Arbitrary-state solutions of the Dirac equation for a Möbius square potential using the Nikiforov-Uvarov method

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Abstract: We inquire into spin and pseudospin symmetries of the Dirac equation under a Möbius square-type potential using the Nikiforov-Uvarov method to calculate the bound state solutions. We numerically discuss the problem and include various explanatory figures.

Key words: Dirac equation, spin symmetry, pseudospin symmetry, Mobius potential

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1 Introduction

The significant impact of spin and pseudospin symmetries of the Dirac equation in hadronic and nuclear spectroscopy go beyond doubt with the experimental results released in the two past decades [1–5]. Consequently, there has been an increasing interest in solving the problem under various interactions [6–11]. Here, we intend to study a potential which appears as the square of the so-called Möbius interaction [12]. The latter resembles the more famous Eckart, Rosen–Morse and Morse-Feshbach potentials and is therefore worth studying [12].

To solve the problem, we choose the analytical Nikiforov-Uvarov (NU) technique [13] which has been used for various wave equations of mathematical physics [14 and references therein]. Readers will find instructive monographs on other analytical techniques of mathematical physics including supersymmetry quantum mechanics, point canonical transformation, Lie groups, Pekeris-type approximations and the ansatz technique, detailed in References [15–24].

2 The Dirac equation

The Dirac equation with scalar potential $s(r)$, vector potential $V(r)$ and a tensor potential $U(r)$ of [1–2]

$$[\vec{\alpha}\cdot\vec{p}+\beta(M+S(r))-\mathrm{i}\beta\vec{\alpha}\cdot\hat{r}U(r)]\psi(\vec{r})=[E-V(r)]\psi(\vec{r}), \tag{1}$$

where E, $\vec{p}=-\mathrm{i}\vec{\nabla}$ and M respectively denote the relativistic energy of the system, three-dimensional momentum operator and mass of the particle. α and β are

$$\alpha=\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2}$$

where I is 2×2 unitary matrix and the spin matrices are

$$\sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2=\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

The total angular momentum operator $\vec{J}$ and the spin–orbit coupling operator $K=(\vec{\sigma}\cdot\vec{L}+1)$, where $\vec{L}$ denotes the orbital angular momentum, of the spherical nucleons commute with the Hamiltonian. The eigenvalues of the spin-orbit coupling operator are $\kappa=\left(j+\frac{1}{2}\right)>0$ and $\kappa=-\left(j+\frac{1}{2}\right)<0$ for unaligned spin $j=\ell-\frac{1}{2}$ and the aligned spin $j=\ell+\frac{1}{2}$, respectively. The set (H^2, K, J^2, J_z) is taken as the complete set of the conservative quantities. Thus, the spinors can be written as [1, 2]

$$\psi_{nk}(\vec{r})=\begin{pmatrix} f_{nk}(\vec{r}) \\ g_{nk}(\vec{r}) \end{pmatrix}=\begin{pmatrix} \dfrac{F_{nk}(\vec{r})}{r}Y^{\ell}_{jm}(\theta,\varphi) \\ \mathrm{i}\dfrac{G_{nk}(\vec{r})}{r}Y^{\tilde{\ell}}_{jm}(\theta,\varphi) \end{pmatrix}, \tag{4}$$

where $f_{nk}(\vec{r})$ is the upper (alternatively called large in the jargon) component and $g_{nk}(\vec{r})$ is the lower (small) component of the Dirac spinors. $Y^{\ell}_{jm}(\theta,\varphi)$ and $Y^{\tilde{\ell}}_{jm}(\theta,\varphi)$ respectively stand for the spin and pseudospin spherical harmonics and m is the projection of the angular momentum along the z-axis. Using

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B})=\vec{A}\cdot\vec{B}+\mathrm{i}\vec{\sigma}\cdot(\vec{A}\times\vec{B}), \tag{5a}$$

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$$(\vec{\sigma}\cdot\vec{p})=(\vec{\sigma}\cdot\hat{r})\left(\hat{r}\cdot\vec{p}+\mathrm{i}\frac{\vec{\sigma}\cdot\vec{L}}{r}\right), \tag{5b}$$

as well as

$$(\vec{\sigma}\cdot\vec{L})Y_{jm}^{\tilde{\ell}}(\theta,\varphi) = (\kappa-1)Y_{jm}^{\tilde{\ell}}(\theta,\varphi), \tag{6a}$$

$$(\vec{\sigma}\cdot\vec{L})Y_{jm}^{\ell}(\theta,\varphi) = -(\kappa-1)Y_{jm}^{\ell}(\theta,\varphi), \tag{6b}$$

$$(\vec{\sigma}\cdot\hat{r})Y_{jm}^{\tilde{\ell}}(\theta,\varphi) = -Y_{jm}^{\ell}(\theta,\varphi), \tag{6c}$$

$$(\vec{\sigma}\cdot\hat{r})Y_{jm}^{\ell}(\theta,\varphi) = -Y_{jm}^{\tilde{\ell}}(\theta,\varphi), \tag{6d}$$

we arrive at the following coupled differential equations for the upper and lower radial wavefunctions [1, 2]

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)F_{nk}(r)=(M+E_{nk}-\Delta(r))G_{nk}(r), \tag{7a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)G_{nk}(r)=(M-E_{nk}+\Sigma(r))F_{nk}(r), \tag{7b}$$

where

$$\Delta(r) = V(r)-S(r), \tag{8a}$$

$$\Sigma(r) = V(r)+S(r). \tag{8b}$$

From Eq. (7), we find [1, 2]

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa}{r}U(r)-\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)\right.\\&\left.+\ \frac{\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}}{M+E_{n\kappa}-\Delta(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)\right\}F_{n\kappa}(r)\\&=(M+E_{n\kappa}-\Delta(r))(M-E_{n\kappa}+\Sigma(r))F_{n\kappa}(r),\end{aligned} \tag{9}$$

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(r)+\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)\right.\\&\left.+\ \frac{\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}}{M-E_{n\kappa}+\Sigma(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)\right\}G_{n\kappa}(r)\\&=(M+E_{n\kappa}-\Delta(r))(M-E_{n\kappa}+\Sigma(r))G_{n\kappa}(r),\end{aligned} \tag{10}$$

where $\kappa(\kappa-1)=\tilde{\ell}(\tilde{\ell}+1)$ and $\kappa(\kappa+1)=\ell(\ell+1)$.

2.1 Pseudospin symmetry limit

Pseudospin symmetry occurs when $\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}=0$ or equivalently $\Sigma(r)=C_{ps}=$Const. [1, 2]. Here, our desired interaction is the so-called Möbius square potential, i.e. [12]

$$\Delta(r)=V_0\left[\frac{A+B\mathrm{e}^{-\alpha r}}{C+D\mathrm{e}^{-\alpha r}}\right]^2, \tag{11}$$

$$U(r)=-\frac{H}{r}, \qquad r\geqslant R_{\mathrm{c}}, \tag{12}$$

$$H=\frac{Z_{\mathrm{a}}Z_{\mathrm{b}}\mathrm{e}^2}{4\pi\varepsilon_0} \tag{13}$$

where $R_{\mathrm{c}}=7.78$ fm is the Coulomb radius, Z_{a} and Z_{b} denote the charges of the projectile particle a and the target nuclei b, respectively [1, 2]. Under this symmetry, from Eqs. (11) to (13) one can obtain

$$\begin{aligned}\delta &= k(k-1)+2kH-H+H^2=(k+H)(k+H-1)\\&=\eta_k(\eta_k-1)\rightarrow\eta_k=k+H.\end{aligned} \tag{14}$$

Now, substitution of the proper approximation (see Fig. 1) [24]

$$\frac{1}{r^2}\approx\frac{C^2\alpha^2}{(C+D\mathrm{e}^{-\alpha r})^2}, \tag{15}$$

where $C=-D$, in Eq. (10) yields

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\delta C^2\alpha^2}{(C+D\mathrm{e}^{-\alpha r})^2}-M^2-MC_{ps}+E_{ps,n\kappa}^2-C_{ps}E_{ps,n\kappa}\right.\\&+\frac{MV_0A^2}{(C+D\mathrm{e}^{-\alpha r})^2}-\frac{E_{ps,n\kappa}V_0A^2}{(C+D\mathrm{e}^{-\alpha r})^2}+\frac{C_{ps}V_0A^2}{(C+D\mathrm{e}^{-\alpha r})^2}\\&+\frac{MV_0B^2\mathrm{e}^{-2\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}-\frac{E_{ps,n\kappa}V_0B^2\mathrm{e}^{-2\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}+\frac{C_{ps}V_0B^2\mathrm{e}^{-2\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}\\&+\frac{2ABV_0M\mathrm{e}^{-\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}-\frac{2ABV_0E_{ps,n\kappa}\mathrm{e}^{-\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}\\&\left.+\frac{2ABV_0C_{ps}\mathrm{e}^{-\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}\right\}G_{ps,n\kappa}(r)=0,\end{aligned} \tag{16}$$

where $\kappa=-\tilde{\ell}$ and $\kappa=\tilde{\ell}+1$ for $\kappa<0$ and $\kappa>0$, respectively.

2.2 Spin symmetry limit

In the spin symmetry limit $\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}=0$ or $\Delta(r)=C_{\mathrm{s}}=$ Const. [1, 2]. As in the previous section, we consider

$$\Sigma(r)=V_0\left[\frac{A+B\mathrm{e}^{-\alpha r}}{C+D\mathrm{e}^{-\alpha r}}\right]^2, \tag{17a}$$

and

$$\begin{aligned}\gamma &= k(k+1)+2kH+H+H^2\\&=(k+H+1)(k+H)=\eta_k(\eta_k-1).\end{aligned} \tag{17b}$$

Substitution of the latter in Eq. (9) gives

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\gamma C^2\alpha^2}{(C+D\mathrm{e}^{-\alpha r})^2}-M^2-\frac{MV_0A^2}{(C+D\mathrm{e}^{-\alpha r})^2}-\frac{MV_0B^2\mathrm{e}^{-2\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}-\frac{2ABV_0M\mathrm{e}^{-\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}+E_{s,n\kappa}^2-\frac{E_{s,n\kappa}V_0A^2}{(C+D\mathrm{e}^{-\alpha r})^2}\right.$$

$$\left.-\frac{E_{s,n\kappa}V_0B^2\mathrm{e}^{-2\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}-\frac{2ABV_0E_{s,n\kappa}\mathrm{e}^{-\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}+C_sM-C_sE_{s,n\kappa}+\frac{C_sV_0A^2}{(C+D\mathrm{e}^{-\alpha r})^2}+\frac{C_sV_0B^2\mathrm{e}^{-2\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}+\frac{2ABV_0C_s\mathrm{e}^{-\alpha r}}{(C+D\mathrm{e}^{-\alpha r})^2}\right\}F_{s,n\kappa}(r)=0, \tag{18}$$

where $\kappa=\ell$ and $\kappa=-\ell-1$ for $\kappa<0$ and $\kappa>0$, respectively.

3 Approximate bound-state solutions

3.1 Approximate bound-state solutions for the pseudospin symmetry limit

From Eq. (16) it can be easily seen that we have to deal with equation

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{(C+D\mathrm{e}^{-\alpha r})^2}(-\delta C^2\alpha^2-M^2C^2-M^2D^2\mathrm{e}^{-2\alpha r}-2M^2CD\mathrm{e}^{-\alpha r}-MC_{ps}C^2-MC_{ps}D^2\mathrm{e}^{-2\alpha r}-2MC_{ps}CD\mathrm{e}^{-\alpha r}\right.$$

$$+E_{ps,n\kappa}^2C^2+E_{ps,n\kappa}^2D^2\mathrm{e}^{-2\alpha r}+2E_{ps,n\kappa}^2CD\mathrm{e}^{-\alpha r}-E_{ps,n\kappa}C_{ps}C^2-E_{ps,n\kappa}C_{ps}D^2\mathrm{e}^{-2\alpha r}-2E_{ps,n\kappa}C_{ps}CD\mathrm{e}^{-\alpha r}+MV_0A^2$$

$$-E_{ps,n\kappa}V_0A^2+C_{ps}V_0A^2+MV_0B^2\mathrm{e}^{-2\alpha r}-E_{ps,n\kappa}V_0B^2\mathrm{e}^{-2\alpha r}+C_{ps}V_0B^2\mathrm{e}^{-2\alpha r}+2ABMV_0\mathrm{e}^{-\alpha r}-2ABE_{ps,n\kappa}V_0\mathrm{e}^{-\alpha r}$$

$$\left.+2ABC_{ps}V_0\mathrm{e}^{-\alpha r})\right\}G_{ps,n\kappa}(r)=0. \tag{19}$$

To obtain a solution of Eq. (19), bearing in mind Eq. (A11), we apply the transformation $z=\mathrm{e}^{-\alpha r}$ to bring Eq. (19) into the form

$$\frac{\mathrm{d}^2G_{n\kappa}(z)}{\mathrm{d}z^2}+\frac{\left(1+\frac{D}{C}z\right)}{z\left(1+\frac{D}{C}z\right)}\frac{\mathrm{d}G_{n\kappa}(z)}{\mathrm{d}z}+\frac{1}{\left(z\left(1+\frac{D}{C}z\right)\right)^2}(kz^2+Lz+f)G_{n\kappa}(z)=0, \tag{20}$$

where

$$k=-\frac{M^2D^2}{\alpha^2C^2}-\frac{MC_{ps}D^2}{\alpha^2C^2}+\frac{E_{ps,n\kappa}^2D^2}{\alpha^2C^2}-\frac{E_{ps,n\kappa}C_{ps}D^2}{\alpha^2C^2}+\frac{MV_0B^2}{\alpha^2C^2}-\frac{E_{ps,n\kappa}V_0B^2}{\alpha^2C^2}+\frac{C_{ps}V_0B^2}{\alpha^2C^2}, \tag{21a}$$

$$L=-\frac{2M^2D}{C\alpha^2}-\frac{2MC_{ps}D}{C\alpha^2}+\frac{2E_{ps,n\kappa}^2D}{C\alpha^2}-\frac{2E_{ps,n\kappa}C_{ps}D}{C\alpha^2}+\frac{2ABMV_0}{\alpha^2C^2}-\frac{2ABE_{ps,n\kappa}V_0}{\alpha^2C^2}+\frac{2ABC_{ps}V_0}{\alpha^2C^2}, \tag{21b}$$

$$f=-\delta-\frac{M^2}{\alpha^2}-\frac{MC_{ps}}{\alpha^2}+\frac{E_{ps,n\kappa}^2}{\alpha^2}-\frac{E_{ps,n\kappa}C_{ps}}{\alpha^2}+\frac{MV_0A^2}{\alpha^2C^2}-\frac{E_{ps,n\kappa}V_0A^2}{\alpha^2C^2}+\frac{C_{ps}V_0A^2}{\alpha^2C^2}. \tag{21c}$$

By comparing Eq. (20) with Eq. (A11), we get

$$\begin{aligned}
\alpha_1 &= 1,\quad \alpha_2=-\frac{D}{C},\quad \alpha_3=-\frac{D}{C},\quad \xi_1=-k,\ \ \xi_2=L,\ \ \xi_3=-f,\\
\alpha_4 &= 0,\quad \alpha_5=\frac{D}{2C},\quad \alpha_6=\frac{D^2}{4C^2}+\xi_1,\\
\alpha_7 &= -\xi_2,\quad \alpha_8=\xi_3,\quad \alpha_9=\frac{D}{C}\xi_2+\frac{D^2}{C^2}\xi_3+\frac{D^2}{4C^2}+\xi_1,\\
\alpha_{10} &= 1+2\sqrt{\xi_3},\quad \alpha_{11}=-2\frac{D}{C}+2\left(\sqrt{\frac{D}{C}\xi_2+\frac{D^2}{C^2}\xi_3+\frac{D^2}{4C^2}+\xi_1}-\frac{D}{C}\sqrt{\xi_3}\right),\\
\alpha_{12} &= \sqrt{\xi_3},\quad \alpha_{13}=\frac{D}{2C}-\left(\sqrt{\frac{D}{C}\xi_2+\frac{D^2}{C^2}\xi_3+\frac{D^2}{4C^2}+\xi_1}-\frac{D}{C}\sqrt{\xi_3}\right),
\end{aligned} \tag{22}$$

from Eq. (A26), we have

$$-\frac{D}{C}n-\frac{(2n+1)D}{2C}+(2n+1)\left(\sqrt{\frac{D}{C}\xi_2+\frac{D^2}{C^2}\xi_3+\frac{D^2}{4C^2}+\xi_1}-\frac{D}{C}\sqrt{\xi_3}\right)-n(n-1)\frac{D}{C}-\xi_2$$
$$-\frac{2D}{C}\xi_3+2\sqrt{\frac{D}{C}\xi_2\xi_3+\frac{D^2}{C^2}\xi_3^2+\frac{D^2}{4C^2}\xi_3+\xi_1\xi_3}=0. \tag{23}$$

For this case, from Eq. (A30), the upper and lower components of the wave function are

$$G_{ps,n\kappa}(r)=\mathrm{e}^{-\alpha r\sqrt{-f}}\left(1+\frac{D}{C}\mathrm{e}^{-\alpha r}\right)^{-\sqrt{-f}-\frac{\frac{D}{2C}-\left(\sqrt{\frac{D}{C}(L)+\frac{D^2}{C^2}(-f)+\frac{D^2}{4C^2}-k}-\frac{D}{C}\sqrt{-f}\right)}{-\frac{D}{C}}}$$
$$\times P_n^{\left(2\sqrt{-f},\frac{-2\frac{D}{C}+2\left(\sqrt{\frac{D}{C}(L)+\frac{D^2}{C^2}(-f)+\frac{D^2}{4C^2}-k}-\frac{D}{C}\sqrt{-f}\right)}{-\frac{D}{C}}-2\sqrt{-f}-2\right)}\left(1+2\frac{D}{C}\mathrm{e}^{-\alpha r}\right), \tag{24a}$$

$$G_{ps,n\kappa}(r)=\mathrm{e}^{-\alpha r\sqrt{-f}}\left(1+\frac{D}{C}\mathrm{e}^{-\alpha r}\right)^{-\sqrt{-f}-\frac{\frac{D}{2C}-\left(\sqrt{\frac{D}{C}(L)+\frac{D^2}{C^2}(-f)+\frac{D^2}{4C^2}-k}-\frac{D}{C}\sqrt{-f}\right)}{-\frac{D}{C}}}\frac{\Gamma(n+1+2\sqrt{-f})}{n!\Gamma(1+2\sqrt{-f})}$$
$$\times{}_2F_1\left(-n,n+2\sqrt{-f}+\frac{-2\frac{D}{C}+2(\sqrt{\frac{D}{C}(L)+\frac{D^2}{C^2}(-f)+\frac{D^2}{4C^2}-k}-\frac{D}{C}\sqrt{-f})}{-\frac{D}{C}}-2\sqrt{-f}-1;2\sqrt{-f}+1;-\frac{D}{C}\mathrm{e}^{-\alpha r}\right), \tag{24b}$$

where $P_n^{(\alpha,\beta)}(1-2z)$ is the Jacobi polynomials or

$$P_n^{(\alpha,\beta)}(1-2z)=\frac{1}{n!}z^{-\alpha}(1-z)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}z^n}[z^{n+\alpha}(1-z)^{n+\beta}], \tag{24c}$$

and the other component can be simply found via

$$F_{ps,n\kappa}(r)=\frac{1}{M-E_{ps,n\kappa}+C_{ps}}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{k}{r}+U(r)\right)G_{ps,n\kappa}(r). \tag{25}$$

3.2 Approximate bound-state solutions for the spin symmetry limit

In this limit, we have to deal with

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{(C+D\mathrm{e}^{-\alpha r})^2}(-\gamma C^2\alpha^2-M^2C^2-M^2D^2\mathrm{e}^{-2\alpha r}-2M^2CD\mathrm{e}^{-\alpha r}-MV_0A^2-MV_0B^2\mathrm{e}^{-2\alpha r}-2ABV_0M\mathrm{e}^{-\alpha r}\right.$$
$$+E_{s,n\kappa}^2C^2+E_{s,n\kappa}^2D^2\mathrm{e}^{-2\alpha r}+2E_{s,n\kappa}^2CD\mathrm{e}^{-\alpha r}-E_{s,n\kappa}V_0A^2-E_{s,n\kappa}V_0B^2\mathrm{e}^{-2\alpha r}-2ABV_0E_{s,n\kappa}\mathrm{e}^{-\alpha r}+C^2C_sM$$
$$+C_sMD^2\mathrm{e}^{-2\alpha r}+2C_sMCD\mathrm{e}^{-\alpha r}-C^2C_sE_{s,n\kappa}-C_sE_{s,n\kappa}D^2\mathrm{e}^{-2\alpha r}-2C_sE_{s,n\kappa}CD\mathrm{e}^{-\alpha r}+C_sV_0A^2+C_sV_0B^2\mathrm{e}^{-2\alpha r}$$
$$\left.+2ABV_0C_s\mathrm{e}^{-\alpha r})\right\}F_{s,n\kappa}(r)=0, \tag{26}$$

or

$$\frac{\mathrm{d}^2F_{n\kappa}(z)}{\mathrm{d}z^2}+\frac{\left(1+\frac{D}{C}z\right)}{z\left(1+\frac{D}{C}z\right)}\frac{\mathrm{d}F_{n\kappa}(z)}{\mathrm{d}z}+\frac{1}{\left(z(1+\frac{D}{C}z)\right)^2}(pz^2+qz+g)F_{n\kappa}(z)=0, \tag{27}$$

where

$$p = -\frac{M^2D^2}{C^2\alpha^2} - \frac{MV_0B^2}{C^2\alpha^2} + \frac{E_{s,n\kappa}^2D^2}{C^2\alpha^2} - \frac{E_{s,n\kappa}V_0B^2}{C^2\alpha^2} + \frac{C_sMD^2}{C^2\alpha^2} - \frac{C_sE_{s,n\kappa}D^2}{C^2\alpha^2} + \frac{C_sV_0B^2}{C^2\alpha^2}, \tag{28a}$$

$$q = -\frac{2M^2D}{C\alpha^2} - \frac{2ABV_0M}{C^2\alpha^2} + \frac{2E_{s,n\kappa}^2D}{C\alpha^2} - \frac{2ABV_0E_{s,n\kappa}}{C^2\alpha^2} + \frac{2C_sMD}{C\alpha^2} - \frac{2C_sE_{s,n\kappa}D}{C\alpha^2} + \frac{2ABV_0C_s}{C^2\alpha^2}, \tag{28b}$$

$$g = -\gamma - \frac{M^2}{\alpha^2} - \frac{MV_0A^2}{C^2\alpha^2} + \frac{E_{s,n\kappa}^2}{\alpha^2} - \frac{E_{s,n\kappa}V_0A^2}{C^2\alpha^2} + \frac{C_sM}{\alpha^2} - \frac{C_sE_{s,n\kappa}}{\alpha^2} + \frac{C_sV_0A^2}{C^2\alpha^2}. \tag{28c}$$

By comparing Eq. (27) with Eq. (A11), we find the correspondence

$$\begin{aligned}
\alpha_1 &= 1, \quad \alpha_2 = -\frac{D}{C}, \quad \alpha_3 = -\frac{D}{C}, \quad \xi_1 = -p, \quad \xi_2 = q, \quad \xi_3 = -g, \\
\alpha_4 &= 0, \quad \alpha_5 = \frac{D}{2C}, \quad \alpha_6 = \frac{D^2}{4C^2} + \xi_1, \\
\alpha_7 &= -\xi_2, \quad \alpha_8 = \xi_3, \quad \alpha_9 = \frac{D}{C}\xi_2 + \frac{D^2}{C^2}\xi_3 + \frac{D^2}{4C^2} + \xi_1, \\
\alpha_{10} &= 1+2\sqrt{\xi_3}, \quad \alpha_{11} = -2\frac{D}{C} + 2\left(\sqrt{\frac{D}{C}\xi_2 + \frac{D^2}{C^2}\xi_3 + \frac{D^2}{4C^2} + \xi_1} - \frac{D}{C}\sqrt{\xi_3}\right), \\
\alpha_{12} &= \sqrt{\xi_3}, \quad \alpha_{13} = \frac{D}{2C} - \left(\sqrt{\frac{D}{C}\xi_2 + \frac{D^2}{C^2}\xi_3 + \frac{D^2}{4C^2} + \xi_1} - \frac{D}{C}\sqrt{\xi_3}\right).
\end{aligned} \tag{29}$$

Therefore, from Eq. (A26), we immediately find

$$\begin{aligned}
&-\frac{D}{C}n - \frac{(2n+1)D}{2C} + (2n+1)\left(\sqrt{\frac{D}{C}\xi_2 + \frac{D^2}{C^2}\xi_3 + \frac{D^2}{4C^2} + \xi_1} - \frac{D}{C}\sqrt{\xi_3}\right) - n(n-1)\frac{D}{C} - \xi_2 - \frac{2D}{C}\xi_3 \\
&+ 2\sqrt{\frac{D}{C}\xi_2\xi_3 + \frac{D^2}{C^2}\xi_3^2 + \frac{D^2}{4C^2}\xi_3 + \xi_1\xi_3} = 0,
\end{aligned} \tag{30}$$

and the upper and lower components of the wave function are

$$\begin{aligned}
F_{s,n\kappa}(r) &= \mathrm{e}^{-\alpha r\sqrt{-g}}\left(1+\frac{D}{C}\mathrm{e}^{-\alpha r}\right)^{-\sqrt{-g} - \frac{\frac{D}{2C} - \left(\sqrt{\frac{D}{C}q + \frac{D^2}{C^2}(-g) + \frac{D^2}{4C^2} - p} - \frac{D}{C}\sqrt{-g}\right)}{-\frac{D}{C}}} \\
&\quad \times P_n^{\left(2\sqrt{-g},\ \frac{-2\frac{D}{C} + 2\left(\sqrt{\frac{D}{C}q + \frac{D^2}{C^2}(-g) + \frac{D^2}{4C^2} - p} - \frac{D}{C}\sqrt{-g}\right)}{-\frac{D}{C}} - 2\sqrt{-g} - 2\right)}\left(1+2\frac{D}{C}\mathrm{e}^{-\alpha r}\right),
\end{aligned} \tag{31a}$$

$$\begin{aligned}
F_{s,n\kappa}(r) &= \mathrm{e}^{-\alpha r\sqrt{-g}}\left(1+\frac{D}{C}\mathrm{e}^{-\alpha r}\right)^{-\sqrt{-g} - \frac{\frac{D}{2C} - (\sqrt{\frac{D}{C}q + \frac{D^2}{C^2}(-g) + \frac{D^2}{4C^2} - p} - \frac{D}{C}\sqrt{-g})}{-\frac{D}{C}}} \frac{\Gamma(n+1+2\sqrt{-g})}{n!\Gamma(1+2\sqrt{-g})} \\
&\quad \times {}_2F_1\left(-n, n+2\sqrt{-g} + \frac{-2\frac{D}{C} + 2(\sqrt{\frac{D}{C}q + \frac{D^2}{C^2}(-g) + \frac{D^2}{4C^2} - p} - \frac{D}{C}\sqrt{-g})}{-\frac{D}{C}} - 2\sqrt{-g} - 1; 2\sqrt{-g}+1; -\frac{D}{C}\mathrm{e}^{-\alpha r}\right),
\end{aligned} \tag{31b}$$

and

$$G_{s,n\kappa}(r)=\frac{1}{M+E_{s,n\kappa}-C_s}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{k}{r}-U(r)\right)F_{s,n\kappa}(r). \quad (32)$$

Figures 2 and Fig. 3 represent energy vs. α for pseudospin and spin symmetry limits, respectively. Figs. 4 and 5 show energy vs. H for both of the two symmetry limits. In Tables 1 and 4 we have portrayed the energy values for various H. Tables 2 and 5 show the energy for different values of C_{ps} and C_s respectively. In Tables 3 and 6 we have reported the energy for various different M.

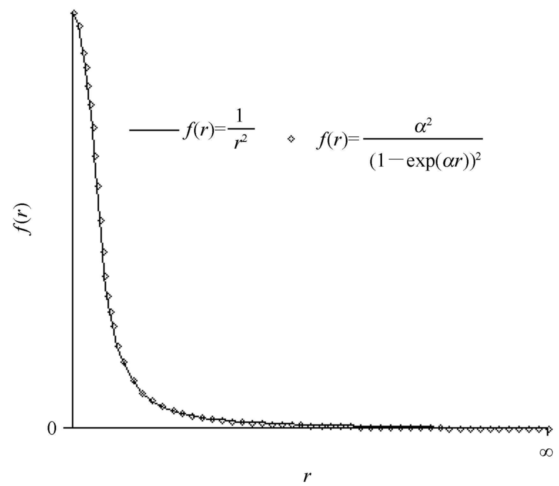

Fig. 1. $\frac{1}{r^2}$ and its approximation.

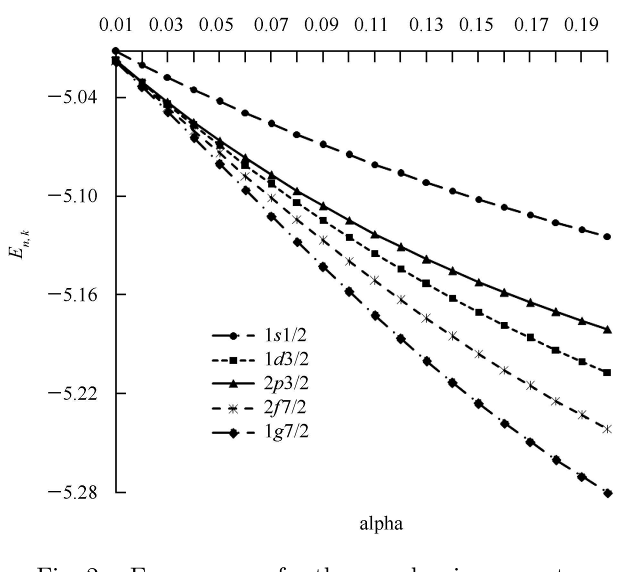

Fig. 2. Energy vs. α for the pseudospin symmetry limit for $H=1$, $M=5\ \mathrm{fm}^{-1}$, $C_{ps}=0$, $V_0=-0.2$, $A=1$, $B=-2$, $C=1$, $D=-1$.

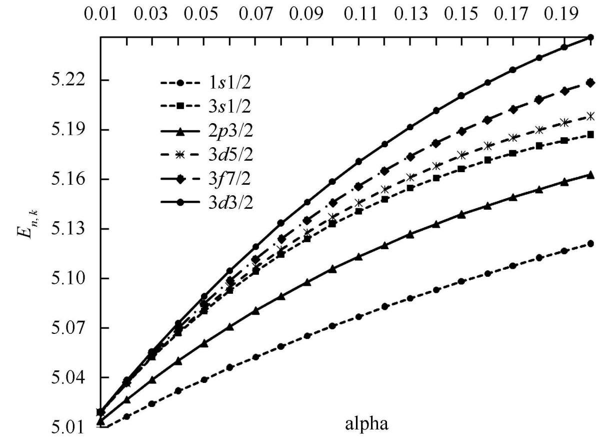

Fig. 3. Energy vs. α for the spin symmetry limit for $H=1$, $M=5\ \mathrm{fm}^{-1}$, $C_s=0$, $V_0=0.2$, $A=1$, $B=-2$, $C=1$, $D=-1$.

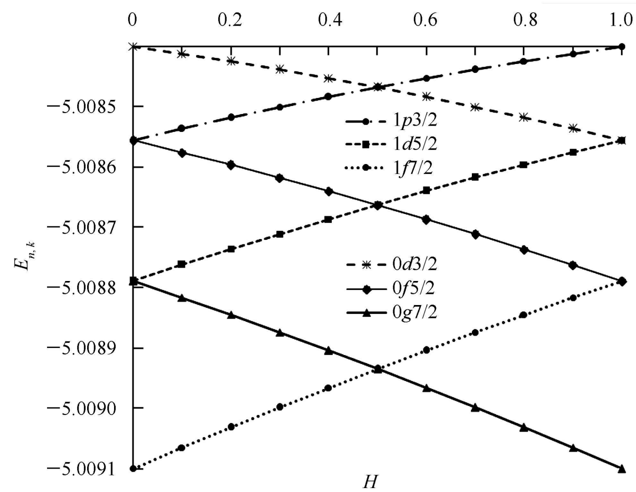

Fig. 4. Energy vs. H for the pseudospin symmetry limit for $\alpha=0.01$, $M=5\ \mathrm{fm}^{-1}$, $C_{ps}=0$, $V_0=-0.2$, $A=1$, $B=-2$, $C=1$, $D=-1$.

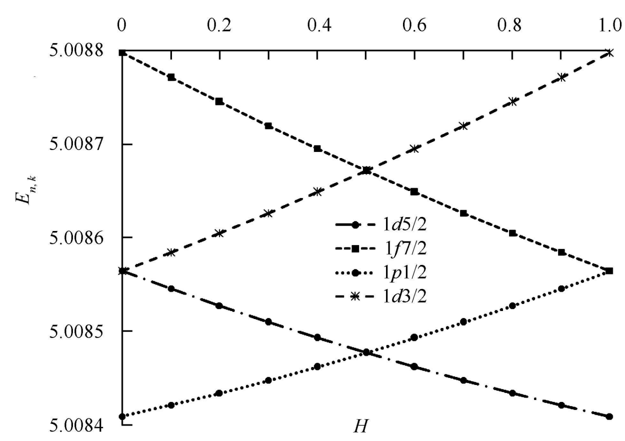

Fig. 5. Energy vs. H for the spin symmetry limit for $\alpha=0.01$, $M=5\ \mathrm{fm}^{-1}$, $C_s=0$, $V_0=0.2$, $A=1$, $B=-2$, $C=1$, $D=-1$.

Table 1. Energy in the pseudospin symmetry limit for α=0.01, M=5 fm^{-1}, C_{ps}=0, V_0=−0.2, A=1, B=−2, C=1, D=−1.

$\tilde{\ell}$	n	κ	(ℓ,j)	$E_{ps,n\kappa}(H=0)$	$E_{ps,n\kappa}(H=0.5)$	$E_{ps,n\kappa}(H=1)$
1	1	−1	$1S_{\frac{1}{2}}$	−5.00840912	−5.00836049	−5.00833131
2	1	−2	$1P_{\frac{3}{2}}$	−5.00856470	−5.00847720	−5.00840912
3	1	−3	$1d_{\frac{5}{2}}$	−5.00879794	−5.00867162	−5.00856470
4	1	−4	$1f_{\frac{7}{2}}$	−5.00910869	−5.00894364	−5.00879794
1	2	−1	$2S_{\frac{1}{2}}$	−5.01378636	−5.01373861	−5.01370995
2	2	−2	$2P_{\frac{3}{2}}$	−5.01393914	−5.01385321	−5.01378636
3	2	−3	$2d_{\frac{5}{2}}$	−5.01416818	−5.01404413	−5.01393914
4	2	−4	$2f_{\frac{7}{2}}$	−5.01447334	−5.01431125	−5.01416818
1	1	2	$0d_{\frac{3}{2}}$	−5.00840912	−5.00847720	−5.00856470
2	1	3	$0f_{\frac{5}{2}}$	−5.00856470	−5.00867162	−5.00879794
3	1	4	$0g_{\frac{7}{2}}$	−5.00879794	−5.00894364	−5.00910869
4	1	5	$0h_{\frac{9}{2}}$	−5.00910869	−5.00929308	−5.00949676
1	2	2	$1d_{\frac{3}{2}}$	−5.01378636	−5.01385321	−5.01393914
2	2	3	$1f_{\frac{5}{2}}$	−5.01393914	−5.01404413	−5.01416818
3	2	4	$1g_{\frac{7}{2}}$	−5.01416818	−5.01431125	−5.01447334
4	2	5	$1h_{\frac{9}{2}}$	−5.01447334	−5.01465440	−5.01485442

Table 2. Energy in the pseudospin symmetry limit for H=1, α=0.01, M=5 fm^{-1}, V_0=−0.2, A=1, B=−2, C=1, D=−1.

C_{ps}	$E_{ps,n\kappa}$/ fm^{-1}				
	$1S_{\frac{1}{2}}$	$2P_{\frac{3}{2}}$	$2f_{\frac{7}{2}}$	$0h_{\frac{9}{2}}$	$1h_{\frac{9}{2}}$
−10	−5.07872142	−5.10806735	−5.12377315	−5.13039911	−5.14576564
−8	−5.01815355	−5.02971793	−5.03148835	−5.02368342	−5.03464681
−6	−5.01302218	−5.02144770	−5.02237218	−5.01587381	−5.02402960
−4	−5.01069513	−5.01765761	−5.01828427	−5.01261785	−5.01940936
−2	−5.00929362	−5.01536478	−5.01583915	−5.01074461	−5.01669141
0	−5.00833131	−5.01378636	−5.01416818	−5.00949676	−5.01485442
2	−5.00761798	−5.01261428	−5.01293386	−5.00859197	−5.01350839
4	−5.00706186	−5.01169934	−5.01197418	−5.00789852	−5.01246838
6	−5.00661250	−5.01095932	−5.01120045	−5.00734584	−5.01163408
8	−5.00623957	−5.01034467	−5.01055947	−5.00689234	−5.01094581
10	−5.00592357	−5.00982351	−5.01001719	−5.00651175	−5.01036555

Table 3. Energy in the pseudospin symmetry limit for H=1, α=0.01, C_{ps}=0, V_0=−0.2, A=1, B=−2, C=1, D=−1.

M/fm^{-1}	$E_{ps,n\kappa}$/fm^{-1}				
	$1S_{\frac{1}{2}}$	$2P_{\frac{3}{2}}$	$2f_{\frac{7}{2}}$	$0h_{\frac{9}{2}}$	$1h_{\frac{9}{2}}$
0	−0.07872142	−0.10806735	−0.12377315	−0.13039911	−0.14576564
2	−2.01302218	−2.02144770	−2.02237218	−2.01587381	−2.02402960
4	−4.00929362	−4.01536478	−4.01583915	−4.01074461	−4.01669141
6	−6.00761798	−6.01261428	−6.01293386	−6.00859197	−6.01350839
8	−8.00661250	−8.01095932	−8.01120044	−8.00734584	−8.01163408
10	−10.0059236	−10.0098235	−10.0100172	−10.0065117	−10.0103655
12	−12.0054136	−12.0089818	−12.0091437	−12.0059047	−12.0094349
14	−14.0050164	−14.0083257	−14.0084648	−14.0054379	−14.0087149

Table 4. Energy in the spin symmetry limit for α=0.01, M=5 fm^{-1}, C_s=0, V_0=0.2, A=1, B=−2, C=1, D=−1.

n	ℓ	κ	(ℓ,j)	$E_{s,n\kappa}(H=0)$	$E_{s,n\kappa}(H=0.5)$	$E_{s,n\kappa}(H=1)$
0	0	−1	$0S_{\frac{1}{2}}$	5.00281555	5.00280564	5.00281555
1	0	−1	$1S_{\frac{1}{2}}$	5.00833131	5.00832158	5.00833131
2	0	−1	$2S_{\frac{1}{2}}$	5.01370995	5.01370040	5.01370995
3	0	−1	$3S_{\frac{1}{2}}$	5.01895534	5.01894596	5.01895534
0	1	−2	$0P_{\frac{3}{2}}$	5.00289480	5.00284527	5.00281555
1	1	−2	$1P_{\frac{3}{2}}$	5.00840912	5.00836049	5.00833131
2	1	−2	$2P_{\frac{3}{2}}$	5.01378636	5.01373861	5.01370995
3	1	−2	$3P_{\frac{3}{2}}$	5.01903039	5.01898349	5.01895534
0	2	−3	$0d_{\frac{5}{2}}$	5.00305326	5.00296414	5.00289480
1	2	−3	$1d_{\frac{5}{2}}$	5.00856470	5.00847720	5.00840912
2	2	−3	$2d_{\frac{5}{2}}$	5.01393914	5.01385321	5.01378636
3	2	−3	$3d_{\frac{5}{2}}$	5.01918044	5.01909604	5.01903039
0	3	−4	$0f_{\frac{7}{2}}$	5.00329082	5.00316216	5.00305326
1	3	−4	$1f_{\frac{7}{2}}$	5.00879794	5.00867162	5.00856470
2	3	−4	$2f_{\frac{7}{2}}$	5.01416818	5.01404413	5.01393914
3	3	−4	$3f_{\frac{7}{2}}$	5.01940539	5.01928356	5.01918044
0	1	1	$0P_{\frac{1}{2}}$	5.00289480	5.00296414	5.003053261
1	1	1	$1P_{\frac{1}{2}}$	5.00840912	5.00847720	5.00856470
2	1	1	$2P_{\frac{1}{2}}$	5.01378636	5.01385321	5.01393914
3	1	1	$3P_{\frac{1}{2}}$	5.01903039	5.01909604	5.01918044
0	2	2	$0d_{\frac{3}{2}}$	5.00305326	5.00316216	5.00329082
1	2	2	$1d_{\frac{3}{2}}$	5.00856470	5.00867162	5.00879794
2	2	2	$2d_{\frac{3}{2}}$	5.01393914	5.01404413	5.01416818
3	2	2	$3d_{\frac{3}{2}}$	5.01918044	5.01928356	5.01940539
0	3	3	$0f_{\frac{5}{2}}$	5.00329082	5.00343921	5.00360732
1	3	3	$1f_{\frac{5}{2}}$	5.00879794	5.00894364	5.00910869
2	3	3	$2f_{\frac{5}{2}}$	5.01416818	5.01431125	5.01447334
3	3	3	$3f_{\frac{5}{2}}$	5.01940539	5.01954591	5.01970511

Table 5. Energy in the spin symmetry limit for H=1, α=0.01, M=5 fm^{-1}, V_0=0.2, A=1, B=−2, C=1, D=−1.

C_s	$E_{s,n\kappa}/\text{fm}^{-1}$				
	$0S_{\frac{1}{2}}$	$2P_{\frac{3}{2}}$	$3d_{\frac{5}{2}}$	$3d_{\frac{3}{2}}$	$2f_{5/2}$
−10	5.00199366	5.00978476	5.01361683	5.01380809	5.01017206
−8	5.00210113	5.01030169	5.01433147	5.01454344	5.01073122
−6	5.00222811	5.01091108	5.01517328	5.01541103	5.01139324
−4	5.00238134	5.01164435	5.01618528	5.01645600	5.01219391
−2	5.00257131	5.01255033	5.01743420	5.01774859	5.01318932
0	5.00281555	5.01370995	5.01903039	5.01940539	5.01447334
2	5.00314606	5.01526983	5.02117320	5.02163805	5.01621819
4	5.00362960	5.01753215	5.02427174	5.02488374	5.01878476
6	5.00443851	5.02126248	5.02935554	5.03025307	5.02310987
8	5.00625258	5.02936250	5.04027031	5.04196425	5.03289657
10	5.04113040	5.10441468	5.12707316	5.13993973	5.13428060

Table 6. Energy in the spin symmetry limit for H=1, α=0.01, C_s=0, V_0=0.2, A=1, B=−2, C=1, D=−1.

M/fm^{-1}	$E_{s,n\kappa}$/fm^{-1}				
	$0S_{\frac{1}{2}}$	$2P_{\frac{3}{2}}$	$3d_{\frac{5}{2}}$	$3d_{\frac{3}{2}}$	$2f_{5/2}$
0	0.04113039	0.10441468	0.12707316	0.13993973	0.13428060
2	2.00443851	2.02126248	2.02935554	2.03025307	2.02310987
4	4.00314606	4.01526983	4.02117320	4.02163805	4.01621819
6	6.00257131	6.01255033	6.01743420	6.01774859	6.01318932
8	8.0022281	8.01091108	8.01517328	8.01541103	8.01139324
10	10.0019937	10.00978476	10.0136168	10.0138081	10.0101721
12	12.0018205	12.00894942	12.0124609	12.0126210	12.0092731

4 Conclusion

We observed that by using an appropriate Pekeris-type approximation as well as the powerful NU technique, the cumbersome problem of the Dirac equation under a complicated exponential interaction, namely the Möbius square potential, can be solved for arbitrary quantum numbers. Our numerical data describe the energy splitting in detail. For H=0, one can clearly see that the degeneracy in the pseudospin doublets, i.e. $ns_{1/2},(n-1)d_{3/2}$ for $\tilde{\ell}=1(\ell=0)$, $np_{3/2},(n-1)f_{5/2}$ for $\tilde{\ell}=2(\ell=1)$, $nd_{5/2},(n-1)g_{7/2}$ for $\tilde{\ell}=3(\ell=2)$, and $nf_{7/2},(n{-}1)h_{9/2}$ for $\tilde{\ell}{=}4(\ell{=}3)$, etc. For $H=0.5$, the degeneracy exists for $(1P_{\frac{3}{2}}{=}0d_{\frac{3}{2}})$, $(1d_{\frac{5}{2}}{=}0f_{\frac{5}{2}})$, $(1f_{\frac{7}{2}}{=}0g_{\frac{7}{2}})$, $(2P_{\frac{3}{2}}=1d_{\frac{3}{2}})$, $(2d_{\frac{5}{2}}=1f_{\frac{5}{2}})$, $(2f_{\frac{7}{2}}=1g_{\frac{7}{2}})$. In the case of H=1, the degenerate states are as follows: $(1d_{\frac{5}{2}}{=}0d_{\frac{3}{2}})$, $(1f_{\frac{7}{2}}{=}0f_{\frac{5}{2}})$, $(2d_{\frac{5}{2}}{=}1d_{\frac{3}{2}})$, $(2f_{\frac{7}{2}}{=}1f_{\frac{5}{2}})$, etc. For spin symmetry limit, from Table 4 we see that for $H=0$, the degeneracy occurs in $(np_{1/2},np_{3/2})$ for $\ell{=}1$, $(nd_{3/2},nd_{5/2})$ for $\ell=2$, $(nf_{5/2},nf_{7/2})$ for $\ell=3$, and $(ng_{7/2},ng_{9/2})$ for $\ell=4$, etc. For $H=0.5$, the states are $(0d_{\frac{5}{2}}=0P_{\frac{1}{2}})$, $(1d_{\frac{5}{2}}=1P_{\frac{1}{2}})$, $(2d_{\frac{5}{2}}=2P_{\frac{1}{2}})$, $(3d_{\frac{5}{2}}=3P_{\frac{1}{2}})$, $(0f_{\frac{7}{2}}=0d_{\frac{3}{2}})$, $(1f_{\frac{7}{2}}=1d_{\frac{3}{2}})$, $(2f_{\frac{7}{2}}=2d_{\frac{3}{2}})$, $(3f_{\frac{7}{2}}=3d_{\frac{3}{2}})$. For $H=1$, the degenerate states are $(0S_{\frac{1}{2}}=0P_{\frac{3}{2}})$, $(1S_{\frac{1}{2}}=1P_{\frac{3}{2}})$, $(2S_{\frac{1}{2}}=2P_{\frac{3}{2}})$, $(3S_{\frac{1}{2}}=3P_{\frac{3}{2}})$, $(0f_{\frac{7}{2}}=0P_{\frac{1}{2}})$, $(1f_{\frac{7}{2}}=1P_{\frac{1}{2}})$, $(2f_{\frac{7}{2}}=2P_{\frac{1}{2}})$, $(3f_{\frac{7}{2}}=3P_{\frac{1}{2}})$. The bound-state solutions can be used for various systems after the requisite fits are done.

Appendix A

The Nikiforov-Uvarov method

In this section, the NU method and its parametric generalization are briefly presented. The NU method, in its general form, solves the ordinary linear second order differential equation, i.e. [13]

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0, \tag{A1}$$

where the prime denotes the derivative w.r.t. the independent variable s. Here, $\sigma(s)$ and $\tilde{\sigma}(s)$ must be polynomials of at most second-degree and $\tilde{\tau}(s)$ is a polynomial with at most first-degree [13]. Using the following transformation

$$\psi(s)=W(s)\Phi(s). \tag{A2}$$

Eq. (A1) is reduced to the well-known hypergeometric-type equation [13]

$$\sigma(s)\Phi''(s)+\tau(s)\Phi'(s)+\lambda\Phi(s)=0, \tag{A3}$$

where

$$\tau(s)=\tilde{\tau}(s)+2\pi(s), \quad \text{with} \quad \tau'(s)<0 \tag{A4}$$

Eq. (A1) has a particular solution when the following relationship is satisfied

$$\lambda=\lambda_n=-n\tau'(s)-\frac{n(n-1)}{2}\sigma''(s). \quad \text{with} \quad n=0,1,2,\cdots \tag{A5}$$

In order to obtain the energy eigenvalues equation, the functions given below are calculated

$$\pi(s)=\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\pm\sqrt{(\frac{\sigma'(s)-\tilde{\tau}(s)}{2})^2-\tilde{\sigma}(s)+k\sigma(s)}, \tag{A6}$$

$$\lambda=k+\pi'(s), \tag{A7}$$

where λ is a constant. The expression under the square root in Eq. (A6) must be the square of a polynomial of the first degree, since $\pi(s)$ is the first-degree polynomial. Then, one can obtain the k values by considering that the discriminant of the square root has to be zero in Eq. (A6) [13]. Consequently, the energy eigenvalues can be obtained by comparing Eq. (A7) with Eq. (A5). The function $\Phi(s)$ given in Eq. (A3) is a hypergeometric-type function whose solution can be written in terms of the polynomials given by the Rodrigues relationship

$$\Phi_n(s)=\frac{C_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[\sigma^n(s)\rho(s)], \tag{A8}$$

where C_n is the normalization constant and the weight function $\rho(s)$ satisfies the condition

$$[\sigma(s)\rho(s)]'=\tau(s)\rho(s). \tag{A9}$$

On the other hand, the other factor $W(s)$ satisfies the following logarithmic equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln W(s)=\frac{\sigma(s)}{\pi(s)}. \tag{A10}$$

Parametric formulation of the Nikiforov-Uvarov method

The NU method, in its parametric form, solves [14]

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\psi_n(s)+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{\mathrm{d}}{\mathrm{d}s}\psi_n(s)+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{[s(1-\alpha_3 s)]^2}\psi_n(s)=0 \tag{A11}$$

Here, we give only the basic ingredients of the generalized NU method. In this case, comparing Eq. (A11) with Eq. (A1), one can obtain [14]

$$\tilde{\tau}(s)=\alpha_1-\alpha_2 s, \tag{A12}$$

$$\sigma(s)=s(1-\alpha_3 s), \tag{A13}$$

$$\tilde{\sigma}(s)=-\xi_1 s^2+\xi_2 s-\xi_3, \tag{A14}$$

Inserting the above equations into Eq. (A6) leads to [14]

$$\pi(s)=\alpha_4+\alpha_5 s\pm\sqrt{(\alpha_6-k\alpha_3)s^2+(\alpha_7+k)s+\alpha_8}, \tag{A15}$$

where

$$\alpha_4=\frac{1}{2}(1-\alpha_1), \tag{A16}$$

$$\alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3), \tag{A17}$$

$$\alpha_6=\alpha_5^2+\xi_1, \tag{A18}$$

$$\alpha_7=2\alpha_4\alpha_5-\xi_2, \tag{A19}$$

$$\alpha_8=\alpha_4^2+\xi_3. \tag{A20}$$

Considering that discriminant of the square root has to be zero in Eq. (A6), we obtain [14]

$$k_{1,2}=-(\alpha_7+2\alpha_3\alpha_8)\pm 2\sqrt{\alpha_8\alpha_9}, \tag{A21}$$

with

$$\alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6, \tag{A22}$$

From Eq. (A15), one can easily see that different k values lead to different $\pi(s)$s. If we take

$$k=-(\alpha_7+2\alpha_3\alpha_8)-2\sqrt{\alpha_8\alpha_9}, \tag{A23}$$

$\pi(s)$ becomes

$$\pi(s)=\alpha_4+\alpha_5 s-[(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})s-\sqrt{\alpha_8}), \tag{A24}$$

and then we find [14]

$$\tau(s)=\alpha_1+2\alpha_4-(\alpha_2-2\alpha_5)s-[(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})s-\sqrt{\alpha_8}]. \tag{A25}$$

The energy eigenvalue equation can be readily obtained by using Eqs. (A5) and (A6) with the above results as follows [14]

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}) + n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0. \tag{A26}$$

In order to obtain the wave functions, one can use the following relationships [14]

$$\rho(s)=s^{\alpha_{10}-1}(1-\alpha_3 s)^{(\alpha_{11}/\alpha_3)-\alpha_{10}-1}, \tag{A27}$$

$$\Phi_n(s)=P_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 s), \tag{A28}$$

$$W(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)}, \tag{A29}$$

$$\Psi_n(s) = s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)} \times P_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 s), \tag{A30}$$

where $P_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 s)$ is the Jacobi polynomials and [14]

$$\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8}, \tag{A31}$$

$$\alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}), \tag{A32}$$

$$\alpha_{12}=\alpha_4+\sqrt{\alpha_8}, \tag{A33}$$

$$\alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}). \tag{A34}$$

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