spins. We first discuss one special case, when $n=1$, i.e. one spin's entanglement entropy S_1 in the ring. In theory, we discuss which factors it is concerned with. Obviously it depends on the parameters a and g, but in fact it also depends on the parameter d, the freedom of one spin which determines the upper limit $\log_2 d$ of the entanglement entropy of one spin. The entanglement entropy of one spin as a function of the parameters a and g is as illustrated in Fig.1, from which we know that the von Neumann entropy takes the upper limit value of 1.58496 i.e. $\log_2 3$ as long as

$$a=\sqrt{\frac{1}{g^2}+g^2+\frac{\sqrt{1+g^2+g^6+g^8}}{g^2}}.$$

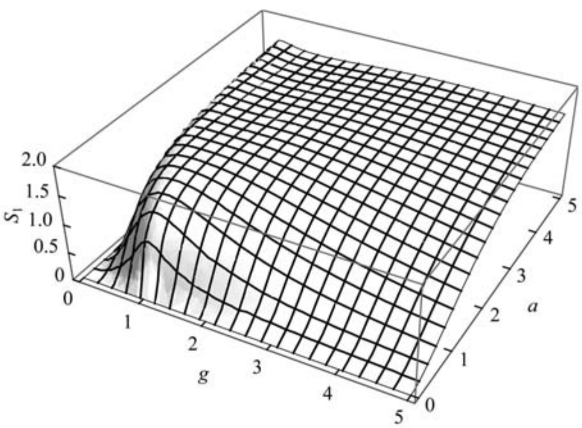

Fig. 1. The entanglement entropy of one spin in the spin-ring, which is taken to be dimensionless, i.e. S_1 as a function of the dimensionless parameters g and a. Note that as long as $a=\sqrt{\frac{1}{g^2}+g^2+\frac{\sqrt{1+g^2+g^6+g^8}}{g^2}}$, the entanglement entropy takes the upper limit value 1.58496, i.e. $\log_2 3$.

Then we discuss the other special case, the entanglement entropy for an asymptotically large block, i.e. the entanglement property of the renormalization group fixed point state $|\Psi_{\mathrm{fp}}\rangle$. Specifically, we resort to the renormalization group approach to characterize the long-wavelength behavior of the specified systems. Similar to the standard Kadanof Blocking scheme, the coarse-graining procedure for matrix product states could be achieved by merging the representative matrices of neighboring sites as $A\rightarrow A_{(\mathrm{pq})}\equiv A_{\mathrm{p}}A_{\mathrm{q}}$ and subsequently performing a fine-grained transformation $A\rightarrow A'$ to select out new representatives [18]. The transfer matrix in every step transforms as $E\rightarrow E'\equiv E^2$ and an iterative process hence leads to a fixed point $E^{\infty}\equiv E_{\mathrm{fp}}$, in which only the vector(s) of the largest eigenvalue(s) can survive. That is to say, the normalized transfer operator of the fixed point is characterized by $E_{\mathrm{fp}}=|\lambda^{\mathrm{R}}_{\max}\rangle\langle\lambda^{\mathrm{L}}_{\max}|$, from which we know that the maximal Hilbert space dimension of one coarse-grained spin in the fixed point is D^2. In terms of the MPS $|\Psi\rangle$ under consideration, the representative matrices of the fixed point state are obtained as

$$A_{\rm fp}^1=\begin{bmatrix}\sqrt{\dfrac{1-g^2+\sqrt{1+(4a^2-2)g^2+g^4}}{2\sqrt{1+(4a^2-2)g^2+g^4}}} & 0\\ 0 & 0\end{bmatrix},$$

$$A_{\rm fp}^2=\begin{bmatrix}0 & \sqrt{\dfrac{a^2}{\sqrt{1+(4a^2-2)g^2+g^4}}}\\ 0 & 0\end{bmatrix},$$

$$A_{\rm fp}^3=\begin{bmatrix}0 & 0\\ \sqrt{\dfrac{g^2}{\sqrt{1+(4a^2-2)g^2+g^4}}} & 0\end{bmatrix}, \tag{6}$$

$$A_{\rm fp}^4=\begin{bmatrix}0 & 0\\ 0 & \sqrt{\dfrac{g^2-1+\sqrt{1+(4a^2-2)g^2+g^4}}{2\sqrt{1+(4a^2-2)g^2+g^4}}}\end{bmatrix}.$$

The depicted renormalization group fixed point state enables a convenient calculation of the entanglement entropy for an asymptotically large block. In detail, it can now be worked out by evaluation of the entanglement entropy of the reduced density matrix of a single site of the fixed point state,

$$S_{\rm fp}=-{\rm Tr}(\rho_{\rm fp}\log_2\rho_{\rm fp}), \tag{7}$$

which not only depends on the parameters mentioned above, but also on the dimension of the representation matrices D, which determines the upper limit $\log_2 D^2=2\log_2 D$ of the maximal value of the asymptotically large block's entanglement entropy. Of course the value $2\log_2 D$ is also the upper limit of the entanglement entropy of any spin block. For the MPS $|\Psi\rangle$ under consideration with $D=2$, the behavior of the entanglement entropy between the bipartite coarse-grained spins of the fixed point, i.e. one half-infinite spin-ring and the other, $S_{\rm fp}=S_\infty$, as a function of the dimensionless parameters g and a, is shown in Fig.2. It is obvious that as long as $g=1$, entanglement entropy takes the upper limit value of 2. Based on the theory of the above sections, we can now realize the maximum of the entanglement entropy of any spin block via choosing the appropriate parameter values.

The analysis of the above two sections indicates that the parameter values of g and a, at which the entanglement entropy of the single site takes the

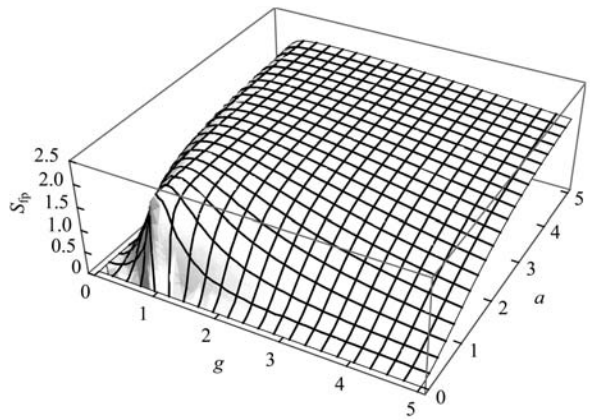

Fig. 2. The entanglement entropy between one half-infinite chain and the other, i.e. $S_{\rm fp}=S_\infty$, as a function of the dimensionless parameters g and a. Note that as long as $g=1$, in terms of the dimension of the representative matrices $D=2$, the entanglement entropy takes the upper limit value of 2.

maximal value, are not completely consistent with the parameter values at which the entanglement entropy for an asymptotically large block takes its maximal value. So in order to gain a comprehensive and deep understanding of the entanglement property of the MPS, we need to consider the entanglement entropy S_n of a spin block varying with its size n as an entanglement measure, and characterize it for large n. In terms of the MPS under consideration for $a=2$ and $g=1$, at which one spin's entanglement entropy and the entanglement entropy for an asymptotically large block take their upper limit value simultaneously, we start off with a description of the calculation according to Eq. (5), then do an analysis and discussion of

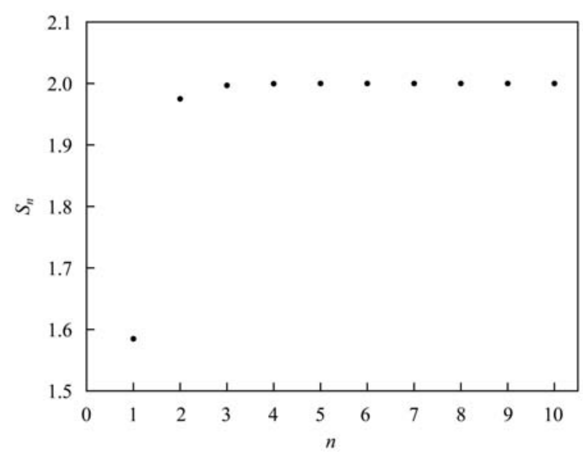

Fig. 3. The entanglement entropy S_n of a spin block varying with its size n for $a=2$ and $g=1$. S_n is monotonically asymptotic to the saturation value, i.e. the upper limit value 2 from 1.58496 as n increases from 1.

the results, and a summary is provided in Fig. 3. S_n is monotonically asymptotic to the saturation value, i.e. the upper limit value 2 from 1.58496 as n increases from 1, which is consistent with the result from the above section about the entanglement entropy $S_{\rm fp}$ of the fixed point. The different entanglement entropy S_n of a spin block as a function of its size n, is responsible for different MPS entanglement, which accounts for the different collective quantum phenomena and quantum-phase transition phenomena at the same time, in the fields of condensed matter [19–24].

2.2 The entanglement dynamics of the MPS

Finally, we undertake a study of the Hamiltonian of the specified system. The reduced density matrix of k adjacent spins according to Eq. (3) has at least d^k-D^2 zero eigenvalues (of course, it is sufficient that the following inequality holds: $d^k > D^2$). We can always construct a local Hamiltonian such that a given MPS is its GS. Therefore, $|\Psi\rangle$ is the GS of any Hamiltonian which is a sum of the local positive operators supported in that null-space. In particular, it is the GS of the Hamiltonian

$$H=\sum_i u_i(P_k), \tag{8}$$

with P_k being the projector onto the null-space of ρ_k and u_i its translation to site i. Here, k takes the value of 2, and one form of the Hamiltonian H under the thermodynamic limit is

$$\begin{aligned} H \;=\; & \sum_i\left(\frac{a^2}{4g^2}-\frac{1}{2g^4}+\frac{1}{2}\right)(S_z^iS_z^{i+1})^2+\left(\frac{a^2}{4g^2}+\frac{1}{2g^4}\right)(S_z^i)^2S_z^{i+1} \\ & +\frac{1}{2g^4}\left((S_z^{i+1})^2+\left(\frac{a^2}{4g^2}+\frac{1}{2}\right)S_z^i(S_z^{i+1})^2+\frac{a^2}{4g^2}S_z^iS_z^{i+1}+\frac{1}{2}((S_z^i)^2-S_z^i)-S_z^{i+1}\right) \\ & -\frac{a}{g^3}(S_z^iS_+^i(S_+^{i+1})^2+S_-^iS_z^i(S_-^{i+1})^2)-\frac{a}{g}((S_+^i)^2S_z^{i+1}S_+^{i+1}+(S_-^i)^2S_-^{i+1}S_z^{i+1}) \\ & +\frac{1}{g^2}(S_+^iS_z^iS_z^{i+1}S_-^{i+1}+S_z^iS_-^iS_+^{i+1}S_z^{i+1}), \end{aligned} \tag{9}$$

where

$$S_z=|+\rangle\langle+|-|-\rangle\langle-|, S_+=\frac{\sqrt{2}}{2}(S_x+{\rm i}S_y)$$

and

$$S_-=\frac{\sqrt{2}}{2}(S_x-{\rm i}S_y).$$

From the above sections we know that when $a=2$ and $g=1$, the system dynamics, H, determine that one spin's entanglement entropy and the entanglement entropy for an asymptotically large block of the MPSs take their upper limit value simultaneously.

3 Conclusions

In conclusion, we investigated the entanglement property of MPSs. For a given MPS, there is a set of representation matrices, which themselves contain no more than the quantity d, the freedom of one spin, the quantity D, the dimension of the representation matrices, and the other control parameters. The quantity d determines the upper limit of the maximal value of one particle's entanglement entropy, while the quantity D determines the upper limit of the maximal value of the multiparticle's entropy. Based on the theory, we can realize the maximum of the entanglement entropy of any spin block by choosing the appropriate parameter values. Considering the case when the entanglement entropy of one spin takes its maximal value, the entanglement entropy of an asymptotically large spin block, i.e. the renormalization group fixed point does not likely take its maximal value, then the entanglement entropy S_n of a spin block varying with size n can fully characterize the spin-ring entanglement feature. Finally, we discuss the entanglement dynamics, i.e. the Hamiltonian of the matrix product system. We believe that our work is helpful for having a comprehensive understanding of matrix product entangled states, and it is of potential direct significance to the preparation of the entangled states of one-dimensional spin lattice models.

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