

Exact results for Wilson loops in orbifold ABJM theory^{*}

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Abstract: We investigate the exact results for circular 1/4 and 1/2 BPS Wilson loops in the $d = 3$ $\mathcal{N} = 4$ super Chern-Simons-matter theory that could be obtained by orbifolding Aharony-Bergman-Jafferis-Maldacena (ABJM) theory. The partition function of the $\mathcal{N} = 4$ orbifold ABJM theory has been computed previously in the literature. In this paper, we re-derive it using a slightly different method. We calculate the vacuum expectation values of the circular 1/4 BPS Wilson loops in fundamental representation and of circular 1/2 BPS Wilson loops in arbitrary representations. We use both the saddle point approach and Fermi gas approach. The results for Wilson loops are in accord with the available gravity results.

Keywords: AdS/CFT correspondence, Wilson loop, Chern-Simons theory, matrix model

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1 Introduction

In the AdS₅/CFT₄ correspondence [1–3], 1/2 BPS Wilson loops in $d = 4$ $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory are dual to fundamental strings in type IIB string theory in AdS₅ × S⁵ spacetime [4–7]. When string theory is weakly coupled and the supergravity approximation is a good one, the dual $d = 4$ $\mathcal{N} = 4$ super Yang-Mills theory is strongly coupled. To compare with the gravity results, one has to know the vacuum expectation values of the Wilson loops at strong coupling. To do this it was proposed in Refs. [8, 9] that $d = 4$ $\mathcal{N} = 4$ super Yang-Mills theory is related to the Gaussian matrix model, and this was proved in Ref. [10] using localization techniques.

There is a similar but more complicated story in the AdS₄/CFT₃ correspondence. M-theory in AdS₄ × S⁷/Z_k spacetime, or type IIA string theory in AdS₄ × CP³ spacetime, is dual to the $d = 3$ $\mathcal{N} = 6$ super Chern-Simons-matter (SCSM) theory with gauge group $U(N) \times U(N)$ and levels $(k, -k)$, which is known as Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [11]. In ABJM theory there are 1/6 BPS [12–14] and 1/2 BPS [15] Wilson

loops. The 1/6 BPS Wilson loops are closely related to the 1/2 BPS Wilson loops in $\mathcal{N} = 2$ SCSM theory in [16]. Localization techniques have been applied to ABJM theory and other SCSM theories with fewer supersymmetries [17–19] and lead to matrix models that are more complicated than the Gaussian matrix model. By using localization, one can calculate the partition function and vacuum expectation values of Wilson loops at both weak coupling and strong coupling [15, 17, 20–22]. The computations in [22] are based on the saddle point solution of the ABJM matrix model at large N limit with finite k , and we will call such a method the saddle point approach. Furthermore, the ABJM matrix model could be reformulated as an ideal Fermi gas with a complicated potential [23], and one can calculate the vacuum expectation values of BPS Wilson loops with fixed winding number using the Fermi gas approach [24]. One can also use the Fermi gas approach to calculate the vacuum expectation values of the 1/2 BPS Wilson loops in arbitrary representations [25].

By Z_r orbifolding the $U(rN) \times U(rN)$ ABJM theory, one can obtain a $d = 3$ $\mathcal{N} = 4$ SCSM theory with gauge group $U(N)^{2r}$ and levels $(k, -k, \dots, k, -k)$. This the-

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ory is dual to M-theory in $\text{AdS}_4 \times \text{S}^7 / (\mathbb{Z}_r \times \mathbb{Z}_{rk})$ spacetime [26–29]. The partition function of the orbifold ABJM theory has been calculated using Fermi gas approach in [30], and in this paper we will re-derive it using a slightly different way. In the orbifold ABJM theory there are 1/4 and 1/2 BPS Wilson loops, and the 1/2 BPS Wilson loops in fundamental representation should be dual to M2-branes with one dimension wrapping on the M-theory circle [31, 32]. In this paper, we will calculate the leading contributions of vacuum expectation values of the Wilson loops using the saddle point approach in the large N limit with k and r being finite. We will also calculate the perturbative part¹⁾ of the vacuum expectation values of the Wilson loops using the Fermi gas approach. The results are in agreement with the available gravity results.

In the $\mathcal{N} = 4$ orbifold ABJM theory with gauge group $U(N)^{2r}$, there are $2r$ linearly independent 1/2 BPS Wilson loops that preserve the same supersymmetries, but there are not so many 1/2 BPS branes in M-theory in $\text{AdS}_4 \times \text{S}^7 / (\mathbb{Z}_r \times \mathbb{Z}_{rk})$ spacetime. It was conjectured that these Wilson loops are 1/2 BPS classically, and only a special linear combination of them is 1/2 BPS quantum mechanically [32]. If all the $2r$ Wilson loops are 1/2 BPS and each of them differs from 1/4 BPS Wilson loop by a Q -exact term quantum mechanically, we can calculate their vacuum expectation values in a matrix model as shown in this paper. If only a special linear combination of the $2r$ Wilson loops is 1/2 BPS quantum mechanically and it differs from an 1/4 BPS Wilson loop by a Q -exact term, we can calculate its vacuum expectation value in the matrix model. If there is no 1/4 BPS Wilson loop that differs from such an 1/2 BPS Wilson loop by a Q -exact term, we cannot calculate the 1/2 BPS Wilson loop’s vacuum expectation value using currently available localization techniques. In this case, a large part of the calculations in the paper are just some results in the matrix model and have nothing to do with vacuum expectations values of Wilson loops.

The rest of the paper is arranged as follows. In Section 2 we review the results in ABJM theory, including the partition function and vacuum expectation values of Wilson loops. In Section 3 we investigate the partition function of the $\mathcal{N} = 4$ orbifold ABJM theory. In Section 4 we review the circular 1/4 and 1/2 BPS half-BPS Wilson loops in the $\mathcal{N} = 4$ orbifold ABJM theory in Euclidean space. In Section 5 we calculate vacuum expectation values of Wilson loops with fixed winding number using the saddle point approach. In Section 6 we calculate vacuum expectation values of Wilson loops in arbitrary representations using the Fermi gas approach. We end with conclusions and discussions in Section 7. In Appendix A we investigate if there are more general 1/2 BPS half-BPS

Wilson loops in $\mathcal{N} = 4$ orbifold ABJM theory other than the ones found in [31, 32]. We find no new ones.

2 Results in ABJM theory

In this section we review some results in ABJM theory. This includes the partition function and vacuum expectation values of circular 1/6 and 1/2 BPS Wilson loops. We focus on what will be used in the following sections, so this is merely a brief review.

2.1 Partition function

The partition function of ABJM theory with gauge group $U(N) \times U(N)$ and levels $(k, -k)$ can be localized to be the ABJM matrix model [17]

$$Z(N) = \frac{1}{N!^2} \int \prod_{i=1}^N \frac{d\mu_i}{2\pi} \frac{d\nu_i}{2\pi} \frac{\prod_{i < j} \left(2 \sinh \frac{\mu_i - \mu_j}{2} \right)^2 \left(2 \sinh \frac{\nu_i - \nu_j}{2} \right)^2}{\prod_{i,j} \left(2 \cosh \frac{\mu_i - \nu_j}{2} \right)^2} \times \exp \left[\frac{ik}{4\pi} \sum_i (\mu_i^2 - \nu_i^2) \right]. \tag{1}$$

The partition function of the matrix model (1) can be written as the canonical partition function $Z(N)$ of N -particle free Fermi gas with the one-particle density matrix being [23]

$$\hat{\rho} = e^{-\hat{H}}, \tag{2}$$

whose explicit form will not be used in this paper. Note that \hat{H} is the one-particle Hamiltonian operator. To calculate $Z(N)$, one can firstly calculate the grand partition function

$$\Xi(\mu) = \sum_{N=0}^{+\infty} z^N Z(N), \tag{3}$$

with $Z(0) = 1$, $z = e^\mu$ being the fugacity and μ being the chemical potential. The grand potential is defined as

$$J(\mu) = \log \Xi(\mu). \tag{4}$$

And then one gets

$$Z(N) = \int_{-\pi i}^{\pi i} \frac{d\mu}{2\pi i} e^{J(\mu) - \mu N}. \tag{5}$$

One can define $j(\mu)$ according to [33]

$$e^{J(\mu)} = \sum_{l=-\infty}^{\infty} e^{j(\mu + 2\pi i l)}, \tag{6}$$

1) By this we mean to include all of the $1/N$ corrections, putting aside the non-perturbative contributions.

and then

$$Z(N) = \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i} e^{j(\mu) - \mu N}. \quad (7)$$

One adopts the phase space formulation of quantum mechanics, and defines

$$n(\mu) = \int \frac{dqdp}{2\pi\hbar} \theta(\mu - \hat{H})_W, \quad (8)$$

with $\hbar = 2\pi k$, $\theta(x)$ being the Heaviside step function, and W being the Wigner transformation. The quantity $n(\mu)$ counts the number of one-particle states whose energy is less than μ . Using the Sommerfeld expansion one can get the expectation value of particle number $N(\mu)$ in the grand canonical ensemble

$$N(\mu) = \pi \partial_\mu \text{csc}(\pi \partial_\mu) n(\mu). \quad (9)$$

It is standard in the grand canonical ensemble that

$$N(\mu) = \frac{\partial J(\mu)}{\partial \mu}, \quad (10)$$

and so we get

$$J(\mu) = \int_{-\infty}^{\mu} N(u) du. \quad (11)$$

We find that when $\mu \rightarrow -\infty$,

$$\partial_\mu^l N(\mu) \rightarrow 0, \quad l = 0, 1, 2, \dots \quad (12)$$

This result is very useful for us. Note that the way from $n(\mu)$ to $N(\mu)$ and then to $J(\mu)$ is a slightly different method of getting $J(\mu)$ to the one in the original paper [23].

In the large μ (i.e. large N) limit, one can split a quantity into the perturbative part and non-perturbative part. The perturbative part is denoted as pt. The non-perturbative part is exponentially suppressed in the large μ (i.e. large N) limit, and it is denoted as np. In this paper we will mainly focus on the perturbative part. It turns out that [23]

$$n_{\text{pt}}(\mu) = C\mu^2 + n_0, \quad (13)$$

with

$$C = \frac{2}{\pi^2 k}, \quad n_0 = -\frac{1}{3k} + \frac{k}{24}. \quad (14)$$

One then gets

$$\begin{aligned} N_{\text{pt}}(\mu) &= C\mu^2 + B, \\ J_{\text{pt}}(\mu) &= \frac{C}{3}\mu^3 + B\mu + A, \end{aligned} \quad (15)$$

where

$$B = n_0 + \frac{\pi^2 C}{3} = \frac{1}{3k} + \frac{k}{24}. \quad (16)$$

Here A appears as an integral constant, and its exact form depends on the full form of $N(\mu)$. One can find the result for A in [34, 35]. One has

$$j_{\text{pt}}(\mu) = J_{\text{pt}}(\mu), \quad (17)$$

and then one gets the perturbative part of the partition function [23, 36]

$$Z_{\text{pt}}(N) = C^{-1/3} e^A \text{Ai}[C^{-1/3}(N - B)], \quad (18)$$

with $\text{Ai}(x)$ being the Airy function.

2.2 Wilson loops

The representations of group $U(N)$ and supergroup $U(N|N)$ can be denoted by Young diagrams. We write a general Young diagram as R , and it can be a representation of $U(N)$ or $U(N|N)$.

We consider the hook representation $R = (a|b)$ with $a + 1$ boxes in the first row and one box in each of the remaining b rows. For both the 1/6 BPS and 1/2 BPS cases, a Wilson loop with winding number n is related to Wilson loops in the hook representations by

$$W^n = \sum_{b=0}^{n-1} (-1)^b W_{(n-1-b|b)}. \quad (19)$$

When $n = 1$, it is just the fundamental representation. In the matrix model (1), the circular 1/6 and 1/2 BPS Wilson loops with winding number n can be written as [15, 17]

$$\begin{aligned} \langle W_{1/6}^n \rangle &= \left\langle \sum_i e^{n\mu_i} \right\rangle, \quad \langle \hat{W}_{1/6}^n \rangle = \left\langle \sum_i e^{n\nu_i} \right\rangle, \\ \langle W_{1/2}^n \rangle &= \left\langle \sum_i [e^{n\mu_i} - (-)^n e^{n\nu_i}] \right\rangle, \end{aligned} \quad (20)$$

with n being the winding number of the loop and the right hand sides being the expectation values in the matrix model. For their expectation values one has the relation

$$\begin{aligned} \langle W_{1/2}^n \rangle &= \langle W_{1/6}^n \rangle - (-)^n \langle \hat{W}_{1/6}^n \rangle \\ &= \langle W_{1/6}^n \rangle - (-)^n \langle W_{1/6}^n \rangle^*, \end{aligned} \quad (21)$$

with $*$ being the complex conjugate.

In the large N limit with finite k , i.e. the M-theory limit, the values μ_i and ν_i at the saddle point can be denoted as a continuous distribution [22]

$$\mu(x) = \sqrt{N}x + i\frac{kx_*}{4\pi}, \quad \nu(x) = \sqrt{N}x - i\frac{kx_*}{4\pi}, \quad (22)$$

with the uniform density

$$\rho(x) = \frac{1}{2x_*}, \quad x \in [-x_*, x_*], \quad x_* = \pi\sqrt{\frac{2}{k}}. \quad (23)$$

In the saddle point approach the Wilson loop vacuum expectation values can be calculated as

$$\begin{aligned} \langle W_{1/6}^n \rangle &\approx N \int_{-x_*}^{x_*} e^{n\mu(x)} \rho(x) dx \approx \frac{i^n k}{2n\pi} \sqrt{\frac{\lambda}{2}} e^{n\pi\sqrt{2\lambda}}, \\ \langle W_{1/2}^n \rangle &\approx N \int_{-x_*}^{x_*} [e^{n\mu(x)} - (-)^n e^{n\nu(x)}] \rho(x) dx \\ &\approx \frac{i^{n-1} k}{4n\pi} e^{n\pi\sqrt{2\lambda}}. \end{aligned} \quad (24)$$

The exponentially suppressed terms are omitted here. Note that one can only get the correct leading contribution of large N in the saddle point approach.

The vacuum expectation values of circular Wilson loops can also be calculated in the Fermi gas approach [24]. One firstly calculates

$$m(\mu) = \int \frac{dqdp}{2\pi\hbar} \theta(\mu - \hat{H})_W e^{\frac{n(q+p)}{k}}, \quad (25)$$

and then using Sommerfeld expansion one gets the 1/6 BPS Wilson loop expectation value in the grand canonical ensemble

$$M(\mu) = \pi \partial_\mu \text{csc}(\pi \partial_\mu) m(\mu). \quad (26)$$

Then the 1/6 BPS Wilson loop expectation value with winding n in the canonical ensemble is

$$\langle W_{1/6}^n \rangle = \frac{1}{Z(N)} \int_{-\pi i}^{\pi i} \frac{d\mu}{2\pi i} e^{J(\mu) - \mu N} M(\mu). \quad (27)$$

Similar to the partition function, one has

$$\langle W_{1/6}^n \rangle_{\text{pt}} = \frac{1}{Z_{\text{pt}}(N)} \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i} e^{J_{\text{pt}}(\mu) - \mu N} M_{\text{pt}}(\mu), \quad (28)$$

with non-perturbative contributions being neglected. It turns out that

$$\begin{aligned} m_{\text{pt}}(\mu) &= (D\mu + E) e^{\frac{2n\mu}{k}}, \\ M_{\text{pt}}(\mu) &= \frac{2\pi n}{k} \text{csc} \frac{2\pi n}{k} e^{\frac{2n\mu}{k}} \\ &\times \left[\left(\mu + \frac{k}{2n} - \pi \cot \frac{2\pi n}{k} \right) D + E \right], \end{aligned} \quad (29)$$

with

$$D = \frac{i^n}{2\pi^2 n}, \quad E = -\frac{i^{n+1} k}{4\pi^2 n} \left(\frac{\pi}{2} - iH_n \right). \quad (30)$$

Here H_n is the harmonic number,

$$H_n = \sum_{d=1}^n \frac{1}{d}, \quad (31)$$

with H_0 being 1. Then the 1/6 BPS Wilson loop vacuum expectation value is

$$\begin{aligned} \langle W_{1/6}^n \rangle_{\text{pt}} &= - \left(\frac{2}{\pi^2 k} \right)^{-1/3} \\ &\times F \frac{\text{Ai}' \left[\left(\frac{2}{\pi^2 k} \right)^{-1/3} \left(N - \frac{k}{24} - \frac{6n+1}{3k} \right) \right]}{\text{Ai} \left[\left(\frac{2}{\pi^2 k} \right)^{-1/3} \left(N - \frac{k}{24} - \frac{1}{3k} \right) \right]} \\ &+ G \frac{\text{Ai} \left[\left(\frac{2}{\pi^2 k} \right)^{-1/3} \left(N - \frac{k}{24} - \frac{6n+1}{3k} \right) \right]}{\text{Ai} \left[\left(\frac{2}{\pi^2 k} \right)^{-1/3} \left(N - \frac{k}{24} - \frac{1}{3k} \right) \right]}, \end{aligned} \quad (32)$$

where

$$\begin{aligned} F &= \frac{2\pi n}{k} \text{csc} \frac{2\pi n}{k} D, \\ G &= \frac{2\pi n}{k} \text{csc} \frac{2\pi n}{k} \left[\left(\frac{k}{2n} - \pi \cot \frac{2\pi n}{k} \right) D + E \right]. \end{aligned} \quad (33)$$

The 1/2 BPS Wilson loop vacuum expectation value is

$$\begin{aligned} \langle W_{1/2}^n \rangle_{\text{pt}} &= \frac{i^{n-1}}{2} \text{csc} \frac{2\pi n}{k} \\ &\times \frac{\text{Ai} \left[\left(\frac{2}{\pi^2 k} \right)^{-1/3} \left(N - \frac{k}{24} - \frac{6n+1}{3k} \right) \right]}{\text{Ai} \left[\left(\frac{2}{\pi^2 k} \right)^{-1/3} \left(N - \frac{k}{24} - \frac{1}{3k} \right) \right]}. \end{aligned} \quad (34)$$

Now we turn to Wilson loops in hook representations based on Ref. [25]. There the density matrix for Fermi gas dual to ABJM theory was obtained as

$$\hat{\rho} = \sqrt{Q} P \sqrt{Q}, \quad \text{with} \quad P = \frac{1}{2 \cosh \frac{p}{2}}, \quad Q = \frac{1}{2 \cosh \frac{q}{2}}. \quad (35)$$

Though it is the same as the matrix in [23] but different from the one in [24], it gives the same partition functions and vacuum expectation values of BPS Wilson loops. One of the key steps in Ref. [25] is the following result

$$\Xi(z) \left\langle \prod_i \frac{f(e^{\mu_i})}{f(e^{\nu_i})} \right\rangle^{\text{GC}} = \text{Det}(1 + z \hat{\rho}_f), \quad (36)$$

where

$$\hat{\rho}_f = \sqrt{Q} \frac{1}{f(-W)} P f(W) \sqrt{Q}, \quad \text{with} \quad W = e^{\frac{q+p}{k}}. \quad (37)$$

The density matrix $\hat{\rho}_f$ with $f(W) = (1 + tW)/(1 - sW)$ can be written as [25]

$$\hat{\rho}_f = \hat{\rho} + (s+t) \sum_{a,b=0}^{\infty} s^a t^b |b\rangle \langle a|, \quad (38)$$

where $|a\rangle$ and $|b\rangle$ are defined in the coordinate q representation as

$$\begin{aligned} \langle q|a\rangle &= \frac{e^{(a+\frac{1}{2})\frac{q}{k} - \frac{\pi i}{k}a(a+1)}}{\sqrt{2\cosh\frac{q}{2}}}, \\ \langle b|q\rangle &= \langle q|b\rangle^* = \frac{e^{(b+\frac{1}{2})\frac{q}{k} + \frac{\pi i}{k}b(b+1)}}{\sqrt{2\cosh\frac{q}{2}}}. \end{aligned} \quad (39)$$

For the half BPS Wilson loop in a hook representation $(a|b)$, the generating function is given by [37]¹⁾

$$\begin{aligned} &1 + (s+t) \sum_{a,b=0}^{\infty} s^a t^b W_{(a|b)} \\ &= \text{Sdet} \left(\frac{1+tU}{1-sU} \right) = \prod_{j=1}^N \frac{(1+te^{\mu_j})(1+se^{\nu_j})}{(1-se^{\mu_j})(1-te^{\nu_j})}, \end{aligned} \quad (40)$$

with $U = \text{diag}(U_\mu, -U_\nu)$, $U_\mu = \text{diag}(e^{\mu_i})$, $U_\nu = \text{diag}(e^{\nu_i})$. Therefore, the grand canonical ensemble expectation value of 1/2 BPS Wilson loop generating function in ABJM theory becomes

$$\begin{aligned} &\left\langle 1 + (s+t) \sum_{a,b=0}^{\infty} s^a t^b W_{(a|b)} \right\rangle^{\text{GC}} = \frac{\det(1+z\hat{\rho}_f)}{\det(1+z\hat{\rho})} \\ &= \det \left(1 + (s+t) \sum_{a,b=0}^{\infty} s^a t^b \frac{z}{1+z\hat{\rho}} |b\rangle\langle a| \right) \\ &= 1 + (s+t) \sum_{a,b=0}^{\infty} s^a t^b \langle a| \frac{z}{1+z\hat{\rho}} |b\rangle. \end{aligned} \quad (41)$$

One gets the relation

$$\begin{aligned} \langle W_{(a|b)} \rangle^{\text{GC}} &= \langle a| \frac{z}{1+z\hat{\rho}} |b\rangle = \text{Tr} \left(\frac{z}{1+z\hat{\rho}} |b\rangle\langle a| \right) \\ &= \text{Tr} \left(\frac{1}{e^{\hat{H}-\mu} + 1} e^{\hat{H}} |b\rangle\langle a| \right). \end{aligned} \quad (42)$$

As discussed in Ref. [25], the perturbative part of the half BPS hook Wilson loop in ABJM theory is determined by the topological vertex of C^3 in Ref. [38]

$$\langle W_{(a|b)} \rangle_{\text{pt}}^{\text{GC}} = \frac{q^{\frac{1}{4}a(a+1) - \frac{1}{4}b(b+1)}}{[a+b+1][a]![b]!} i^{a+b+1} e^{\frac{2(a+b+1)\mu}{k}}, \quad (43)$$

with $q = e^{\frac{4\pi i}{k}}$ and $[n] = q^{\frac{n}{2}} - q^{-\frac{n}{2}}$.

Let us consider the circular half BPS Wilson loops in non-hook representations. One can decompose the Young diagram for a non-hook representation into hooks from the upper left to the lower right to get $(a_1|b_1), \dots, (a_s|b_s)$. This general representation will be denoted as $R = (a_1 \cdots a_s | b_1 \cdots b_s)$. The Giambelli formula states that

$$W_{(a_1 a_2 \cdots a_s | b_1 b_2 \cdots b_s)}(e^{\mu_i}, e^{\nu_j}) = \det_{p,q} W_{(a_p | b_q)}(e^{\mu_i}, e^{\nu_j}). \quad (44)$$

The authors of Ref. [25] considered the following generating function

$$W(N) = \langle \det_{p,q} [\delta_{pq} + tW_{(a_p | b_q)}(e^{\mu_i}, e^{\nu_j})] \rangle. \quad (45)$$

The computations in Ref. [25] give

$$W(N) = \frac{1}{N!} \int \prod_i [d\nu_i] \mathbf{Det} \rho_f(\nu_i, \nu_j), \quad (46)$$

where

$$\begin{aligned} [d\mu_i] &= \frac{d\mu_i}{2\pi} \exp\left(\frac{ik\mu_i^2}{4\pi}\right), \quad [d\nu_i] = \frac{d\nu_i}{2\pi} \exp\left(-\frac{ik\nu_i^2}{4\pi}\right), \\ \rho_f(\nu_i, \nu_j) &= \int [d\mu] \frac{1}{2\cosh\frac{\mu-\nu_j}{2}} \left(\frac{1}{2\cosh\frac{\nu_i-\mu}{2}} \right. \\ &\quad \left. + t \sum_{p=1}^s e^{(b_p+1/2)\nu_i} e^{(a_p+1/2)\mu} \right). \end{aligned} \quad (47)$$

Then one has

$$\sum_{N=0}^{\infty} z^N W(N) = \mathbf{Det}(1+z\rho_f). \quad (48)$$

The multiplication between boldface variables is understood as matrix multiplication with indices μ, ν and summation being replaced by integration with measures $[d\mu], [d\nu]$. Then by introducing

$$\begin{aligned} Q(\mu, \nu) &= \frac{1}{2\cosh\frac{\mu-\nu}{2}}, \quad P(\nu, \mu) = \frac{1}{2\cosh\frac{\nu-\mu}{2}}, \\ \rho &= \sqrt{Q} P \sqrt{Q}, \quad \left(\langle a| \frac{1}{\sqrt{Q}} \right) (\mu) = e^{(a+1/2)\mu}, \\ \left(\frac{1}{\sqrt{Q}} |b\rangle \right) (\nu) &= e^{(b+1/2)\nu}, \end{aligned} \quad (49)$$

one can get

$$\begin{aligned} &\mathbf{Det} \left[1 + z \left(P + t \sum_{p=1}^s \frac{1}{\sqrt{Q}} |b_p\rangle\langle a_p| \frac{1}{\sqrt{Q}} \right) Q \right] \\ &= \mathbf{Det}(1+z\rho) \det_{p,q} [\delta_{pq} + zt \langle a_p | (1+z\rho)^{-1} | b_q \rangle]. \end{aligned} \quad (50)$$

Taking $t=0$ in the above results, one gets

$$\Xi(z) = \mathbf{Det}(1+z\rho), \quad (51)$$

and then one has

$$\langle \det_{p,q} (\delta_{pq} + tW_{(a_p | b_q)}) \rangle^{\text{GC}} = \det_{p,q} [\delta_{pq} + zt \langle a_p | (1+z\rho)^{-1} | b_q \rangle]. \quad (52)$$

1) In the remainder of this section, we will only discuss the half BPS Wilson loop and omit 1/2 in the subscript.

The coefficient of t^s in both sides of the above equation gives

$$\langle W_{(a_1 a_2 \dots a_s | b_1 b_2 \dots b_s)} \rangle^{\text{GC}} = \det_{p,q} [z \langle \mathbf{a}_p | (1+z\rho)^{-1} | \mathbf{b}_q \rangle]. \quad (53)$$

Restricted to hook representation cases, one has

$$\langle W_{(a|b)} \rangle^{\text{GC}} = z \langle \mathbf{a} | (1+z\rho)^{-1} | \mathbf{b} \rangle. \quad (54)$$

So finally one gets

$$\langle W_{(a_1 a_2 \dots a_s | b_1 b_2 \dots b_s)} \rangle^{\text{GC}} = \det_{p,q} \langle W_{(a_p | b_q)} \rangle^{\text{GC}}. \quad (55)$$

This shows that, for the non-hook representation cases, the expectation values of half BPS Wilson loops in the grand canonical ensemble can be written as the determinant of the expectation values of Wilson loops in the hook representations. In other words, they are Giambelli compatible.

3 Partition function

The computation of the partition function of the $\mathcal{N} = 4$ orbifold ABJM theory can be localized to the matrix model [17]¹⁾

$$\begin{aligned} Z_r(N) &= \frac{1}{N!^{2r}} \int \prod_{\ell=0}^{r-1} \prod_{i=1}^N \frac{d\mu_{\ell,i}}{2\pi} \frac{d\nu_{\ell,i}}{2\pi} \\ &\times \frac{\prod_{i < j} \left(2 \sinh \frac{\mu_{\ell,i} - \mu_{\ell,j}}{2} \right)^2 \left(2 \sinh \frac{\nu_{\ell,i} - \nu_{\ell,j}}{2} \right)^2}{\prod_{\ell} \prod_{i,j} \left(2 \cosh \frac{\mu_{\ell,i} - \nu_{\ell,j}}{2} \right) \left(2 \cosh \frac{\nu_{\ell+1,i} - \mu_{\ell,j}}{2} \right)} \\ &\times \exp \left[\frac{ik}{4\pi} \sum_{\ell,i} (\mu_{\ell,i}^2 - \nu_{\ell,i}^2) \right]. \quad (56) \end{aligned}$$

When $r = 1$ it is reduced to the ABJM matrix model (1). It can be written as canonical ensemble partition function of an N -particle Fermi gas with one-particle density matrix [23]

$$\hat{\rho}_r = e^{-r\hat{H}}, \quad (57)$$

with \hat{H} being the same as that of ABJM theory in (2).

We calculate the partition function in the Fermi gas approach. We firstly have

$$\begin{aligned} n_r(\mu) &= \int \frac{dqdp}{2\pi\hbar} \theta(\mu - r\hat{H})_W \\ &= \int \frac{dqdp}{2\pi\hbar} \theta\left(\frac{\mu}{r} - \hat{H}\right)_W = n\left(\frac{\mu}{r}\right), \end{aligned} \quad (58)$$

with $n(\mu)$ being the same function as (8). Then

$$N_r(\mu) = \pi \partial_\mu \csc(\pi \partial_\mu) n_r(\mu) = \frac{\sin(r\pi \partial_\mu)}{r \sin(\pi \partial_\mu)} N\left(\frac{\mu}{r}\right). \quad (59)$$

Then using (12) we can get

$$J_r(\mu) = \int_{-\infty}^{\mu} N_r(u) du = \frac{\sin(r\pi \partial_\mu)}{\sin(\pi \partial_\mu)} J\left(\frac{\mu}{r}\right). \quad (60)$$

Note that we have the following expansion

$$\begin{aligned} \frac{\sin(r\pi \partial_\mu)}{r \sin(\pi \partial_\mu)} &= 1 - \frac{\pi^2(r^2-1)}{6} \partial_\mu^2 \\ &+ \frac{\pi^4(3r^4-10r^2+7)}{360} \partial_\mu^4 + O(\partial_\mu^6). \end{aligned} \quad (61)$$

Formula (60) is a convenient way to get the grand potential $J_r(\mu)$ with Hamiltonian $r\hat{H}$ from the grand potential $J(\mu)$ with Hamiltonian \hat{H} , including both the perturbative and non-perturbative parts.

From the results of ABJM theory we have

$$\begin{aligned} n_r^{\text{pt}}(\mu) &= C_r \mu^2 + n_0^r, \quad N_r^{\text{pt}}(\mu) = C_r \mu^2 + B_r, \\ J_r^{\text{pt}}(\mu) &= \frac{C_r}{3} \mu^3 + B_r \mu + A_r, \end{aligned} \quad (62)$$

with

$$\begin{aligned} C_r &= \frac{C}{r^2} = \frac{2}{\pi^2 r^2 k}, \quad n_0^r = n_0 = -\frac{1}{3k} + \frac{k}{24}, \\ B_r &= B - \frac{\pi^2 C(r^2-1)}{3r^2} = -\frac{r^2-2}{3r^2 k} + \frac{k}{24}, \quad A_r = rA. \end{aligned} \quad (63)$$

Then we have the perturbative part of the partition function

$$\begin{aligned} Z_r^{\text{pt}}(N) &= \left(\frac{C}{r^2}\right)^{-1/3} e^{rA} \\ &\times \text{Ai} \left[\left(\frac{C}{r^2}\right)^{-1/3} \left(N - B + \frac{\pi^2 C(r^2-1)}{3r^2}\right) \right]. \end{aligned} \quad (64)$$

This is in accordance with the result in Ref. [30], and here we re-derive it in a different way.

The non-perturbative part of the grand potential for ABJM theory $J_{\text{np}}(\mu)$ is a summation of terms of the form [23]

$$(a\mu^2 + b\mu + c)e^{-d\mu}, \quad (65)$$

with a, b, c, d being constants and $d > 0$. Correspondingly in the grand potential of the $\mathcal{N} = 4$ SCSM theory $J_r^{\text{np}}(\mu)$ there is the term

$$\frac{\sin(r\pi \partial_\mu)}{\sin(\pi \partial_\mu)} \left(\frac{a}{r^2} \mu^2 + \frac{b}{r} \mu + c \right) e^{-\frac{d}{r}\mu} = (a_r \mu^2 + b_r \mu + c_r) e^{-\frac{d}{r}\mu}, \quad (66)$$

1) Supersymmetric localization in $d = 3$ $\mathcal{N} = 2$ SCSM theories was first studied in Refs. [18, 19].

with

$$\begin{aligned} a_r &= \frac{a}{r^2} f_r \left(\frac{d}{r} \right), \quad b_r = \frac{b}{r} f_r \left(\frac{d}{r} \right) - \frac{2a}{r^2} f_r' \left(\frac{d}{r} \right), \\ c_r &= c f_r \left(\frac{d}{r} \right) - \frac{b}{r} f_r' \left(\frac{d}{r} \right) + \frac{a}{r^2} f_r'' \left(\frac{d}{r} \right). \end{aligned} \quad (67)$$

Here we have defined the function

$$f_r(x) = \frac{\sin(r\pi x)}{\sin(\pi x)}. \quad (68)$$

Note that when $x=l$ is an integer we have

$$\begin{aligned} f_r(l) &= r(-)^{(r-1)l}, \quad f_r'(l) = 0, \\ f_r''(l) &= -\frac{r\pi^2}{3}(-)^{(r-1)l}(r^2-1). \end{aligned} \quad (69)$$

4 Circular BPS Wilson loops

In this section we review the circular 1/4 and 1/2 BPS Wilson loops for the $\mathcal{N}=4$ orbifold ABJM theory in Euclidean space [31, 32]. This theory is an SCSM theory with gauge groups $U(N)^{2r}$ and levels $(k, -k, \dots, k, -k)$.¹⁾ In $d=3$ Euclidean space we use the convention of spinors in Ref. [39], and especially we have the coordinates $x^\mu = (x^1, x^2, x^3)$ and the gamma matrices

$$\gamma_\alpha^{\mu\beta} = (-\sigma^2, \sigma^1, \sigma^3), \quad (70)$$

with $\sigma^{1,2,3}$ being the Pauli matrices. The circle is parameterized as $x^\mu = (\cos\tau, \sin\tau, 0)$.

Using every gauge field $A_\mu^{(2\ell+1)}$ with $\ell=0, 1, \dots, r-1$ and matter that couples to it, one can define the 1/4 BPS Wilson loop

$$\begin{aligned} W_{1/4}^{(2\ell+1)} &= \text{Tr} \mathcal{P} \exp \left(-i \oint d\tau \mathcal{A}^{(2\ell+1)}(\tau) \right), \\ \mathcal{A}^{(2\ell+1)} &= A_\mu^{(2\ell+1)} \dot{x}^\mu + \frac{2\pi}{k} \left(M_j^i \phi_i^{(2\ell+1)} \bar{\phi}_{(2\ell+1)}^j \right. \\ &\quad \left. + M_j^i \phi_i^{(2\ell)} \bar{\phi}_{(2\ell)}^j \right) |\dot{x}|, \\ M_j^i &= M_{\hat{j}}^{\hat{i}} = \text{diag}(i, -i). \end{aligned} \quad (71)$$

The conserved supersymmetries can be denoted as

$$\begin{aligned} \vartheta^{1\hat{1}} &= i\gamma_3 \theta^{1\hat{1}}, \quad \vartheta^{2\hat{2}} = -i\gamma_3 \theta^{2\hat{2}}, \\ \vartheta^{1\hat{2}} &= \theta^{2\hat{1}} = \vartheta^{1\hat{2}} = \vartheta^{2\hat{1}} = 0, \end{aligned} \quad (72)$$

where the spinors $\theta^{i\hat{i}}$ and $\vartheta^{i\hat{i}}$ with $i=1, 2$ and $\hat{i}=\hat{1}, \hat{2}$ denote the parameters of Poincaré and conformal supersymmetries, respectively.

Also using every gauge field $\hat{A}_\mu^{(2\ell)}$ with $\ell=0, 1, \dots, r-1$ and matter that couples to it, one can define the 1/4 BPS

Wilson loop

$$\begin{aligned} \hat{W}_{1/4}^{(2\ell)} &= \text{Tr} \mathcal{P} \exp \left(-i \oint d\tau \hat{\mathcal{A}}^{(2\ell)}(\tau) \right), \\ \hat{\mathcal{A}}^{(2\ell)} &= \hat{A}_\mu^{(2\ell)} \dot{x}^\mu + \frac{2\pi}{k} \left(N_i^j \bar{\phi}_{(2\ell-1)}^i \phi_j^{(2\ell-1)} \right. \\ &\quad \left. + N_i^j \bar{\phi}_{(2\ell)}^i \phi_j^{(2\ell)} \right) |\dot{x}|, \\ N_i^j &= N_{\hat{i}}^{\hat{j}} = \text{diag}(i, -i). \end{aligned} \quad (73)$$

This kind of 1/4 BPS Wilson loop preserves the same supersymmetries as the previous one (72). The 1/4 BPS Wilson loops (71) and (73) can be combined to give a 1/4 BPS Wilson loop

$$\begin{aligned} W_{1/4}^{(\ell)} &= \text{Tr} \mathcal{P} \exp \left(-i \oint d\tau L_{1/4}^{(\ell)}(\tau) \right), \\ L_{1/4}^{(\ell)} &= \begin{pmatrix} \mathcal{A}^{(2\ell+1)} & \\ & \hat{\mathcal{A}}^{(2\ell)} \end{pmatrix}. \end{aligned} \quad (74)$$

Using two adjacent gauge groups $\hat{A}_\mu^{(2\ell)}$, $A_\mu^{(2\ell+1)}$ in the quiver diagram and matter that couples to them, one can define the 1/2 BPS Wilson loop

$$\begin{aligned} W_{1/2}^{(\ell)} &= \text{Tr} \mathcal{P} \exp \left(-i \oint d\tau L_{1/2}^{(\ell)}(\tau) \right), \\ L_{1/2}^{(\ell)} &= \begin{pmatrix} \mathcal{A}^{(2\ell+1)} & \bar{f}_1^{(2\ell)} \\ f_2^{(2\ell)} & \hat{\mathcal{A}}^{(2\ell)} \end{pmatrix}, \\ \mathcal{A}^{(2\ell+1)} &= A_\mu^{(2\ell+1)} \dot{x}^\mu + \frac{2\pi}{k} \left(M_j^i \phi_i^{(2\ell+1)} \bar{\phi}_{(2\ell+1)}^j \right. \\ &\quad \left. + M_j^i \phi_i^{(2\ell)} \bar{\phi}_{(2\ell)}^j \right) |\dot{x}|, \\ \hat{\mathcal{A}}^{(2\ell)} &= \hat{A}_\mu^{(2\ell)} \dot{x}^\mu + \frac{2\pi}{k} \left(N_i^j \bar{\phi}_{(2\ell-1)}^i \phi_j^{(2\ell-1)} \right. \\ &\quad \left. + N_i^j \bar{\phi}_{(2\ell)}^i \phi_j^{(2\ell)} \right) |\dot{x}|, \\ \bar{f}_1^{(2\ell)} &= \sqrt{\frac{2\pi}{k}} \bar{\eta}^{(2\ell)} \psi_{(2\ell)}^1 |\dot{x}|, \quad f_2^{(2\ell)} = \sqrt{\frac{2\pi}{k}} \bar{\psi}_1^{(2\ell)} \eta_{(2\ell)} |\dot{x}|, \\ M_j^i &= N_i^j = \text{diag}(i, -i), \quad M_{\hat{j}}^{\hat{i}} = N_{\hat{i}}^{\hat{j}} = \text{diag}(-i, -i), \\ \bar{\eta}^{(2\ell)\alpha} &= \bar{\beta}(e^{i\tau/2}, e^{-i\tau/2}), \quad \eta_{(2\ell)\alpha} = (e^{-i\tau/2}, e^{i\tau/2})\beta, \quad \bar{\beta}\beta = i. \end{aligned} \quad (75)$$

Note that $\bar{\beta}$ and β are Grassmann even constants. The conserved supersymmetries are

$$\vartheta^{1\hat{i}} = i\gamma_3 \theta^{1\hat{i}}, \quad \vartheta^{2\hat{i}} = -i\gamma_3 \theta^{2\hat{i}}, \quad \hat{i} = \hat{1}, \hat{2}. \quad (76)$$

In the terminology of [32], the above 1/2 BPS Wilson loop is called the ψ_1 -loop, since it is coupled to fields $\psi_{(2\ell)}^1$ and $\bar{\psi}_1^{(2\ell)}$.

Similarly there is a 1/2 BPS ψ_2 -loop that is coupled to fields $\psi_{(2\ell)}^2$ and $\bar{\psi}_2^{(2\ell)}$ [32]. In the conventions of [31]

1) Without loss of generality, we assume k to be positive.

such a 1/2 BPS Wilson loop can be constructed as

$$\begin{aligned} \tilde{W}_{1/2}^{(\ell)} &= \text{Tr} \mathcal{P} \exp \left(-i \oint d\tau \tilde{L}_{1/2}^{(\ell)}(\tau) \right), \\ \tilde{L}_{1/2}^{(\ell)} &= \begin{pmatrix} \mathcal{A}^{(2\ell+1)} & \bar{f}_1^{(2\ell)} \\ f_2^{(2\ell)} & \hat{\mathcal{A}}^{(2\ell)} \end{pmatrix}, \\ \mathcal{A}^{(2\ell+1)} &= A_\mu^{(2\ell+1)} \dot{x}^\mu + \frac{2\pi}{k} \left(M_{ij}^i \phi_i^{(2\ell+1)} \bar{\phi}_{(2\ell+1)}^{\bar{j}} \right. \\ &\quad \left. + M_{ij}^i \phi_i^{(2\ell)} \bar{\phi}_{(2\ell)}^{\bar{j}} \right) |\dot{x}|, \\ \hat{\mathcal{A}}^{(2\ell)} &= \hat{A}_\mu^{(2\ell)} \dot{x}^\mu + \frac{2\pi}{k} \left(N_i^j \bar{\phi}_{(2\ell-1)}^{\bar{i}} \phi_j^{(2\ell-1)} \right. \\ &\quad \left. + N_i^j \bar{\phi}_{(2\ell)}^{\bar{i}} \phi_j^{(2\ell)} \right) |\dot{x}|, \\ \bar{f}_1^{(2\ell)} &= \sqrt{\frac{2\pi}{k}} \bar{\eta}^{(2\ell)} \psi_{(2\ell)}^2 |\dot{x}|, \quad f_2^{(2\ell)} = \sqrt{\frac{2\pi}{k}} \bar{\psi}_2^{(2\ell)} \eta_{(2\ell)} |\dot{x}|, \\ M_j^i &= N_i^j = \text{diag}(i, -i), \quad M_j^i = N_i^j = \text{diag}(i, i), \\ \bar{\eta}^{(2\ell)\alpha} &= \bar{\beta}(e^{i\tau/2}, -e^{-i\tau/2}), \quad \eta_{(2\ell)\alpha} = (e^{-i\tau/2}, -e^{i\tau/2})\beta, \\ \bar{\beta}\beta &= i. \end{aligned} \tag{77}$$

The ψ_2 -loop preserves the same supersymmetries as the ψ_1 -loop (76).

It has been checked that the difference of 1/4 and 1/2 BPS Wilson loops is Q -exact with Q being some supercharge preserved by both the 1/4 and 1/2 BPS Wilson loops [31, 32]. This applies to both the ψ_1 -loop (75) and ψ_2 -loop (77), and explicitly one has

$$W_{1/2}^{(\ell)} - W_{1/4}^{(\ell)} = QV^{(\ell)}, \quad \tilde{W}_{1/2}^{(\ell)} - W_{1/4}^{(\ell)} = Q\tilde{V}^{(\ell)}, \tag{78}$$

with $V^{(\ell)}$ and $\tilde{V}^{(\ell)}$ being some operators. It was conjectured that the $2r$ Wilson loops $W_{1/2}^{(\ell)}$ and $\tilde{W}_{1/2}^{(\ell)}$ with $\ell = 0, 1, \dots, r-1$ are 1/2 BPS classically, and only a special linear combination of them is 1/2 BPS quantum mechanically [32]. If it is the case, we may denote such a true 1/2 BPS Wilson loop as

$$W_{1/2}^{\text{qm}} = \sum_{\ell=0}^{r-1} \left(c_\ell W_{1/2}^{(\ell)} + \tilde{c}_\ell \tilde{W}_{1/2}^{(\ell)} \right), \tag{79}$$

with c_ℓ and \tilde{c}_ℓ being some to-be-determined constants. Here superscript qm means that the Wilson loop is 1/2 BPS quantum mechanically. We do not know if Wilson loops (71), (73) and (74) are still BPS quantum mechanically, but we expect that at least there is the 1/4 BPS Wilson loop

$$W_{1/4}^{\text{qm}} = \sum_{\ell=0}^{r-1} (c_\ell + \tilde{c}_\ell) W_{1/4}^{(\ell)}. \tag{80}$$

In this case, equations (78) would also be spoiled. We expect that

$$W_{1/2}^{\text{qm}} - W_{1/4}^{\text{qm}} = QV, \quad \text{with} \quad V = \sum_{\ell=0}^{r-1} (c_\ell V^{(\ell)} + \tilde{c}_\ell \tilde{V}^{(\ell)}). \tag{81}$$

Now we have three possibilities. The first is that Wilson loops (75) and (77) are 1/2 BPS quantum mechanically, and equations (78) also hold quantum mechanically. In this case the Wilson loops (75) and (77) have the same vacuum expectation values. The second possibility is that only Wilson loop (79) is 1/2 BPS quantum mechanically, and (81) holds quantum mechanically. The third possibility is that Wilson loop (79) is 1/2 BPS quantum mechanically, but (81) does not hold.

5 Wilson loops in saddle point approach

In this section, we compute the vacuum expectation values of Wilson loops with fixed winding number based on the saddle point approach.

If equation (78) holds quantum mechanically, we have the relations between vacuum expectation values of Wilson loops and expectation values in the matrix model (56)

$$\begin{aligned} \langle W_{1/4}^{(2\ell+1),n} \rangle &= \left\langle \sum_i e^{n\mu_{\ell,i}} \right\rangle, \quad \langle \hat{W}_{1/4}^{(2\ell),n} \rangle = \left\langle \sum_i e^{n\nu_{\ell,i}} \right\rangle, \\ \langle W_{1/2}^{(\ell),n} \rangle &= \langle \tilde{W}_{1/2}^{(\ell),n} \rangle = \left\langle \sum_i [e^{n\mu_{\ell,i}} - (-)^n e^{n\nu_{\ell,i}}] \right\rangle, \end{aligned} \tag{82}$$

with n being the winding number. From Z_r symmetry of the matrix model (56), we have

$$\begin{aligned} \langle W_{1/4}^{(2\ell+1),n} \rangle &= \left\langle \sum_i e^{n\mu_{0,i}} \right\rangle, \quad \langle \hat{W}_{1/4}^{(2\ell),n} \rangle = \left\langle \sum_i e^{n\nu_{0,i}} \right\rangle, \\ \langle W_{1/2}^{(\ell),n} \rangle &= \langle \tilde{W}_{1/2}^{(\ell),n} \rangle = \left\langle \sum_i [e^{n\mu_{0,i}} - (-)^n e^{n\nu_{0,i}}] \right\rangle. \end{aligned} \tag{83}$$

If quantum mechanically we have (81), we get

$$\langle W_{1/2}^{\text{qm},n} \rangle = c \left\langle \sum_i [e^{n\mu_{0,i}} - (-)^n e^{n\nu_{0,i}}] \right\rangle, \tag{84}$$

with

$$c = \sum_{\ell=0}^{r-1} (c_\ell + \tilde{c}_\ell). \tag{85}$$

It is possible that (81) is not true quantum mechanically. But it is still an interesting problem in its own right to calculate the expectation values in the matrix model

$$\left\langle \sum_i e^{n\mu_{0,i}} \right\rangle, \quad \left\langle \sum_i [e^{n\mu_{0,i}} - (-)^n e^{n\nu_{0,i}}] \right\rangle. \tag{86}$$

We calculate the Wilson loop expectation values in the saddle point approach. For the matrix model (56), at the saddle point we have [22]

$$\mu_\ell(x) = \mu(x), \quad \nu_\ell(x) = \nu(x), \tag{87}$$

with $\mu(x)$ and $\nu(x)$ being the same as (22). Then for the 1/4 and 1/2 BPS Wilson loops there are leading contributions of the vacuum expectation values which are the same as in the ABJM case

$$\langle W_{1/4}^{(2\ell+1),n} \rangle \approx \frac{i^n k}{2n\pi} \sqrt{\frac{\lambda}{2}} e^{n\pi\sqrt{2\lambda}}, \quad (88)$$

$$\langle W_{1/2}^{(\ell),n} \rangle = \langle \tilde{W}_{1/2}^{(\ell),n} \rangle \approx \frac{i^{n-1} k}{4n\pi} e^{n\pi\sqrt{2\lambda}}.$$

Note that there is no ℓ or r dependence in this result. If the Wilson loops (75) and (77) are not 1/2 BPS quantum mechanically, we cannot use the matrix model to calculate their vacuum expectation values [32]. If (81) holds quantum mechanically we can repeat the above process for the true 1/2 BPS Wilson loops (79), and we have

$$\langle W_{1/2}^{\text{qm},n} \rangle \approx \frac{i^{n-1} ck}{4n\pi} e^{n\pi\sqrt{2\lambda}}, \quad (89)$$

with constant c being (85). The Wilson loops in the fundamental representation are those with winding number $n = 1$. In Refs. [31, 32] it was shown that a suitably positioned M2-brane in M-theory in $\text{AdS}_4 \times \text{S}^7 / (\mathbb{Z}_r \times \mathbb{Z}_{rk})$ spacetime can be 1/2 BPS, and for the regularized on-shell action of the M2-brane in Euclidean space one has

$$e^{-S_{M2}} \sim e^{\pi\sqrt{2\lambda}}. \quad (90)$$

We find matches of the matrix model and gravity results.

If both (78) and (81) are spoiled by quantum corrections, the matrix model calculations here would have nothing to do with vacuum expectations values of Wilson loops.

6 Wilson loops in Fermi gas approach

In this section we use the Fermi gas approach, and study vacuum expectation values of Wilson loops with fixed winding number and of 1/2 BPS Wilson loops in both hook and non-hook representations.

6.1 Wilson loops with fixed winding number

We calculate the Wilson loops expectation values in Fermi gas approach. We firstly calculate

$$m_r(\mu) = \int \frac{dqdp}{2\pi\hbar} \theta(\mu - r\hat{H})_w e^{\frac{n(q+p)}{k}}$$

$$= \int \frac{dqdp}{2\pi\hbar} \theta\left(\frac{\mu}{r} - \hat{H}\right)_w e^{\frac{n(q+p)}{k}} = m\left(\frac{\mu}{r}\right). \quad (91)$$

And then we can get

$$m_r^{\text{pt}}(\mu) = \left(\frac{D}{r}\mu + E\right) e^{\frac{2n\mu}{rk}},$$

$$M_r^{\text{pt}}(\mu) = \frac{2\pi n}{rk} \csc \frac{2\pi n}{rk}$$

$$\cdot \left[\left(\mu + \frac{rk}{2n} - \pi \cot \frac{2\pi n}{rk}\right) \frac{D}{r} + E \right] e^{\frac{2n\mu}{rk}}, \quad (92)$$

with D and E being the same as (30). Then the 1/4 BPS Wilson loop expectation value is

$$\langle W_{1/4}^{(2\ell+1),n} \rangle_{\text{pt}}$$

$$= - \left(\frac{2}{\pi^2 r^2 k} \right)^{-1/3}$$

$$\cdot F_r \frac{\text{Ai}' \left[\left(\frac{2}{\pi^2 r^2 k} \right)^{-1/3} \left(N - \frac{k}{24} + \frac{r^2 - 6nr - 2}{3r^2 k} \right) \right]}{\text{Ai} \left[\left(\frac{2}{\pi^2 r^2 k} \right)^{-1/3} \left(N - \frac{k}{24} + \frac{r^2 - 2}{3r^2 k} \right) \right]}$$

$$+ G_r \frac{\text{Ai} \left[\left(\frac{2}{\pi^2 r^2 k} \right)^{-1/3} \left(N - \frac{k}{24} + \frac{r^2 - 6nr - 2}{3r^2 k} \right) \right]}{\text{Ai} \left[\left(\frac{2}{\pi^2 r^2 k} \right)^{-1/3} \left(N - \frac{k}{24} + \frac{r^2 - 2}{3r^2 k} \right) \right]}, \quad (93)$$

where

$$F_r = \frac{2\pi n}{rk} \csc \frac{2\pi n}{rk} \frac{D}{r},$$

$$G_r = \frac{2\pi n}{rk} \csc \frac{2\pi n}{rk} \left[\left(\frac{rk}{2n} - \pi \cot \frac{2\pi n}{rk} \right) \frac{D}{r} + E \right]. \quad (94)$$

The 1/2 BPS Wilson loops expectation values are

$$\langle W_{1/2}^{(\ell),n} \rangle_{\text{pt}}$$

$$= \langle \tilde{W}_{1/2}^{(\ell),n} \rangle_{\text{pt}} = \frac{i^{n-1}}{2r} \csc \frac{2\pi n}{rk}$$

$$\cdot \frac{\text{Ai} \left[\left(\frac{2}{\pi^2 r^2 k} \right)^{-1/3} \left(N - \frac{k}{24} + \frac{r^2 - 6nr - 2}{3r^2 k} \right) \right]}{\text{Ai} \left[\left(\frac{2}{\pi^2 r^2 k} \right)^{-1/3} \left(N - \frac{k}{24} + \frac{r^2 - 2}{3r^2 k} \right) \right]}. \quad (95)$$

If the Wilson loops (75) and (77) are not 1/2 BPS but (81) holds, we can still get vacuum expectation value of the true 1/2 BPS Wilson loop (79) as

$$\langle W_{1/2}^{\text{qm},n} \rangle_{\text{pt}}$$

$$= \frac{i^{n-1} c}{2r} \csc \frac{2\pi n}{rk}$$

$$\times \frac{\text{Ai} \left[\left(\frac{2}{\pi^2 r^2 k} \right)^{-1/3} \left(N - \frac{k}{24} + \frac{r^2 - 6nr - 2}{3r^2 k} \right) \right]}{\text{Ai} \left[\left(\frac{2}{\pi^2 r^2 k} \right)^{-1/3} \left(N - \frac{k}{24} + \frac{r^2 - 2}{3r^2 k} \right) \right]}, \quad (96)$$

with c being (85). If (81) is not true, we only have some matrix model results.

We expand the above results in the limit $N \gg k \gg 1$ with r being fixed, and now for the 't Hooft coupling $\lambda = N/k$ there is

$$\lambda \gg 1. \tag{97}$$

We make expansion of large λ and large k . For the 1/4 BPS Wilson loop we have

$$\begin{aligned} & \langle W_{1/4}^{(2\ell+1),n} \rangle_{\text{pt}} \\ &= \frac{i^n k}{2n\pi} \sqrt{\frac{\lambda}{2}} e^{n\pi\sqrt{2\lambda}} \left[\left(1 + \frac{2n^2\pi^2}{3r^2k^2} + O\left(\frac{1}{k^4}\right) \right) \right. \\ & \quad - \left(\frac{12\pi i + n\pi^2 + 24H_n}{2\pi} \right. \\ & \quad + \frac{n\pi(12n\pi i + 24 + n^2\pi^2 - 12r^2 + 36rn - 48r + 24nH_n)}{3r^2k^2} \\ & \quad \left. \left. + O\left(\frac{1}{k^4}\right) \right) \frac{1}{12\sqrt{2\lambda}} + O\left(\frac{1}{\lambda}\right) \right]. \end{aligned} \tag{98}$$

For the 1/2 BPS Wilson loops we have

$$\begin{aligned} \langle W_{1/2}^{(\ell),n} \rangle_{\text{pt}} &= \langle \tilde{W}_{1/2}^{(\ell),n} \rangle_{\text{pt}} \\ &= \frac{i^{n-1}k}{4n\pi} e^{n\pi\sqrt{2\lambda}} \left[\left(1 + \frac{2n^2\pi^2}{3r^2k^2} + O\left(\frac{1}{k^4}\right) \right) \right. \\ & \quad - \left(\frac{1}{2} + \frac{n^2\pi^2 + 36rn - 12r^2 + 24}{3r^2k^2} + O\left(\frac{1}{k^4}\right) \right) \\ & \quad \left. \cdot \frac{n\pi}{12\sqrt{2\lambda}} + O\left(\frac{1}{\lambda}\right) \right]. \end{aligned} \tag{99}$$

These are in accord with the results in saddle point approach (88). Note that for the leading contribution of large k , i.e. the genus zero part, there is no r dependence.

6.2 1/2 BPS Wilson loops in hook representations

Now we turn to half BPS Wilson loops¹⁾ in hook representations based on Ref. [25], where the density matrix for Fermi gas dual to ABJM theory was obtained as (35). For the $\mathcal{N} = 4$ orbifold ABJM theory, we have

$$\hat{\rho}_r = \hat{\rho}^r. \tag{100}$$

Similar to the calculation in Ref. [25], we obtain the following result in $\mathcal{N} = 4$ orbifold ABJM theory,

$$\Xi_r(z) \left\langle \prod_i \frac{f(e^{\mu_i})}{f(-e^{\nu_i})} \right\rangle^{\text{GC}} = \text{Det}(1 + z\hat{\rho}_{f,r}), \tag{101}$$

1) We have the ψ_1 -loops $W_{1/2}^{(\ell)}$ (75) and ψ_2 -loops $\tilde{W}_{1/2}^{(\ell)}$ (77) with $\ell = 0, 1, \dots, r-1$. If all of them are half BPS quantum mechanically, we can calculate their vacuum expectation values in the matrix model as shown in this subsection. Due to the Z_r symmetry of the theory, the results are independent of ℓ . From now on we can omit the index ℓ and subscript 1/2. We add subscript r to some quantities of the $\mathcal{N} = 4$ orbifold ABJM theory to distinguish them from their counterparts in ABJM theory. Also the results are the same for the ψ_1 -loops and ψ_2 -loops, and so we will not write the same results twice. If only a special combination of the ψ_1 -loops and ψ_2 -loops (79) is half BPS quantum mechanically and (81) holds, the following calculations still apply provided that a constant c (85) is added to the result. In the worst condition (81) does not hold quantum mechanically, and the calculations here are just matrix model results.

where

$$\hat{\rho}_{f,r} = \hat{\rho}_f \hat{\rho}^{r-1}, \tag{102}$$

with $\hat{\rho}_f$ being the same as (37).

The generating function for the half BPS Wilson loop in hook representations ($a|b$) was given in Ref. [37]

$$\begin{aligned} & 1 + (s+t) \sum_{a,b=0}^{\infty} s^a t^b W_{r,(a|b)} \\ &= \text{Sdet} \left(\frac{1+tU}{1-sU} \right) = \prod_{j=1}^N \frac{(1+te^{\mu_j})(1+se^{\nu_j})}{(1-se^{\mu_j})(1-te^{\nu_j})}. \end{aligned} \tag{103}$$

Therefore, the grand canonical ensemble expectation value of a circular 1/2 BPS Wilson loop $W_{r,(a|b)}$ in the $\mathcal{N} = 4$ orbifold ABJM theory in Euclidean space becomes

$$\begin{aligned} & \left\langle 1 + (s+t) \sum_{a,b=0}^{\infty} s^a t^b W_{r,(a|b)} \right\rangle^{\text{GC}} = \frac{\det(1 + z\hat{\rho}_f \hat{\rho}^{r-1})}{\det(1 + z\hat{\rho}^r)} \\ &= \det \left(1 + (s+t) \sum_{a,b=0}^{\infty} s^a t^b \frac{z\hat{\rho}^{r-1}}{1+z\hat{\rho}^r} |b\rangle \langle a| \right) \\ &= 1 + (s+t) \sum_{a,b=0}^{\infty} s^a t^b \langle a| \frac{z\hat{\rho}^{r-1}}{1+z\hat{\rho}^r} |b\rangle, \end{aligned} \tag{104}$$

with the states $|a\rangle$ and $\langle b|$ being defined the same as (39). We get the relation

$$\begin{aligned} \langle W_{r,(a|b)} \rangle^{\text{GC}} &= \langle a| \frac{z\hat{\rho}^{r-1}}{1+z\hat{\rho}^r} |b\rangle \\ &= \text{Tr} \left(\frac{z\hat{\rho}^{r-1}}{1+z\hat{\rho}^r} |b\rangle \langle a| \right) = \text{Tr} \left(\frac{1}{e^{r\hat{H}-\mu} + 1} e^{\hat{H}} |b\rangle \langle a| \right). \end{aligned} \tag{105}$$

Using Sommerfeld expansion we get

$$\langle W_{r,(a|b)} \rangle^{\text{GC}} = \pi \partial_{\mu} \csc(\pi \partial_{\mu}) m_r(\mu), \tag{106}$$

where

$$\begin{aligned} m_r(\mu) &= \text{Tr}(\theta(\mu - r\hat{H}) e^{\hat{H}} |b\rangle \langle a|) \\ &= \text{Tr}(\theta(\mu/r - \hat{H}) e^{\hat{H}} |b\rangle \langle a|) = m(\mu/r). \end{aligned} \tag{107}$$

Note that for a circular half BPS Wilson loop in ABJM theory $W_{(a|b)}$ there is

$$\langle W_{(a|b)} \rangle^{\text{GC}} = \pi \partial_{\mu} \csc(\pi \partial_{\mu}) m(\mu), \tag{108}$$

and then we have

$$\langle W_{r,(a|b)} \rangle^{\text{GC}}(\mu) = \frac{\sin(r\pi\partial_\mu)}{r\sin(\pi\partial_\mu)} \langle W_{(a|b)} \rangle^{\text{GC}}(\mu/r). \quad (109)$$

Then we can use (43) and get

$$\langle W_{r,(a|b)} \rangle_{\text{pt}}^{\text{GC}} = \frac{q^{\frac{1}{4}a(a+1) - \frac{1}{4}b(b+1)} i^{n-1}}{[a]![b]!} \frac{2\pi n}{2r} \csc \frac{2\pi n}{rk} e^{\frac{2n\mu}{rk}}, \quad (110)$$

where $n = a + b + 1$ is the number of boxes of Young diagram $(a|b)$. The 1/2 BPS Wilson loops expectation values in the canonical ensemble are

$$\langle W_{r,(a|b)} \rangle_{\text{pt}} = \frac{q^{\frac{1}{4}a(a+1) - \frac{1}{4}b(b+1)} i^{n-1}}{[a]![b]!} \csc \frac{2\pi n}{rk} \quad (111)$$

$$\times \frac{\text{Ai} \left[\left(\frac{2}{\pi^2 r^2 k} \right)^{-1/3} \left(N - \frac{k}{24} + \frac{r^2 - 6nr - 2}{3r^2 k} \right) \right]}{\text{Ai} \left[\left(\frac{2}{\pi^2 r^2 k} \right)^{-1/3} \left(N - \frac{k}{24} + \frac{r^2 - 2}{3r^2 k} \right) \right]}.$$

In the large N limit, the expectation values scale as

$$\langle W_{r,(a|b)} \rangle_{\text{pt}} \sim e^{n\pi\sqrt{2\lambda}}. \quad (112)$$

6.3 1/2 BPS Wilson loops in non-hook representations

Let us consider the half BPS Wilson loops in general representations $R = (a_1 \cdots a_s | b_1 \cdots b_s)$. The Giambelli formula states that

$$W_{r,(a_1 a_2 \cdots a_s | b_1 b_2 \cdots b_s)}(e^{\mu_i}, e^{\nu_j}) = \det_{p,q} W_{r,(a_p | b_q)}(e^{\mu_i}, e^{\nu_j}). \quad (113)$$

As in Ref. [25], we consider the following generating function

$$W_r(N) = \langle \det_{p,q}(\delta_{pq} + tW_{r,(a_p | b_q)}(e^{\mu_i}, e^{\nu_j})) \rangle. \quad (114)$$

Similar to computations in Ref. [25], with the definitions (47) and (49) we can get

$$Z_r(N) = \frac{1}{N!} \int \prod_i [d\nu_i] \det_{ij} \rho^r(\nu_i, \nu_j), \quad (115)$$

$$W_r(N) = \frac{1}{N!} \int \prod_i [d\nu_i] \det_{ij} (\rho_f(\mathbf{PQ})^{r-1})(\nu_i, \nu_j).$$

Note that the multiplication between boldface variables is understood as matrix multiplication with indices μ, ν and summation being replaced by integration with measures $[d\mu], [d\nu]$ in Eq. (47). We then have

$$\Xi_r(z) = \sum_{N=0}^{\infty} z^N Z_r(N) = \mathbf{Det}(1 + z\rho^r),$$

$$\sum_{N=0}^{\infty} z^N W_r(N) = \mathbf{Det}(1 + z\rho_f(\mathbf{PQ})^{r-1}). \quad (116)$$

Using the relation

$$\mathbf{Det}(1 + z\rho_f(\mathbf{PQ})^{r-1}) = \mathbf{Det}(1 + z\rho^r) \quad (117)$$

$$\times \det_{p,q} (\delta_{pq} + zt\langle \mathbf{a}_p | (1 + z\rho^r)^{-1} \rho^{r-1} | \mathbf{b}_q \rangle),$$

we can get

$$\langle \det_{p,q}(\delta_{pq} + tW_{r,(a_p | b_q)}) \rangle^{\text{GC}}$$

$$= \det_{p,q} (\delta_{pq} + zt\langle \mathbf{a}_p | (1 + z\rho^r)^{-1} \rho^{r-1} | \mathbf{b}_q \rangle). \quad (118)$$

The coefficient of t^s in both sides of the above equation gives

$$\langle W_{r,(a_1 a_2 \cdots a_s | b_1 b_2 \cdots b_s)} \rangle^{\text{GC}} = \det_{p,q} (z\langle \mathbf{a}_p | (1 + z\rho^r)^{-1} \rho^{r-1} | \mathbf{b}_q \rangle). \quad (119)$$

Restricted to hook representation cases, we have

$$\langle W_{r,(a|b)} \rangle^{\text{GC}} = z\langle \mathbf{a} | (1 + z\rho^r)^{-1} \rho^{r-1} | \mathbf{b} \rangle. \quad (120)$$

So finally we get

$$\langle W_{r,(a_1 a_2 \cdots a_s | b_1 b_2 \cdots b_s)} \rangle^{\text{GC}} = \det_{p,q} \langle W_{r,(a_p | b_q)} \rangle^{\text{GC}}. \quad (121)$$

This shows that the grand canonical ensemble expectation values of the circular 1/2 BPS Wilson loops are Giambelli compatible at least in the matrix model sense.

7 Discussion and conclusions

In this paper, we have calculated the vacuum expectation values of the circular BPS Wilson loops in arbitrary representations in the $\mathcal{N} = 4$ orbifold ABJM theory. We used both the saddle point approach in Ref. [22] and the Fermi gas approach in Refs. [23, 24], and the results agree with the available gravity results in Refs. [31, 32]. It will be quite interesting to study the string/M theory dual of the Wilson loops in the higher dimensional representations.

There are other $\mathcal{N} = 4$ SCSM theories [29, 40–43] and $\mathcal{N} = 3$ SCSM theories [44–47]. Recently there have been investigations of partition functions of these theories in the Fermi gas approach [48, 49]. There are also 1/2 BPS Wilson loops in general $\mathcal{N} = 4$ SCSM theories [32]. It would be interesting to investigate the vacuum expectation values of supersymmetric Wilson loops in these theories.

Expectation values of the 1/4 and 1/2 BPS Wilson loops of orbifold ABJM theory in weak coupling can be calculated directly in the matrix model, like the ABJM theory case in Ref. [17]. Also one can calculate the vacuum expectation values of Wilson loops perturbatively using Feynman rules in the orbifold ABJM theory in

weak coupling, like the ABJM theory case in Refs. [12–14, 50–52]. It would be nice to compare the results of the matrix model to the results of Feynman rules. In fact it was proposed in Refs. [32] that a perturbative calculation of expectation values of the 1/2 BPS Wilson loops using Feynman rules would be helpful in fixing the coefficients in the true 1/2 BPS Wilson loop (79).

Appendix A More general 1/2 BPS Wilson loops in orbifold ABJM theory?

For two adjacent gauge fields in the quiver diagram and matter fields that couple to them of the $\mathcal{N} = 4$ orbifold ABJM theory, one can define two kinds of 1/2 BPS Wilson loops, i.e. the ψ_1 -loop [31, 32] and the ψ_2 -loop [32]. In this appendix we will investigate if there is a more general 1/2 BPS Wilson loop that preserves the same supersymmetries as the ψ_1 -loop and ψ_2 -loop.

There is no spacelike BPS Wilson loop in Minkowski spacetime [39]. The BPS Wilson loops along straight lines in Euclidean space are just the timelike BPS Wilson loops of straight lines along Minkowski spacetime after Wick rotation. The circular BPS Wilson loops in Euclidean space can be obtained by the conformal transformation of the BPS Wilson loops along infinite straight lines. Also, for straight lines the cases of Poincaré supersymmetries and conformal supersymmetries are separated and very similar. So it is enough to just consider the Poincaré supersymmetries of the 1/2 BPS Wilson loops along timelike infinite straight lines in Minkowski spacetime.

We use the conventions in Refs. [31, 39]. Especially we choose the coordinates $x^\mu = (x^0, x^1, x^2)$, and we use the gamma matrices

$$\gamma^\mu{}_\alpha{}^\beta = (i\sigma^2, \sigma^1, \sigma^3), \quad (\text{A1})$$

with $\sigma^{1,2,3}$ being the Pauli matrices. For the infinite straight line $x^\mu = \tau \delta^\mu_0$, we want to get a 1/2 BPS Wilson loop that preserves the Poincaré supersymmetries

$$\begin{aligned} \gamma_0 \theta^{1\hat{i}} &= i\theta^{1\hat{i}}, & \gamma_0 \theta^{2\hat{i}} &= -i\theta^{2\hat{i}}, \\ \bar{\theta}_{1\hat{i}} \gamma_0 &= i\bar{\theta}_{1\hat{i}}, & \bar{\theta}_{2\hat{i}} \gamma_0 &= -i\bar{\theta}_{2\hat{i}}, \end{aligned} \quad (\text{A2})$$

with $\hat{i} = \hat{1}, \hat{2}$. We only use the gauge fields $A_\mu^{(2\ell+1)}$ and $\hat{A}_\mu^{(2\ell)}$ and matter fields that couple to them. A general Wilson loop would be of the form

$$\begin{aligned} W_{1/2}^{(\ell)} &= \mathcal{P} \exp \left(-i \int d\tau L_{1/2}^{(\ell)}(\tau) \right), \\ L_{1/2}^{(\ell)} &= \begin{pmatrix} \mathcal{A}^{(2\ell+1)} & \bar{f}_1^{(2\ell)} \\ f_2^{(2\ell)} & \hat{\mathcal{A}}^{(2\ell)} \end{pmatrix}, \\ \mathcal{A}^{(2\ell+1)} &= A_\mu^{(2\ell+1)} \dot{x}^\mu + \frac{2\pi}{k} \left(M_j^i \phi_i^{(2\ell+1)} \bar{\phi}_j^{(2\ell+1)} \right. \\ &\quad \left. + M_j^{\hat{i}} \phi_{\hat{i}}^{(2\ell)} \bar{\phi}_{(2\ell)}^{\hat{j}} \right) |\dot{x}|, \\ \hat{\mathcal{A}}^{(2\ell)} &= \hat{A}_\mu^{(2\ell)} \dot{x}^\mu + \frac{2\pi}{k} \left(N_i^j \bar{\phi}_{(2\ell-1)}^i \phi_j^{(2\ell-1)} \right. \end{aligned} \quad (\text{A3})$$

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$$\begin{aligned} &+ N_i^{\hat{j}} \bar{\phi}_{(2\ell)}^{\hat{i}} \phi_{\hat{j}}^{(2\ell)} \Big) |\dot{x}|, \\ \bar{f}_1^{(2\ell)} &= \sqrt{\frac{2\pi}{k}} \bar{\eta}_{\hat{i}}^{(2\ell)} \psi_{(2\ell)}^{\hat{i}} |\dot{x}|, \\ f_2^{(2\ell)} &= \sqrt{\frac{2\pi}{k}} \psi_{\hat{i}}^{(2\ell)} \eta_{(2\ell)}^{\hat{i}} |\dot{x}|, \end{aligned}$$

with $\bar{\eta}_{\hat{i}}^{(2\ell)}$ and $\eta_{(2\ell)}^{\hat{i}}$ being Grassmann even spinors. To make the loop BPS we must find $\bar{g}_1^{(2\ell)}$ and $g_2^{(2\ell)}$ that satisfy [53]

$$\begin{aligned} \delta \mathcal{A}^{(2\ell+1)} &= i(\bar{f}_1^{(2\ell)} g_2^{(2\ell)} - \bar{g}_1^{(2\ell)} f_2^{(2\ell)}), \\ \delta \hat{\mathcal{A}}^{(2\ell)} &= i(f_2^{(2\ell)} \bar{g}_1^{(2\ell)} - g_2^{(2\ell)} \bar{f}_1^{(2\ell)}), \\ \delta \bar{f}_1^{(2\ell)} &= \mathcal{D}_\tau \bar{g}_1^{(2\ell)} \equiv \partial_\tau \bar{g}_1^{(2\ell)} + i\mathcal{A}^{(2\ell+1)} \bar{g}_1^{(2\ell)} - i\bar{g}_1^{(2\ell)} \hat{\mathcal{A}}^{(2\ell)}, \\ \delta f_2^{(2\ell)} &= \mathcal{D}_\tau g_2^{(2\ell)} \equiv \partial_\tau g_2^{(2\ell)} + i\hat{\mathcal{A}}^{(2\ell)} g_2^{(2\ell)} - i g_2^{(2\ell)} \mathcal{A}^{(2\ell+1)}. \end{aligned} \quad (\text{A4})$$

Because of the form of $\bar{f}_1^{(2\ell)}$ and $f_2^{(2\ell)}$, terms with fields $\psi_{(2\ell+1)}^{\hat{i}}$ and $\bar{\psi}_{\hat{i}}^{(2\ell+1)}$ should cancel in the variation of $\mathcal{A}^{(2\ell+1)}$. Similarly, terms with $\psi_{(2\ell-1)}^{\hat{i}}$ and $\bar{\psi}_{\hat{i}}^{(2\ell-1)}$ should cancel in the variation of $\hat{\mathcal{A}}^{(2\ell)}$. This forces us to choose

$$M_j^i = N_i^{\hat{j}} = \text{diag}(-1, 1). \quad (\text{A5})$$

If we take the ansatz

$$\bar{\eta}_{\hat{i}}^{(2\ell)} = \bar{\eta}^{(2\ell)} \delta_{\hat{i}}^1, \quad \eta_{(2\ell)}^{\hat{i}} = \eta_{(2\ell)} \delta_{\hat{i}}^1, \quad (\text{A6})$$

we get the ψ_1 -loop with

$$\begin{aligned} M_j^{\hat{i}} &= N_{\hat{i}}^{\hat{j}} = \text{diag}(1, 1), \\ \bar{\eta}^{(2\ell)\alpha} &= \bar{\beta}(-i, 1), \quad \eta_{(2\ell)\alpha} = (i, 1)\beta, \quad \bar{\beta}\beta = -i. \end{aligned} \quad (\text{A7})$$

Or if we take the ansatz

$$\bar{\eta}_{\hat{i}}^{(2\ell)} = \bar{\eta}^{(2\ell)} \delta_{\hat{i}}^2, \quad \eta_{(2\ell)}^{\hat{i}} = \eta_{(2\ell)} \delta_{\hat{i}}^2, \quad (\text{A8})$$

we get the ψ_2 -loop with

$$\begin{aligned} M_j^{\hat{i}} &= N_{\hat{i}}^{\hat{j}} = \text{diag}(-1, -1), \\ \bar{\eta}^{(2\ell)\alpha} &= \bar{\beta}(i, 1), \quad \eta_{(2\ell)\alpha} = (-i, 1)\beta, \quad \bar{\beta}\beta = -i. \end{aligned} \quad (\text{A9})$$

We wonder if there is a more general 1/2 BPS Wilson loop that preserves the same supersymmetries (A2) as the ψ_1 -loop and ψ_2 -loop, at least classically. One of the consequences of (A4) is that

$$\bar{g}_1^{(2\ell)} = \bar{g}_1^{(2\ell)\hat{i}} \phi_{\hat{i}}^{(2\ell)}, \quad g_2^{(2\ell)} = g_{2\hat{i}}^{(2\ell)} \bar{\phi}_{(2\ell)}^{\hat{i}}, \quad (\text{A10})$$

with $\bar{g}_1^{(2\ell)\hat{i}}$ and $g_{2\hat{i}}^{(2\ell)}$ being Grassmann odd and having no color index or spinor index. From (A4) and the variation of $\mathcal{A}^{(2\ell+1)}$ we must have

$$\begin{aligned} \gamma_0\theta^{i\hat{i}} &= -iM_j^{\hat{i}}\theta^{i\hat{j}} - i\sqrt{\frac{k}{8\pi}}\eta_{(2\ell)}^{\hat{i}}\bar{g}_1^{(2\ell)\hat{i}}, \\ \bar{\theta}_{i\hat{i}}\gamma_0 &= -iM_j^{\hat{j}}\bar{\theta}_{i\hat{j}} + i\sqrt{\frac{k}{8\pi}}\bar{\eta}_i^{(2\ell)}g_{2\hat{i}}^{(2\ell)}. \end{aligned} \quad (\text{A11})$$

Then from (A2) we have

$$\begin{aligned} \theta^{1\hat{i}} &= -M_j^{\hat{i}}\theta^{1\hat{j}} - \sqrt{\frac{k}{8\pi}}\eta_{(2\ell)}^1\bar{g}_1^{(2\ell)\hat{i}}, \\ -\theta^{2\hat{i}} &= -M_j^{\hat{j}}\theta^{2\hat{j}} - \sqrt{\frac{k}{8\pi}}\eta_{(2\ell)}^2\bar{g}_1^{(2\ell)\hat{i}}, \\ \bar{\theta}_{1\hat{i}} &= -M_i^{\hat{j}}\bar{\theta}_{1\hat{j}} + \sqrt{\frac{k}{8\pi}}\bar{\eta}_1^{(2\ell)}g_{2\hat{i}}^{(2\ell)}, \\ -\bar{\theta}_{2\hat{i}} &= -M_i^{\hat{j}}\bar{\theta}_{2\hat{j}} + \sqrt{\frac{k}{8\pi}}\bar{\eta}_2^{(2\ell)}g_{2\hat{i}}^{(2\ell)}. \end{aligned} \quad (\text{A12})$$

Note that $\theta^{1\hat{i}}$ and $\theta^{2\hat{i}}$ are nonvanishing, general and linearly independent, and similarly $\bar{\theta}_{1\hat{i}}$ and $\bar{\theta}_{2\hat{i}}$ are nonvanishing, general and linearly independent. First of all, $\bar{g}_1^{(2\ell)\hat{i}}$ and $g_{2\hat{i}}^{(2\ell)}$ cannot be vanishing, otherwise there would be no solutions for the matrix $M_j^{\hat{i}}$. Then we must have $\eta_{(2\ell)}^1 = 0$ or $\eta_{(2\ell)}^2 = 0$, as well as $\bar{\eta}_1^{(2\ell)} = 0$ or $\bar{\eta}_2^{(2\ell)} = 0$. When $\eta_{(2\ell)}^1 = 0$, we have $M_j^{\hat{i}} = -\delta_j^{\hat{i}}$, and then there is $\bar{\eta}_1^{(2\ell)} = 0$. This gives the ψ_2 -loop. When $\eta_{(2\ell)}^2 = 0$, we have $M_j^{\hat{i}} = \delta_j^{\hat{i}}$, and then there is $\bar{\eta}_2^{(2\ell)} = 0$. This gives the ψ_1 -loop.

In summary we have no choices other than the ψ_1 -loop and ψ_2 -loop that satisfies the following conditions.

- 1) It is constructed by two adjacent gauge fields $A_\mu^{(2\ell+1)}$ and $\hat{A}_\mu^{(2\ell)}$ in quiver diagrams and fields that couple to them in the general form (A3).
- 2) It preserves the same supersymmetries as the ψ_1 -loop and ψ_2 -loop (A2), at least classically.

This result can be taken as a small step towards classification of BPS Wilson loops in $\mathcal{N} = 4$ SCSM theories.

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