Self duality solution with a Higgs field

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Abstract: The self-duality concept for the Higgs field is handled in the presence of contact geometry in 5 dimensions. A non-trivial $SO(3)$ Higgs field lives only on the fifth dimension of the contact manifold because of the contact structure, while the **SD** Yang-Mills field lives in the 4-dimensional hyperplane of the contact manifold. The Higgs and **SD** Yang-Mills fields do not interact with one another.

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1 Introduction

The self-duality (SD) solutions to the Yang-Mills equations in four dimensions and higher are related to the behavior of the Lie algebra-valued 1- and 2-forms under the Hodge duality, interpreted as the gauge potential and gauge field strength, respectively. These solutions are well defined for even-dimensional manifolds, and very well-known pioneering examples are given in Refs. [1–3]. However, defining the SD concept in odd dimensions is not a simple task.

A nice example in this context can be found in the Baraglia and Hekmati's paper on the moduli space for instantons on 5-dimensional contact manifolds [4]. Because the geometry of a contact manifold runs in odd dimensions in both real and complex cases, the contact structure in 5 dimensions yields a 4-dimensional hyperplane of the tangent bundle on the contact manifold, and the SD notion for 2-forms is defined on this hyperplane.

The SD concept in Yang-Mills theories is also considered as a vacuum solution, and the gauge potential solving the vacuum Yang-Mills equation includes the Higgs or monopole field embedded inside one of its components. Therefore, the SD notion in a 5-dimensional contact setting is unrelated to such solutions. However, the SD notion in this paper is evaluated together with the Higgs field, so that a non-trivial $SO(3)$ Higgs field lives only on the fifth dimension, owing to the contact structure of the manifold, while the SD Yang-Mills field lives on the 4-dimensional hyperplane. An interesting case of our ansatz is that in which the Higgs field and SD Yang-Mills potential do not interact.

Let A be a gauge potential of a gauge group G on a smooth 5-dimensional manifold $M=\mathbb{R}^5$ with local coordinates $\{x^\mu\}\in\mathbb{R}^5$. This gauge potential is considered as a Lie algebra-valued 1-form, $A=A_\mu \mathrm{d}x^\mu\in\Lambda^1(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of the gauge group G and $A_\mu : C^\infty(M)\to\mathfrak{g}$. The covariant derivative is $\nabla=d+[A,\cdot]$, so that the gauge field strength is given such that $F=\nabla A=dA+A\wedge A\in\Lambda^2(\mathfrak{g})$ as a Lie algebra-valued 2-form $F=F_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$. Furthermore, the Bianchi identity of the gauge field strength is $\nabla F=dF+A\wedge F-F\wedge A=0$. Therefore, the extremum of the Yang-Mills action integral $\int_M \mathfrak{tr}\|F\|^2$ gives the following vacuum Yang-Mills equation:

$$\nabla * F=0. \tag{1}$$

If the solution to eq. (1) in 4 dimensions is (anti-) self-dual ((A)SD), i.e.,

$$*F=\pm F, \tag{2}$$

then this solution is known as an (anti-) instanton [5–7].

Because the SD/ASD concept is also considered in dimensions higher than four [1–3, 8, 9], a generalized SD/ASD concept for 2-forms in 5 dimensions is given by

$$*F=\lambda\Sigma\wedge F,\ \ \Sigma\in\Lambda^1(M), \tag{3}$$

where $\lambda=+1$ for SD and $\lambda=-1$ for ASD, and Σ is an auxiliary form. Using the Bianchi identity, the Yang-Mills equation becomes

$$\nabla * F=\lambda \mathrm{d}\Sigma\wedge F=0. \tag{4}$$

Here, there are two points worth noting:

1) If Σ is in a closed form, then this equation reduces automatically to the vacuum Yang-Mills equation $\nabla * F=0$.

2) If Σ is in a non-closed form, then the behavior of the Yang-Mills equation is controlled by the eigenvalues λ.

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Therefore, for the ASD with $\lambda=-1$, Eq. (4) is satisfied in higher dimensions, as in Refs. [4], [10], [11], [12]. In addition, see the equation below (22). Conversely, for the SD case with $\lambda=+1$, Eq. (4) is not satisfied if Σ is non-closed. However, an SD solution in 6 dimensions for closed Σ is presented in [13].

However, the aim of this study is to find an answer to that question of how to obtain an SD concept if Σ is in a non-closed form. We will attempt to answer this question in the frame of a 5-dimensional contact manifold. Because Eq. (4) will give $\nabla * F=\lambda \mathrm{d}\Sigma\wedge F\neq$ for an SD gauge potential (again see Eq. (22)), we will add the Higgs field to the action integral to solve an SD equation in a 5-dimensional contact setting.

We consider the Higgs field as a Lie algebra-valued 0-form: $\phi: C^\infty(M)\to\mathfrak{g}$. Therefore, we define the following gauge invariant action integral, also called the Yang-Mills-Higgs (YMH) action integral:

$$S[\phi,A]=\int_M \mathfrak{tr}\{\chi\|F\|^2+\|\nabla\phi\|^2-\mathcal{V}(\phi)\}, \tag{5}$$

where $\|\alpha\|^2=\alpha\wedge*\alpha$ for $\alpha\in\Lambda^r(\mathbb{R}^5)$, $\mathfrak{tr}$ is the trace operator, χ is the coupling constant, and $\mathcal{V}(\phi)\in\Lambda^5(M)$ is the potential form for Higgs field ϕ. The term $\|\nabla\phi\|^2$ is interpreted as the kinetic energy of the Higgs field ϕ. Therefore, the action integral (5) gives the following field equations with respect to the variables A and ϕ, respectively:

$$\chi\nabla(*F)+[\phi,*\nabla\phi]\ =\ 0, \tag{6}$$

$$\nabla(*\nabla\phi)-\frac{1}{2}\frac{\delta\mathcal{V}(\phi)}{\delta\phi}\ =\ 0, \tag{7}$$

where $[,]$ is the Lie bracket.

We have mentioned above that Eq. (6) for the ASD, $*F=-\Sigma\wedge F$, automatically reduces to vacuum Yang-Mills equation, and so the anti-Yang-Mills instanton is obtained. Therefore, we take the SD concept $\lambda=+1$ together with a non-trivial Higgs field ϕ. Thus, we have the following identity:

$$\nabla*F=\mathrm{d}\Sigma\wedge F. \tag{8}$$

Comparing this identity with the field equation (6), we obtain

$$\chi\mathrm{d}\Sigma\wedge F+[\phi,*\nabla\phi]=0. \tag{9}$$

We will see that the non-trivial $SO(3)$ Higgs field on a contact 5-manifold satisfies this equation if $\chi=0$. This means that an SD gauge potential and Higgs field on a contact 5-manifold do not interact.

2 SD/ASD concept on contact 5-manifolds

We provide the contact manifold definition from Blair's famous book [14]. Let M be a 5-dimensional Riemannian manifold. Take a 1-form $\eta\in\Lambda^1(M)$ on this manifold and a vector field $\xi\in\Gamma(M)$. These will be called the contact 1-form and its characteristic vector field (also known as the Reeb vector field), respectively. A manifold $(M=\mathbb{R}^5,\eta,\xi)$ is called a contact manifold if it satisfies the following conditions:

$$\eta\wedge\mathrm{d}\eta\wedge\mathrm{d}\eta\neq0,\ \eta(\xi)=1, \tag{10}$$

In particular, $\eta\wedge\mathrm{d}\eta\wedge\mathrm{d}\eta$ is a volume element of the manifold M, and therefore contact manifolds are orientable.

Let $\mathcal{H}=\mathrm{Ker}(\eta)\subset TM$ be a hyperplane defined as a subbundle of a tangent bundle TM on the contact manifold (M,η,ξ), where $\mathrm{Ker}(\eta)$ denotes the kernel of the 1-form η. Therefore, the decomposition of this tangent bundle is written as

$$TM=\mathcal{H}\oplus\mathbb{R}\xi, \tag{11}$$

where $\mathcal{H}$ is also called the horizontal part of the tangent bundle, and $\mathbb{R}\xi$ is the complement. Because $\dim(TM)=5$, it follows that $\dim(\mathcal{H})=4$.

Now, we can choose the following contact 1-form η and its characteristic vector field ξ with respect to the standard Cartesian coordinates $(x^1,\ldots,x^5)$:

$$\eta=\frac{1}{2}(\mathrm{d}x^5-x^2\mathrm{d}x^1-x^4\mathrm{d}x^3),\ \xi=2\partial_5, \tag{12}$$

$$\eta\wedge\mathrm{d}\eta\wedge\mathrm{d}\eta=\frac{1}{4}\mathrm{d}x^{12345}. \tag{13}$$

Therefore, the metric that is compatible with the contact structure is given by

$$G_{\mu\nu}=\tfrac{1}{4}\begin{pmatrix}1+(x^2)^2 & 0 & x^2x^4 & 0 & -x^2\\ 0 & 1 & 0 & 0 & 0\\ x^2x^4 & 0 & 1+(x^4)^2 & 0 & -x^4\\ 0 & 0 & 0 & 1 & 0\\ -x^2 & 0 & -x^4 & 0 & 1\end{pmatrix}, \tag{14}$$

$\det(G_{\mu\nu})=1.$

Considering the decomposition in Eq. (11) given by $TM=\mathcal{H}\oplus\mathbb{R}\xi$, the characteristic vector field ξ of the contact structure η defines a 1-dimensional foliation on the manifold. Thus, one can consider the transverse geometry relating to this foliation. For details, see Ref. [4]. This foliation has codimension 4 on a contact 5-manifold, and the SD/ASD concept is constructed with respect to this transverse geometry.

First, we define a transverse Hodge duality notion on a contact manifold M. From the decomposition in Eq. (11), the bundle of k-forms spanned by the coframes on the hyperplane $\mathcal{H}$ is given by

$$\Lambda^k_{\mathcal{H}}(M)=\{\alpha\in\Lambda^k(M)\,|\,\imath_\xi(\alpha)=0\}, \tag{15}$$

where $\imath_\xi(\alpha)$ denotes the inner derivative of the form α with respect to the vector field ξ. Therefore, $\alpha\in\Lambda^k_{\mathcal{H}}(M)$

is called the transverse form with respect to the characteristic vector field ξ. If we consider the SD/ASD notion in Eq. (3), the transverse duality notion for $\alpha\in\Lambda^k_{\mathcal{H}}(M)$ with respect to the auxiliary form $\Sigma\in\Lambda^1(M)$ is presented by

$$*(\Sigma\wedge\alpha)=(-1)^k\imath_\xi(*\alpha). \tag{16}$$

Here, we must explain the meaning of $\imath_\xi(*\alpha)$. If we want to investigate the SD/ASD concept on contact manifolds, then we need a transverse direction with respect to the contact 1-form η. For example, if we take any 2-form $\alpha\in\Lambda^2(M)$ on the contact 5-manifold M with η and ξ, as in (12), then some components of $*\alpha$ are spanned by $\mathrm{d}x^{ij5}$, where $i<j=1,\ldots,4$. Thus, because $\xi=\partial_5$, the transverse duality $\imath_\xi(*\alpha)$ maps the 2-form α to $\Lambda^2_{\mathcal{H}}(M)$, and so when $\pm\alpha=\imath_\xi(*\alpha)$ we can say that α is an SD/ASD 2-form.

Of course, we can choose the auxiliary form Σ as the contact 1-form of a contact 5-manifold:

$$\Sigma=\eta. \tag{17}$$

Therefore, the SD concept in Eq. (3) for the 2-form F is rewritten on the contact 5-manifold Ref. [9] as follows:

$$*F=\lambda\eta\wedge F=\lambda*\imath_\xi(*F). \tag{18}$$

Eventually, the SD/ASD concept under a linear map $*_{\mathcal{H}}:\Lambda^2_{\mathcal{H}}(M)\to\Lambda^2_{\mathcal{H}}(M)$ can be given as

$$*_{\mathcal{H}}\omega=*(\eta\wedge\omega)=\imath_\xi(*\omega), \tag{19}$$

Therefore, if

$$\omega=\lambda*(\eta\wedge\omega)=\lambda\imath_\xi(*\omega), \tag{20}$$

then we say that ω is an ASD 2-form for $\lambda=-1$ and an SD 2-form for $\lambda=+1$. Then, an SD(ASD) 2-form on a contact 5-manifold is written with respect to the decomposition in Eq. (11) and the SD(ASD) concept given in Eq. (20) as follows:

$$\begin{aligned}\omega=&w_{34}(\lambda\mathrm{d}x^{12}+\mathrm{d}x^{34})+w_{24}(-\lambda\mathrm{d}x^{13}+\mathrm{d}x^{24})\\&+w_{23}(\lambda\mathrm{d}x^{14}+\mathrm{d}x^{23}).\end{aligned} \tag{21}$$

The exterior product of this SD/ASD form with the exterior derivative $\mathrm{d}\eta$ of the contact 1-form η gives the following expression:

$$\mathrm{d}\eta\wedge\omega=(\lambda+1)w_{34}\mathrm{d}x^{1234}. \tag{22}$$

Therefore, if $\lambda=-1$, then the 2-form ω is ASD. This appears as an important key point in defining an anti-instanton model for the vacuum Yang-Mills equation on a contact 5-dimensional manifold, because $\nabla*F=\mathrm{d}\eta\wedge F=0$. Of course, we do not deal with this equation in this paper, because our aim is only the SD concept with the Higgs field. Consequently, we have the following decompositions for the bundle of 2-forms:

$$\Lambda^2(M)=\Lambda^2_{\mathcal{H}}(M)\oplus(\eta\wedge\Lambda^1_{\mathcal{H}}(M)), \tag{23}$$

where

$$\Lambda^2_{\mathcal{H}}(M)=\Lambda^2_{\mathcal{H}}(M)^+\oplus\Lambda^2_{\mathcal{H}}(M)^-. \tag{24}$$

Thus, the bundle $\Lambda^2(M)$ and its subbundles are spanned by the following bases:

$$\left.\begin{aligned}\Lambda^1_{\mathcal{H}}(M)\ &=\{\mathrm{d}x^1,\mathrm{d}x^2,\mathrm{d}x^3,\mathrm{d}x^4\},\\ \Lambda^2(M)\ &=\{\mathrm{d}x^{12},\mathrm{d}x^{13},\mathrm{d}x^{14},\mathrm{d}x^{15},\mathrm{d}x^{23},\\ &\quad\ \mathrm{d}x^{24},\mathrm{d}x^{25},\mathrm{d}x^{34},\mathrm{d}x^{35},\mathrm{d}x^{45}\},\\ \Lambda^2_{\mathcal{H}}(M)\ &=\{\mathrm{d}x^{12},\mathrm{d}x^{13},\mathrm{d}x^{14},\mathrm{d}x^{23},\mathrm{d}x^{24},\mathrm{d}x^{34}\},\\ \Lambda^2_{\mathcal{H}}(M)^+\ &=\{(\mathrm{d}x^{12}+\mathrm{d}x^{34}),(-\mathrm{d}x^{13}+\mathrm{d}x^{24}),(\mathrm{d}x^{14}+\mathrm{d}x^{23})\},\\ \Lambda^2_{\mathcal{H}}(M)^-\ &=\{(-\mathrm{d}x^{12}+\mathrm{d}x^{34}),(\mathrm{d}x^{13}+\mathrm{d}x^{24}),(-\mathrm{d}x^{14}+\mathrm{d}x^{23})\},\\ \eta\wedge\Lambda^1_{\mathcal{H}}(M)\ &=\{\mathrm{d}x^{15},\mathrm{d}x^{25},\mathrm{d}x^{35},\mathrm{d}x^{45}\}.\end{aligned}\right\} \tag{25}$$

3 Self-duality equations

Because $\Lambda^2(\mathbb{R}^m)=\mathfrak{so}(m)$, the decompositions of some Lie algebras with respect to Eq. (23) are given by

$$\mathfrak{so}(5)=\mathfrak{so}(4)\oplus\mathfrak{m},\ \mathfrak{so}(4)=\mathfrak{so}(3)\oplus\mathfrak{so}(3)=\mathfrak{su}(2)\oplus\mathfrak{su}(2). \tag{26}$$

Thus, we have that

$$\Lambda^2_{\mathcal{H}}(M)^\pm=\mathfrak{so}(3)\cong\mathfrak{su}(2),\ \eta\wedge\Lambda^1_{\mathcal{H}}(M)=\mathfrak{m}. \tag{27}$$

Because of these, we can choose group $SO(3)$ with the Lie algebra $\mathfrak{g}=\mathfrak{so}(3)$ as the gauge group. Thus, an $\mathfrak{so}(5)$-valued 2-form on a contact 5-manifold decomposes such that

$$F=F^+_{\mathcal{H}}+F^-_{\mathcal{H}}+F_\eta, \tag{28}$$

where + and − label **SD** and **ASD**, respectively, $F^\pm_{\mathcal{H}}\in\Lambda^2_{\mathcal{H}}(M)^\pm(\mathfrak{g})$, and $F_\eta\in\eta\wedge\Lambda^1_{\mathcal{H}}(M)(\mathfrak{m})$.

We chose $F^-_{\mathcal{H}}=0$ and $F_\eta=0$ to obtain the SD concept with a Higgs field. Let $g\in G$ for a gauge group G. Then, the Maurer-Cartan (MC) 1-form is given together with the MC equation such that

$$\omega=g^{-1}\mathrm{d}g,\ \mathrm{d}\omega+\omega\wedge\omega=0. \tag{29}$$

On the other hand, let $f\in\mathcal{C}^\infty(M)$. Therefore, we write the following gauge potentials and their vanishing gauge field strengths:

$$A^-_{\mathcal{H}}=\omega,\quad F^-_{\mathcal{H}}=\mathrm{d}A^-_{\mathcal{H}}+A^-_{\mathcal{H}}\wedge A^-_{\mathcal{H}}=0, \tag{30}$$

$$A_\eta=\omega+\mathrm{d}f,\ F_\eta=\mathrm{d}A_\eta+A_\eta\wedge A_\eta=0. \tag{31}$$

Thus, an SD $SO(5)$ gauge potential on a contact 5-manifold becomes that of $SO(3)$ on the hyperplane $\mathcal{H}$ of this manifold. According to this case, the SD configuration of the $G=SO(3)$ gauge group on a contact 5-manifold M is $(A^+_{\mathcal{H}},F^+_{\mathcal{H}},\phi)$, so that the 2-forms on this bundle are spanned by the bases

$$\Lambda^2_{\mathcal{H}}(M)^+=\{(\mathrm{d}x^{12}+\mathrm{d}x^{34}),(-\mathrm{d}x^{13}+\mathrm{d}x^{24}),(\mathrm{d}x^{14}+\mathrm{d}x^{23})\}. \tag{32}$$

We can choose the following gauge potential together with its covariant derivative and gauge field strength:

$$\nabla^+=d+A^+,\quad A^+=f\omega, \tag{33}$$

$$F^+=\nabla^+A^+=\mathrm{d}A^++A^+\wedge A^+. \tag{34}$$

Hereafter, we will use the shorthand $X^\pm$ instead of $X^\pm_{\mathcal{H}}$. Now, rewrite equation (9) and (7), respectively, as

$$\chi\mathrm{d}\eta\wedge F^++[\phi,*\nabla^+\phi]\ =\ 0, \tag{35}$$

$$\nabla^+(*\nabla^+\phi)-\frac{1}{2}\frac{\delta V(\phi)}{\delta\phi}\ =\ 0, \tag{36}$$

where $*:\Lambda^p(M)\to\Lambda^{5-p}(M)$. We have that

$$\begin{aligned}F^+\ &=\ F^+_{34}(\mathrm{d}x^{12}+\mathrm{d}x^{34})+F^+_{24}(-\mathrm{d}x^{13}+\mathrm{d}x^{24})\\&\quad+F^+_{23}(\mathrm{d}x^{14}+\mathrm{d}x^{23}),\end{aligned} \tag{37}$$

$$\mathrm{d}\eta\wedge F^+\ =\ 2F^+_{34}\mathrm{d}x^{1234}. \tag{38}$$

Because the hyperbundle $\mathcal{H}$ is spanned by ∂_a, where $a=1,\ldots,4$, we write $\nabla^+\phi=\nabla^+_a\phi\mathrm{d}x^a$. By using Eq. (22), for the SD gauge potential this must be $F^+_{34}\neq0$. In this case, the other components are $F^+_{24}\neq0$ and $F^+_{23}\neq0$. Thus, the field Eq. (35) is expressed in the terms of the local coordinates such that

$$2\chi F^+_{34}+[\phi,\nabla_5\phi]=0, \tag{39}$$

where

$$\nabla_\mu\phi=\partial_\mu\phi+[A_\mu,\phi], \tag{40}$$

and for the other base $\mathrm{d}x^{ijk5}$, $(i<j<k=1,2,3,4)$,

$$\nabla^+_i\phi=0. \tag{41}$$

The meaning of this is that the Higgs field ϕ is covariant free on the hyperplane $\mathcal{H}$ or with respect to the direction ∂_i. On the other hand, from Eq. (37) we have also the following SD equations on the hyperbundle $\mathcal{H}$:

$$F^+_{34}-F^+_{12}=0,\ F^+_{24}+F^+_{13}=0,\ F^+_{23}-F^+_{14}=0. \tag{42}$$

This SD equations run on the 4-dimensional subspace of our contact 5-manifold. Therefore, the coordinates are $(x^1,\ldots,x^4)$. In order to solve these equations, we consider the gauge potential $A^a_i:\mathbb{R}^4\to\mathfrak{g}$, which can be written as follows:

$$A^a_i=f(s)\varepsilon^a_{ij}\frac{x^j}{s^2}=f(s)\omega^a_i, \tag{43}$$

where $\omega=\omega^a_i\tau_a\mathrm{d}x^i$ is the MC 1-form satisfying the MC equation $\mathrm{d}\omega+\omega\wedge\omega=0$, ε^a_{ij} is a skew symmetric tensor with constant components, and

$$s^2=G_{ij}x^ix^j, \tag{44}$$

with respect to the metric tensor G_{ij} on the subspace of the base manifold. The gauge field strength is such that

$$F^a_{ij}=-\frac{2(f-f^2)}{s^2}\varepsilon^a_{ij}+\frac{x^k}{s^3}\left(f'-\frac{2(f-f^2)}{s}\right)\left(b_i\varepsilon^a_{jk}-b_j\varepsilon^a_{ik}\right). \tag{45}$$

where

$$b_i=x_i-\frac{1}{2}(\partial_iG_{kl})x^kx^l. \tag{46}$$

In order to preserve the tensorial structure of the gauge field strength, it must hold that $f'-\frac{2(f-f^2)}{s}=0$. The solution to this equation is

$$f(s)=\frac{s^2}{r_0^2+s^2}, \tag{47}$$

where r_0 is a constant. Consequently, we have that

$$F^a_{ij}=\frac{2r_0^2}{(r_0^2+s^2)^2}\varepsilon^a_{ij}.\quad A^a_i=\frac{(r_0^2+s^2)}{2r_0^2}F^a_{ij}x^j. \tag{48}$$

We can write the gauge potential as a Lie algebra-valued 1-form in 5 dimensions separable as 4+1, such that

$$A^a=A^a_i\mathrm{d}x^i+A^a_5\mathrm{d}x^5, \tag{49}$$

where

$$s^2=G_{ij}x^ix^j,\ r^2=G_{\mu\nu}x^\mu x^\nu, \tag{50}$$

and we define A^a_5 as follows:

$$A^a_5=\frac{(r_0^2+r^2)}{2r_0^2}F^a_{5j}x^j. \tag{51}$$

Consider an $SO(3)$ gauge potential, its gauge field strength, and the Higgs field, respectively, as skew-symmetric matrices:

$$A^a_i=\begin{pmatrix}0&A^1_i&A^3_i\\-A^1_i&0&A^2_i\\-A^3_i&-A^2_i&0\end{pmatrix},\quad F^a_{ij}=\begin{pmatrix}0&F^1_{ij}&F^3_{ij}\\-F^1_{ij}&0&F^2_{ij}\\-F^3_{ij}&-F^2_{ij}&0\end{pmatrix},$$
$$\phi=\begin{pmatrix}0&\phi^1&\phi^3\\-\phi^1&0&\phi^2\\-\phi^3&-\phi^2&0\end{pmatrix}. \tag{52}$$

The self-duality equations of an $SO(3)$ gauge potential in 4 dimensions were given in Ref. [15], such that

$$F^1_{12}=F^1_{34}=q^1,\ F^2_{13}=-F^2_{24}=q^2,\ F^3_{14}=F^3_{23}=q^3, \tag{53}$$

where $q^1,q^2,q^3\in\mathcal{C}^\infty(\mathbb{R}^4)$ are independent from x^5. The matrix components of the gauge field strength become

$$\begin{aligned}F^1&=q^1\mathcal{L}^1_{ij}\mathrm{d}x^{ij}=q^1(\mathrm{d}x^{12}+\mathrm{d}x^{34}),\\F^2&=q^2\mathcal{L}^2_{ij}\mathrm{d}x^{ij}=q^2(\mathrm{d}x^{13}-\mathrm{d}x^{24}),\\F^3&=q^3\mathcal{L}^3_{ij}\mathrm{d}x^{ij}=q^3(\mathrm{d}x^{14}+\mathrm{d}x^{23}),\end{aligned}\tag{54}$$

Where the $\mathcal{L}^a$'s define a quaternionic structure on $\mathbb{R}^4$:

$$\mathcal{L}^1=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\quad\mathcal{L}^2=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix},$$
$$\mathcal{L}^3=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}.\tag{55}$$

These satisfy

$$(\mathcal{L}^a)^2=-\mathbb{I}_{4\times4},\ \mathcal{L}^a\mathcal{L}^b=-\mathcal{L}^b\mathcal{L}^a,\ a,b=1,2,3.\tag{56}$$

Therefore, the components of the SD $SO(3)$-gauge potential on $\mathbb{R}^4$ are

$$A^a_j=\frac{q^a(r_0^2+s^2)}{2r_0^2}\mathcal{L}^a_{ij}x^j.\tag{57}$$

Furthermore, the component A^a_5 is given as follows:

$$A^a_5=\frac{(r_0^2+r^2)}{2r_0^2}F^a_{5j}x^j,\tag{58}$$

where there is no summation for $q^a\mathcal{L}^a_{ij}$ over the index a.

On the other hand, we use the following Hodge gauge-fixing condition, named after Uhlenbeck [16]

$$\mathrm{d}{*}A=\left(\partial_\mu A^\mu+\Gamma^\lambda_{\lambda\mu}A^\mu\right)\mathrm{dVol},\tag{59}$$

where $\Gamma^\lambda_{\lambda\mu}=\partial_\mu\ln|\sqrt{\det(G_{\mu\nu})}|$, $A^\mu=G^{\mu\nu}A_\nu$, and we used the expression given in [17]. For the metric tensor (14), because $\det(G_{\mu\nu})=1$, we obtain the Hodge gauge-fixing as

$$\partial_\mu A^\mu=0.\tag{60}$$

Using the metric tensor (14), the Hodge gauge-fixing condition (59) reduces to the following equations:

$$\partial_5A^a_j+\partial_jA^a_5=0,\ \partial_iA^a_j=0.\tag{61}$$

The solution to the first equation is

$$F^a_{5j}=\frac{1}{(r^2+r_0^2)}D^a_j,\quad A^a_5=\frac{1}{2r_0^2}D^a_jx^j,\quad\partial_5A^a_5=0,\tag{62}$$

where D^a_j are constants. The solution to the second equation is

$$q^a=\frac{C^a}{(s^2+r_0^2)},\tag{63}$$

where C^a are constants. Therefore, we obtain

$$A^a_i=\frac{C^a}{2r_0^2}\mathcal{L}^a_{ij}x^j,\quad F^a_{ij}=\frac{C^a}{(s^2+r_0^2)}\mathcal{L}^a_{ij},\tag{64}$$

where there is no summation over the index a.

Solution to $2\chi F^+_{34}+\left[\phi,\nabla^+_5\phi\right]=0$:

From Eq. (54), we have that

$$F^1_{34}=q_1,\quad F^2_{34}=0,\quad F^3_{34}=0.\tag{65}$$

Then, the SD Eq. (39) is expanded as

$$\begin{aligned}&A^1_5\frac{\phi^2}{\phi^3}+A^1_5\frac{\phi^3}{\phi^2}-A^2_5\frac{\phi^1}{\phi^3}-A^3_5\frac{\phi^1}{\phi^2}+\partial_5\ln\left|\frac{\phi^3}{\phi^2}\right|=-\frac{2\chi}{\phi^2\phi^3}q^1,\\&A^1_5\frac{\phi^3}{\phi^2}+A^2_5\frac{\phi^3}{\phi^1}-A^3_5\frac{\phi^1}{\phi^2}-A^3_5\frac{\phi^2}{\phi^1}+\partial_5\ln\left|\frac{\phi^2}{\phi^1}\right|=0,\\&A^1_5\frac{\phi^2}{\phi^3}-A^2_5\frac{\phi^1}{\phi^3}-A^2_5\frac{\phi^3}{\phi^1}+A^3_5\frac{\phi^2}{\phi^1}-\partial_5\ln\left|\frac{\phi^1}{\phi^3}\right|=0.\end{aligned}\tag{66}$$

When we sum these equations, we obtain $\chi q^1=0$. Because it cannot hold that $q^1\neq0$, the coupling constant must be

$$\chi=0.\tag{67}$$

Thus, we see that the SD gauge potential that can be defined on the hyperplane $\mathcal{H}$ does not interact with the Higgs field ϕ. Then, the SD equation (39) becomes

$$[\phi,\nabla_5\phi]=0.\tag{68}$$

Therefore, when we consider Eq. (41), the Higgs field ϕ satisfies the following covariance equation on the contact 5-manifold with respect to the SD gauge potential:

$$\left[\phi,\nabla^+_\mu\phi\right]=0.\tag{69}$$

In order to solve this equation, we use the following mechanism. The Higgs field $\phi\in\mathfrak{so}(3)$ satisfies

$$\phi^3+\|\phi\|^2\phi=0,\tag{70}$$

where

$$\|\phi\|^2=(\phi^1)^2+(\phi^2)^2+(\phi^3)^2.\tag{71}$$

If we define a new field on $\mathfrak{so}(3)$ such that

$$\hat{f}=\frac{\phi}{\|\phi\|},\tag{72}$$

then this field satisfies the following structure equation:

$$\hat{f}^3+\hat{f}=0,\tag{73}$$

also called the f-structure, defined by Yano [18]. Furthermore $\hat{f}$ satisfies

$$(\hat{f}^1)^2+(\hat{f}^2)^2+(\hat{f}^3)^2=1.\tag{74}$$

The covariant derivative of the Higgs field in the direction x^5 is written with respect to $\hat{f}$ as follows:

$$\nabla_\mu\phi=(\partial_\mu\|\phi\|)\hat{f}++\|\phi\|\partial_\mu\hat{f}.\tag{75}$$

On the other hand, we obtain the following identity:

$$[\phi,\nabla_\mu\phi]=\|\phi\|^2[\hat{f},\nabla_\mu\hat{f}].\tag{76}$$

Therefore, Eq. (68) can also be written as

$$[\hat{f},\nabla_\mu \hat{f}]=0 \tag{77}$$

and we find the following solution to this equation:

$$\hat{f}=\frac{1}{\sqrt{1+2f_0^2}}\begin{pmatrix}0 & 1 & f_0\\ -f_0 & 0 & f_0\\ -f_0 & -f_0 & 0\end{pmatrix}, \tag{78}$$

where f_0=Constant. Thus, $\hat{f}$ becomes covariant free in all directions

$$\nabla_\mu \hat{f}=0. \tag{79}$$

When we use Eq. (72), we obtain

$$\nabla_\mu\phi=(\partial_\mu \ln\|\phi\|)\phi. \tag{80}$$

Therefore, Eq. (68) becomes $[\phi,\phi]=0$. On the other hand, because $\nabla_i^+\phi=0$, this is also equivalent to $\nabla_i\hat{f}=0$. Therefore, we obtain

$$\nabla_i\phi=(\partial_i \ln\|\phi\|)\phi=0. \tag{81}$$

Then, we see that $\|\phi\|$ is independent of the coordinates x^i. Consequently,

$$\nabla_\mu\phi=\nabla_5\phi=(\partial_5 \ln\|\phi\|)\phi. \tag{82}$$

On the other hand, we obtain the following identity:

$$\nabla(*\nabla\phi)=\left\{\partial_5^2\ln\|\phi\|+(\partial_5\ln\|\phi\|)^2\right\}\phi\mathrm{dVol}. \tag{83}$$

Consider the following potential form for the massless $SO(3)$ Higgs field as a ϕ^4 field theory:

$$V(\phi)=\lambda\phi^4\mathrm{dVol}, \tag{84}$$

where λ is real parameter. Using Eq. (70), we have that

$$\frac{\delta V}{\delta\phi}=4\lambda\phi^4=-4\lambda\|\phi\|^2\phi\mathrm{dVol} \tag{85}$$

Then, Eq. (36) becomes

$$\partial_5^2\ln\|\phi\|+(\partial_5\ln\|\phi\|)^2+2\lambda\|\phi\|^2=0. \tag{86}$$

This implies that

$$x^5=t,\ \|\phi\|=\sigma,\ \partial_5\sigma=\dot{\sigma}. \tag{87}$$

After some arrangements, this equation is rewritten as follows:

$$\ddot{\sigma}+2\lambda\sigma^2=0. \tag{88}$$

We make two fundamental ansatzes for solutions to this equation. One of these is the monopole case: $V(\phi)=0$ when one chooses $\lambda=0$. The equation (68) is a simple consequence of the SD equation with a Higgs field in higher dimensions. Although the ASD concept in higher dimensions is a exact vacuum Yang-Mills case, the SD one becomes a Yang-Mills-Higgs system. On the other hand, if the potential form is set as $V(\phi)=0$, in the simplest interpretation this system represents a monopole notion on a contact 5-manifold. Therefore, if we try the solution model

$$\sigma=\sigma_0\exp(\alpha(t)), \tag{89}$$

then the monopole solution is given as follows:

$$\|\phi\|_{\lambda=0}=\sigma_0\exp\left(\alpha_0-\frac{1}{2}\beta_0\exp(-2t)\right), \tag{90}$$

where σ_0, $\alpha_0>0$, and $\beta_0>0$ are constants. This solution has the following stability situations:

$$t\to0 \quad \|\phi\|_{\lambda=0}\to\sigma_0\exp\left(\alpha_0-\frac{1}{2}\beta_0\right), \tag{91}$$

$$t\to\infty \quad \|\phi\|_{\lambda=0}\to\sigma_0\exp(\alpha_0) \tag{92}$$

Because the Eq. (88) is a second-order nonlinear ordinary differential equation, we adopt the following method. Let $p(t)$ and $q(t)$ be two arbitrary scalars that satisfy the following equation:

$$(p(t)\dot{\sigma})^{\cdot}+q(t)\sigma^2=0. \tag{93}$$

This equations can be expressed as

$$\ddot{\sigma}+\frac{\dot{p}}{p}\dot{\sigma}+q\sigma^2=0. \tag{94}$$

When we set

$$\frac{\dot{p}}{p}\dot{\sigma}+q\sigma^2=2\lambda\sigma^2, \tag{95}$$

we obtain the following solution to this equation:

$$\|\phi\|=\frac{1}{C+\displaystyle\int\frac{1}{\dot{p}/p}(q(t)-2\lambda)\mathrm{d}t}<\infty. \tag{96}$$

It is easily seen that this $SO(3)$ solution cannot be reduced to the monopole solution (90) at the limit $\lambda\to\infty$ or for $\lambda=0$.

4 Conclusion

We dealt with the self-duality concept with a Higgs field on a 5-dimensional contact manifold. A non-trivial $SO(3)$ Higgs field lives only on the fifth dimension of the contact manifold, owing to the contact structure, while the SD Yang-Mills field lives on the 4-dimensional hyperplane of the tangent bundle on the contact manifold. In our solution, the gauge potential and its gauge field strength do not include any singularities as long as $r_0\neq0$. On the other hand, the $SO(3)$ Higgs field yields a structure on the Lie algebra $\mathfrak{so}(3)$ such that $\hat{f}^3+\hat{f}=0$, and it does not interact with the SD gauge potential, because the coupling constant vanishes owing to Eq. (67). Namely, the Higgs and SD Yang-Mills fields do not interact with one another. Thus, our (massless) solution on a contact 5-manifold is summarized as follows.

SD $SO(3)$-gauge potential and its gauge field strength:

$$A_i^a = \frac{C^a}{2r_0^2}\mathcal{L}_{ij}^a x^j, \quad A_5^a = \frac{1}{2r_0^2}D_j^a x^j,$$

$$F_{ij}^a = \frac{C^a}{(s^2+r_0^2)}\mathcal{L}_{ij}^a, \quad F_{5j}^a = \frac{1}{(r^2+r_0^2)}D_j^a.$$

Massless $SO(3)$-Higgs ($\lambda \neq 0$) and monopole ($\lambda = 0$) fields:

$$\hat{f} = \frac{1}{\sqrt{1+2f_0^2}}\begin{pmatrix} 0 & 1 & f_0 \\ -f_0 & 0 & f_0 \\ -f_0 & -f_0 & 0 \end{pmatrix},$$

$$\hat{f}^3 + \hat{f} = 0,$$

$$\phi = \|\phi\|\hat{f},$$

$$\|\phi\|_{\lambda\neq 0} = \frac{1}{C + \int \frac{1}{\dot{p}/p}(q(t)-2\lambda)\mathrm{d}t}, \quad V(\phi) = \lambda\phi^4 \mathrm{dVol},$$

$$\|\phi\|_{\lambda=0} = \sigma_0 \exp\left(\alpha_0 - \frac{1}{2}\beta_0 \exp(-2t)\right), \quad V(\phi) = 0,$$

where C^a and D_j^a are constants, and there is no summation over the index a. Furthermore, because the Higgs field does not interact with the SD gauge potential, it holds that

$$\chi = 0.$$

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