Relationship between quark-antiquark potential and quark-antiquark free energy in hadronic matter *

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Abstract: In high-temperature quark-gluon plasma and its subsequent hadronic matter created in a high-energy nucleus-nucleus collision, the quark-antiquark potential depends on the temperature. The temperature-dependent potential is expected to be derived from the free energy obtained in lattice gauge theory calculations. This requires one to study the relationship between the quark-antiquark potential and the quark-antiquark free energy. When the system's temperature is above the critical temperature, the potential of a heavy quark and a heavy antiquark almost equals the free energy, but the potential of a light quark and a light antiquark, of a heavy quark and a light antiquark and of a light quark and a heavy antiquark is substantially larger than the free energy. When the system's temperature is below the critical temperature, the quark-antiquark free energy can be taken as the quark-antiquark potential. This allows one to apply the quark-antiquark free energy to study hadron properties and hadron-hadron reactions in hadronic matter.

Key words: potential, free energy, Hadronic matter

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1 Introduction

The quark-antiquark free energy F is defined as the quark-antiquark internal energy U minus the product of the temperature T and the quark-antiquark entropy S. The internal energy of a quark and an antiquark at rest is the quark-antiquark potential. When the temperature is above the critical temperature T_c, the quark-antiquark free energy cannot be identified as the quark-antiquark potential [1]. From the correlator of a very heavy quark-antiquark pair a time-dependent potential has been obtained in quenched lattice QCD [2, 3]. Only at very large times and at $T<T_c$ does the potential agree with the free energy in the Coulomb gauge. At $T>T_c$ the potential at any time deviates from the free energy. In determining the potential there is uncertainty from the form of the correlator. The potential at very large times is thought to be the quark-antiquark potential. Hence, the relation between the quark-antiquark potential and the quark-antiquark free energy is not well understood. The medium created in high-energy nucleus-nucleus collisions is hadronic matter and the quark-gluon plasma. On the one hand hadron properties and hadron-hadron reactions change considerably from vacuum to hadronic matter with the change of the quark potential [4, 5], on the other hand the most striking medium effect is the QCD phase transition in which quark behavior and quark confinement change [6]. Hence, we need a change of the quark potential with respect to vacuum, hadronic matter, and the quark-gluon plasma, i.e. the temperature dependence of the quark potential. At present, the temperature-dependent potential is expected to be derived from the free energy obtained in lattice gauge theory calculations. Therefore, we must study the relation between the quark-antiquark potential and the quark-antiquark free energy.

In thermodynamics the entropy indicates the disorder of random motion of particles in a system that consists of a large number of particles. It equals the negative of the derivative of the system free energy with respect to the temperature when the system volume V is fixed. While V is fixed, the distance r between any quark and any antiquark always changes. Then, the quark-antiquark entropy cannot be taken as the negative of the derivative of the quark-antiquark free energy $F(T,r)$ with respect to the temperature while r is fixed. How to get the quark-antiquark entropy from the quark-antiquark free energy is a problem in lattice QCD. A formula for calculating the entropy from the QCD partition function with additional heavy quarks is given in Ref. [7]. The partition function contains the QCD action and an operator that is proportional to the product of two thermal Wilson

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lines. One needs to calculate the expectation values of the operator, of the derivative of the action with respect to the temperature and of the product of the operator and the derivative of the action with respect to the temperature. In addition, a constant needs to be fixed through renormalization. The formula is thus complicated and, in particular, not suitable for lattice simulations.

We must find another way to calculate the quark-antiquark entropy. The entropy is the thermodynamic quantity that characterizes the disorder of thermal motion of particles in a system. It is obtained from observables in thermodynamics. In the present work we evaluate the quark-antiquark entropy from energy densities and pressure of particle systems. A quark-gluon plasma created in a high-energy nucleus-nucleus collision expands while its temperature decreases, and the plasma hadronizes at the critical temperature to produce hadronic matter. Hadronic matter expands while its temperature decreases toward the freeze-out temperature. Quarks and antiquarks in the quark-gluon plasma and the pions that dominate hadronic matter are the particle systems with which we are concerned.

2 Free energy and potential

The quark-antiquark internal energy is related to the quark-antiquark free energy by

$$U(T,r)=F(T,r)+TS, \tag{1}$$

where r is the distance between the quark and the antiquark. The free energy is given by the Polyakov loop correlation function in lattice QCD. If TS is small, the quark-antiquark free energy can be considered as the quark-antiquark potential. If the free energy is of a heavy quark and a heavy antiquark, we calculate the entropy of this heavy quark-antiquark pair. If the free energy is of a light quark and a light antiquark, we calculate the entropy of this light quark-antiquark pair. In the following two subsections we show that TS is generally comparable to the free energy in a quark-gluon plasma, but is small in hadronic matter.

The quark-gluon plasma and hadronic matter contain a multitude of quark-antiquark pairs. Different quark-antiquark pairs may have different free energies, and the free energies depend on gluon states that propagate between the quark and the antiquark [8, 9]. The free energy is an extensive variable, and the quark-antiquark free energy is the system's free energy divided by the number of quark-antiquark pairs.

2.1 In the case of massless particles

In a quark-gluon plasma with three massless flavors, the quark energy density is [10]

$$\epsilon_{\rm q}=g_{\rm Q}\frac{7\pi^2}{240}T^4, \tag{2}$$

with the color-spin-flavor degeneracy factor $g_{\rm Q}=18$, and the antiquark energy density is

$$\epsilon_{\bar{\rm q}}=g_{\bar{\rm Q}}\frac{7\pi^2}{240}T^4, \tag{3}$$

with $g_{\bar{\rm Q}}=18$. The pressure due to quarks is $P_{\rm q}=\frac{1}{3}\epsilon_{\rm q}$, and the pressure due to antiquarks is $P_{\bar{\rm q}}=\frac{1}{3}\epsilon_{\bar{\rm q}}$. The quark entropy density is

$$s_{\rm q}=\frac{\epsilon_{\rm q}+P_{\rm q}}{T}=\frac{4}{3}\frac{\epsilon_{\rm q}}{T}, \tag{4}$$

and the antiquark entropy density is

$$s_{\bar{\rm q}}=\frac{\epsilon_{\bar{\rm q}}+P_{\bar{\rm q}}}{T}=\frac{4}{3}\frac{\epsilon_{\bar{\rm q}}}{T}. \tag{5}$$

The quark number density is [10]

$$n_{\rm q}=g_{\rm Q}\frac{3\zeta(3)}{4\pi^2}T^3, \tag{6}$$

with $\zeta(3)=1.20205$, and the antiquark number density is

$$n_{\bar{\rm q}}=g_{\bar{\rm Q}}\frac{3\zeta(3)}{4\pi^2}T^3. \tag{7}$$

There is one quark and one antiquark in the volume $V_{\rm c}=1/n_{\rm q}=1/n_{\bar{\rm q}}$. The dimensionless entropy of the quark-antiquark pair is proportional to

$$S_{{\rm q}\bar{\rm q}}=s_{\rm q}V_{\rm c}+s_{\bar{\rm q}}V_{\rm c}=\frac{14\pi^4}{135\zeta(3)}\simeq 8.408, \tag{8}$$

which is independent of temperature. On average the spatial separation of the quark and the antiquark is

$$\sqrt[3]{V_{\rm c}}=\frac{1}{T}\sqrt[3]{\frac{4\pi^2}{3g_{\rm Q}\zeta(3)}}. \tag{9}$$

At $T=1.01T_{\rm c}$ with the critical temperature $T_{\rm c}=0.175$ GeV [11, 12], the spatial separation is 0.946 fm where the free energy given in the lattice calculations is about 0.123 GeV [11]. The free energy is only for a quark-antiquark pair in a flavor and in the color singlet. However, the quark (antiquark) entropy density $s_{\rm q}$ ($s_{\bar{\rm q}}$) used in Eq. (8) corresponds to the three flavors and the three colors. $\frac{1}{3}s_{\rm q}$ $\left(\frac{1}{3}s_{\bar{\rm q}}\right)$ is the quark (antiquark) entropy density for a flavor or for a color. Hence, $S_{{\rm q}\bar{\rm q}}$ is multiplied by $\frac{1}{3}$ to get the entropy of the quark-antiquark pair in a flavor, and by another $\frac{1}{3}$ to get the entropy of the quark-antiquark pair in the color singlet. $\frac{1}{3}\frac{1}{3}S_{{\rm q}\bar{\rm q}}$ is then the entropy of the quark-antiquark pair in a flavor and in the color singlet. $\frac{1}{3}\frac{1}{3}TS_{{\rm q}\bar{\rm q}}$ is then compared to the free energy given in the lattice calculations. $\frac{1}{3}\frac{1}{3}TS_{\mathrm{q\bar{q}}}$ takes the value 0.165 GeV and is larger than the free energy 0.123 GeV. The quark-antiquark internal energy is larger than the free energy by 0.165 GeV. By subtracting the total kinetic energy of the quark and the antiquark, which is 0.062 GeV in the volume V_{c}, we get the quark-antiquark potential, which is larger than the free energy by 0.103 GeV. In the evolution of the quark-gluon plasma entropy is conserved. When temperature increases from T_{c}, the free energy decreases [11], the ratio of $\frac{1}{3}\frac{1}{3}TS_{\mathrm{q\bar{q}}}$ to the free energy increases, and the difference between the quark-antiquark potential and the free energy becomes larger and larger. Therefore, the quark-antiquark free energy cannot be taken as the quark-antiquark potential in the quark-gluon plasma in the case of massless quarks and antiquarks.

The energy density of a system of massless pions is [10]

$$\epsilon_{\pi}=g_{\pi}\frac{\pi^2}{30}T^4, \tag{10}$$

with $g_{\pi}=3$ for π^+, π^0, and π^-. The pressure due to massless pions is $P_{\pi}=\frac{1}{3}\epsilon_{\pi}$. The pion entropy density is

$$s_{\pi}=\frac{\epsilon_{\pi}+P_{\pi}}{T}=\frac{4}{3}\frac{\epsilon_{\pi}}{T}. \tag{11}$$

At T_{c} the quark and the antiquark in the volume V_{c} transit into a pion. The dimensionless pion entropy in this volume is proportional to

$$S_{\pi}=s_{\pi}V_{\mathrm{c}}=\frac{8\pi^4 g_{\pi}}{135 g_{\mathrm{Q}}\zeta(3)}\approx 0.8, \tag{12}$$

which is independent of temperature. The entropy that corresponds to the quark-antiquark free energy is $\frac{1}{3}S_{\pi}$ for one isospin component. The free energy slightly below T_{c} is about 0.29 GeV [11]. The ratio of $\frac{1}{3}TS_{\pi}=0.046$ GeV with $T=0.99T_{\mathrm{c}}$ to the free energy is about 0.16. The quark-antiquark internal energy is larger than the free energy by 0.046 GeV. By subtracting the total kinetic energy of the quark and the antiquark, which is 0.0128 GeV in the center-of-mass frame of the pion, we get the quark-antiquark potential which is larger than the free energy by 0.0332 GeV. While hadronic matter expands, entropy is conserved, temperature decreases, the free energy increases [11], the ratio of $\frac{1}{3}TS_{\pi}$ to the free energy decreases, and the difference between the quark-antiquark potential and the free energy becomes smaller and smaller. Hence, we conclude that $U(T,r)\approx F(T,r)$ in hadronic matter and the free energy can be taken as the quark-antiquark potential.

2.2 In the case of massive particles

The grand partition function for a quark system is

$$\Xi=\prod_{l}(1+\mathrm{e}^{\beta(\mu-\varepsilon_l)})^{\omega_l}, \tag{13}$$

where $\beta=1/T$, μ is the quark chemical potential, and ω_l is the number of states corresponding to the quark energy ε_l. In the relativistic case a quark has the energy $\varepsilon_l=\sqrt{\vec{p}^2+m^2}$. Quark, antiquark and gluon distributions in the quark-gluon plasma are described by the Jüttner distribution with the fugacity $\lambda=\exp(\mu/T)$. The evolution of the quark-gluon plasma can be determined by a set of master rate equations which give us the time dependence of the temperature and fugacities [13]. Initial values of the temperature, quark fugacity, antiquark fugacity, and gluon fugacity of the quark-gluon plasma are provided in [14]. Similar initial temperatures but different initial fugacities were obtained in [13] via free streaming. Solving the master rate equations with the initial values, we obtain 0.64, 0.64, and 0.76 individually for the quark fugacity, antiquark fugacity, and gluon fugacity at $T=1.01T_{\mathrm{c}}$ for central Au-Au collisions at $\sqrt{s_{\mathrm{NN}}}=200$ GeV and 1, 1, and 1 for central Pb-Pb collisions at $\sqrt{s_{\mathrm{NN}}}=5.5$ TeV. The corresponding chemical potentials are -0.078, 0.078, and -0.048 for the quark, the antiquark, and the gluon at the RHIC energy and 0, 0, and 0 at the LHC energy. Therefore, in a volume V of the quark-gluon plasma created in heavy ion collisions from the highest RHIC energy to the highest LHC energy, the quark chemical potential is nearly or exactly zero, and we have

$$\ln\Xi=\frac{4\pi g_{\mathrm{Q}}V}{(2\pi)^3}\int_0^{\infty}\mathrm{d}|\vec{p}|\vec{p}^2\ln(1+\mathrm{e}^{-\beta\sqrt{\vec{p}^2+m^2}}). \tag{14}$$

With the variable $z=\beta\sqrt{\vec{p}^2+m^2}$ the total energy of the quark system is

$$\begin{aligned}E_{\mathrm{q}} &= -\frac{\partial}{\partial\beta}\ln\Xi=\frac{4\pi g_{\mathrm{Q}}V}{(2\pi)^3\beta^4}\int_{m\beta}^{\infty}\mathrm{d}z z^2\sqrt{z^2-m^2\beta^2}/(\mathrm{e}^z+1)\\ &= \frac{12\pi g_{\mathrm{Q}}V}{(2\pi)^3\beta^4}\int_{m\beta}^{\infty}\mathrm{d}z z\sqrt{z^2-m^2\beta^2}\ln(1+\mathrm{e}^{-z})\\ &\quad +\frac{4\pi g_{\mathrm{Q}}m^2V}{(2\pi)^3\beta^2}\int_{m\beta}^{\infty}\mathrm{d}z\frac{z\ln(1+\mathrm{e}^{-z})}{\sqrt{z^2-m^2\beta^2}},\end{aligned} \tag{15}$$

and the pressure of the quark system is

$$\begin{aligned}P_{\mathrm{q}} &= \frac{1}{\beta}\frac{\partial}{\partial V}\ln\Xi=\frac{4\pi g_{\mathrm{Q}}}{(2\pi)^3\beta^4}\int_{m\beta}^{\infty}\mathrm{d}z z\sqrt{z^2-m^2\beta^2}\ln(1+\mathrm{e}^{-z})\\ &= \frac{1}{3}\frac{E_{\mathrm{q}}}{V}-\frac{4\pi g_{\mathrm{Q}}m^2}{3(2\pi)^3\beta^2}\int_{m\beta}^{\infty}\mathrm{d}z\frac{z\ln(1+\mathrm{e}^{-z})}{\sqrt{z^2-m^2\beta^2}}.\end{aligned} \tag{16}$$

The quark entropy density is

$$s_{\rm q}=\frac{1}{T}\left(\frac{E_{\rm q}}{V}+P_{\rm q}\right). \tag{17}$$

The total energy, pressure and entropy density of the antiquark system are written similarly. For a quark and an antiquark in the volume $V_{\rm c}$, their entropy is proportional to

$$S_{\rm q\bar{q}}=s_{\rm q}V_{\rm c}+s_{\bar{\rm q}}V_{\rm c}. \tag{18}$$

Below we show three examples of the system of quarks and antiquarks. In the first example we consider charm quarks and antiquarks. If they are thermalized, the above formulae can be applied with $g_{\rm Q}=6$ for the charm quark and $g_{\bar{\rm Q}}=6$ for the charm antiquark. For the color singlet of the charm quark and antiquark, $S_{\rm c\bar{c}}$ is multiplied by $\frac{1}{3}$ to get the entropy of the color singlet, and $\frac{1}{3}TS_{\rm c\bar{c}}=1.72\times10^{-3}$ GeV where $T=1.01T_{\rm c}$. This indicates that the potential of the charm quark and the charm antiquark almost equals the free energy. In the second example we consider charm quarks and down antiquarks. For the color singlet of a charm quark and a massless down antiquark (a down antiquark with a mass of 0.32 GeV [4, 5]), $\frac{1}{3}TS_{\rm c\bar{d}}=0.0834$ GeV (0.0594 GeV). By comparison with the free energy 0.123 GeV and the subtraction of the total kinetic energy of the quark and antiquark from the internal energy, we realize that the free energy cannot be taken as the potential of the charm quark and the down antiquark. In the third example we consider bottom (top) quarks and down antiquarks. For the color singlet of a bottom (top) quark and a down antiquark with a mass of 0 or 0.32 GeV, $\frac{1}{3}TS_{\rm b\bar{d}}=0.0825$ GeV or 0.0591 GeV $\left(\frac{1}{3}TS_{\rm t\bar{d}}=0.0825\text{ GeV or }0.0591\text{ GeV}\right)$. We conclude that the free energy cannot be taken as the potential of the bottom (top) quark and the down antiquark.

The grand partition function for a meson system is

$$\varXi=\prod_l(1-{\rm e}^{\beta(\mu-\varepsilon_l)})^{-\omega_l}, \tag{19}$$

where μ is the meson chemical potential, and ω_l is the number of states corresponding to the meson energy ε_l. A coalescence of quarks and antiquarks forms mesons at $T_{\rm c}$. Pions, as the dominant species of hadronic matter, have a chemical potential close to zero from the highest RHIC energy to the highest LHC energy. In the relativistic case and at $\mu=0$,

$$\ln\varXi=-\frac{4\pi g_{\rm M}V}{(2\pi)^3}\int_0^\infty {\rm d}|\vec{p}|\vec{p}^2\ln(1-{\rm e}^{-\beta\sqrt{\vec{p}^2+m^2}}), \tag{20}$$

where $g_{\rm M}$ is the spin-isospin degeneracy factor. With the variable $z=\beta\sqrt{\vec{p}^2+m^2}$ the total energy of the meson system is

$$\begin{aligned}E_{\rm m} &= -\frac{\partial}{\partial\beta}\ln\varXi=\frac{4\pi g_{\rm M}V}{(2\pi)^3\beta^4}\int_{m\beta}^\infty {\rm d}zz^2\sqrt{z^2-m^2\beta^2}/({\rm e}^z-1)\\ &= -\frac{12\pi g_{\rm M}V}{(2\pi)^3\beta^4}\int_{m\beta}^\infty {\rm d}zz\sqrt{z^2-m^2\beta^2}\ln(1-{\rm e}^{-z})\\ &\quad -\frac{4\pi g_{\rm M}m^2V}{(2\pi)^3\beta^2}\int_{m\beta}^\infty {\rm d}z\frac{z\ln(1-{\rm e}^{-z})}{\sqrt{z^2-m^2\beta^2}},\end{aligned} \tag{21}$$

and the pressure of the meson system is

$$\begin{aligned}P_{\rm m}&=\frac{1}{\beta}\frac{\partial}{\partial V}\ln\varXi=-\frac{4\pi g_{\rm M}}{(2\pi)^3\beta^4}\int_{m\beta}^\infty {\rm d}zz\sqrt{z^2-m^2\beta^2}\ln(1-{\rm e}^{-z})\\ &=\frac{1}{3}\frac{E_{\rm m}}{V}+\frac{4\pi g_{\rm M}m^2}{3(2\pi)^3\beta^2}\int_{m\beta}^\infty {\rm d}z\frac{z\ln(1-{\rm e}^{-z})}{\sqrt{z^2-m^2\beta^2}}.\end{aligned} \tag{22}$$

The meson entropy density is

$$s_{\rm m}=\frac{1}{T}\left(\frac{E_{\rm m}}{V}+P_{\rm m}\right), \tag{23}$$

and the meson entropy in the volume $V_{\rm c}$ is proportional to

$$S_{\rm m}=s_{\rm m}V_{\rm c}. \tag{24}$$

To get an impression of $TS_{\rm m}$, we take the unrealistic case that J/ψ is thermalized in hadronic matter. With the J/ψ mass 2.85 GeV/c^2 at $T=0.99\ T_{\rm c}$ [5], we obtain $TS_{\rm J/\psi}=1.98\times10^{-6}$ GeV at $g_{\rm M}=3$ while $S_{\rm J/\psi}$ is the J/ψ entropy in the volume $V_{\rm c}$. Therefore, the quark-antiquark potential in a J/ψ meson equals the quark-antiquark free energy. For the system of massless pions we have already obtained the result that the free energy can be taken as the potential. While the meson mass increases from 0, $E_{\rm m}$, $P_{\rm m}$, $s_{\rm m}$, and $TS_{\rm m}$ decrease. Then, the result is also true in a meson with a nonzero mass.

While the quark chemical potential and the meson chemical potential are not zero, they are negative [13], which corresponds to chemical nonequilibrium of matter created in relativistic heavy ion collisions below the highest RHIC energy. The grand partition function for the quark system in Eq. (13) and the one for the meson system in Eq. (19) are reduced. $E_{\rm q}$, $P_{\rm q}$, $s_{\rm q}$, $S_{\rm q\bar{q}}$, $E_{\rm m}$, $P_{\rm m}$, $s_{\rm m}$, and $S_{\rm m}$ decrease. Then, we can infer that the free energy can be taken as the potential in the meson system, the free energy cannot be taken as the potential of a heavy quark (antiquark) and a light antiquark (quark) in the quark-gluon plasma, and the potential of a heavy quark and a heavy antiquark almost equals the free energy.

Finally, we note that we cannot use the entropy density obtained in lattice calculations to deal with the relation between the quark-antiquark potential and the quark-antiquark free energy. This entropy density includes not only that of quarks and antiquarks but also that of gluons. Nobody has separated the entropy density of quarks and antiquarks from the entropy density obtained in lattice calculations.

3 Summary

The entropy of a particle system depends on the particle mass. When the system's temperature is larger than the critical temperature, the product of the temperature and the entropy of a massless quark-antiquark pair is comparable to the quark-antiquark free energy, and the free energy cannot be taken as the potential of a massless quark and a massless antiquark. This is also true while the quark or the antiquark is massless, but not while both the quark and the antiquark are heavy. When the system's temperature is smaller than the critical temperature, the product of the temperature and a meson's entropy is small or negligible in comparison with the quark-antiquark free energy, and the free energy can be taken as the quark-antiquark potential to a good approximation, which is in agreement with the result of the lattice calculations in Refs. [2, 3].

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