Shear and bulk viscosity of high-temperature gluon plasma *

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Abstract: We calculate the shear viscosity (η) and bulk viscosity (ζ) to entropy density (s) ratios η/s and ζ/s of a gluon plasma system in kinetic theory, including both the elastic $gg\leftrightarrow gg$ forward scattering and the inelastic soft gluon bremsstrahlung $gg\leftrightarrow ggg$ processes. Due to the suppressed contribution to η and ζ in the $gg\leftrightarrow gg$ forward scattering and the effective $g\leftrightarrow gg$ gluon splitting, Arnold, Moore and Yaffe (AMY) and Arnold, Dogan and Moore (ADM) have got the leading order computations for η and ζ in high-temperature QCD matter. In this paper, we calculate the correction to η and ζ in the soft gluon bremsstrahlung $gg\leftrightarrow ggg$ process with an analytic method. We find that the contribution of the collision term from the $gg\leftrightarrow ggg$ soft gluon bremsstrahlung process is just a small perturbation to the $gg\leftrightarrow gg$ scattering process and that the correction is at $\sim$5% level. Then, we obtain the bulk viscosity of the gluon plasma for the number-changing process. Furthermore, our leading-order result for bulk viscosity is the formula $\zeta\propto\frac{\alpha_s^2T^3}{\ln\alpha_s^{-1}}$ in high-temperature gluon plasma.

Keywords: shear viscosity, bulk viscosity, entropy density, high-temperature gluon plasma, soft gluon bremsstrahlung

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1 Introduction

It is believed that the hot and dense quark-gluon plasma (QGP) found at the Relativistic Heavy Ion Collider (RHIC) seem to be a near-perfect fluid [1, 2]. The remanent of the non-central collisions shows collective motion (elliptic flow) with the shear viscosity to entropy density ratio $\eta/s=0.1\pm0.1$ (theory) ±0.08 (experiment) [3]. This η/s ratio is close to a conjectured minimum bound $1/4\pi$ [4], which is motivated by the uncertainty principle and gauge/string duality.

It is known that the parametric behavior is $\eta\propto\frac{T^3}{\ln\alpha_s^2\alpha_s^{-1}}$ in the QGP [5–9]. Arnold, Moore and Yaffe (AMY) calculated complete results both at leading logarithmic order [10] and full leading order [11] in the QCD coupling α_s. However, a recent perturbative QCD calculation of η and η/s of a gluon plasma by XG (Z. Xu and Greiner) [12] and Q. Wang [9, 13] considered the inelastic number changing process with $gg\leftrightarrow ggg$ soft gluon bremsstrahlung in gluon plasma. The bulk viscosity ζ has been considered to be zero for a long time [5, 8]. Now, however, it is believed that bulk viscosity is related to the conformal variance of a system: in a conformally invariant system, the bulk viscosity is definitely zero; but in a conformally non-invariant system, the bulk viscosity is not zero but a function of the velocity of sound v_s. Therefore, we must consider the contribution of the process of particle number violation. Arnold, Dogan and Moore (ADM) considered the contributions of quark or gluon splitting process for particle number violation. They give the leading-log order of bulk viscosity in the massless high-temperature QCD matter, $\zeta\sim\frac{\alpha_s^2T^3}{\ln\alpha_s^{-1}}$ [13, 14]. The bulk viscosity ζ is much smaller than η, $\zeta/\eta\propto\alpha_s^4\ll1$ [13, 14]. In lattice QCD, however, the bulk viscosity may be divergent [15–18] and ζ/s is large [19–23] at the chiral phase transition point. This has also been observed in PNJL model calculations [24].

In this paper, we will use an analytic method to solve the Boltzmann equation to get shear and bulk viscosity, including the inelastic gluon bremsstrahlung $gg\leftrightarrow ggg$ process, in a high-temperature gluon plasma system. The structure of this paper is as follows. Section 2 introduces the effective kinetic theory and the form of η and ζ in the variational method. The computation of the collision term is described in Section 3. Section 4 concludes the paper along with some discussions of the result.

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2 Effective kinetic theory

In a local equilibrium state, the stress-energy tensor can be expanded into an equilibrium term and dissipative term:

$$T^{\mu\nu}=<T^{\mu\nu}>_{eq}+\triangle T^{\mu\nu}, \tag{1}$$

where,

$$\triangle T_{ij}=-\eta[\nabla_i u_j+\nabla_j u_i-\frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}]-\zeta\delta_{ij}\nabla^l u_l. \tag{2}$$

η is the shear viscosity, and ζ is the bulk viscosity. This expression is written in the local rest frame, and $u^\mu u^\nu\triangle T^{\mu\nu}=0$.

On the other side, in kinetic theory the stress-energy tensor is:

$$T^{\mu\nu}=\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{p^\mu p^\nu}{p^0}f(\boldsymbol{p},x), \tag{3}$$

where $f(\boldsymbol{p},x)$ is the color-averaged gluon distribution function in gluon-plasma.

In a high temperature and weak coupling gluon plasma system, the gluon distribution function satisfies the Boltzmann equation in the usual form

$$\left[\frac{\partial}{\partial_t}+\boldsymbol{v_p}\cdot\frac{\partial}{\partial_{\boldsymbol{x}}}\right]f(\boldsymbol{x},\boldsymbol{p},t)=-C[f(\boldsymbol{p},\boldsymbol{x},t)]. \tag{4}$$

$\boldsymbol{v_p}=\hat{\boldsymbol{p}}\equiv\boldsymbol{p}/|\boldsymbol{p}|$ means the spatial velocity, which is a unit vector. Here, we do not consider the external field. $C[f(\boldsymbol{p},\boldsymbol{x},t)]$ denotes the collision term. This equation is a differential integral equation, and is very hard to be solved analytically. We consider only the $2\leftrightarrow2$ collision term and $2\leftrightarrow3$ collision term [10]:

$$C[f]=C^{2\leftrightarrow2}[f]+C^{2\leftrightarrow3}[f], \tag{5}$$

Here,

$$\begin{aligned}C^{2\leftrightarrow2}[f]\ =\ &\frac{1}{4|\boldsymbol{p}_1|N_g}\int_{\boldsymbol{p_2},\boldsymbol{p_3},\boldsymbol{p_4}}|\mathcal{M}^{gg}_{gg}(\boldsymbol{p}_1,\boldsymbol{p_2};\boldsymbol{p_3},\boldsymbol{p_4})|^2\\&\times\{f(\boldsymbol{p}_1)f(\boldsymbol{p}_2)[1+f(\boldsymbol{p}_3)][1+f(\boldsymbol{p}_4)]\\&-[1+f(\boldsymbol{p}_1)][1+f(\boldsymbol{p}_2)]f(\boldsymbol{p}_3)f(\boldsymbol{p}_4)\}\\&\times(2\pi)^4\delta^{(4)}(P_1+P_2-P_3-P_4),\end{aligned} \tag{6}$$

and

$$\begin{aligned}C^{2\leftrightarrow3}[f]\ =\ &\frac{1}{8|\boldsymbol{p}_1|N_g}\int_{\boldsymbol{p_2},\boldsymbol{p_3},\boldsymbol{p_4},\boldsymbol{k}}|\mathcal{M}^{gg}_{ggg}(\boldsymbol{p}_1,\boldsymbol{p_2};\boldsymbol{p_3},\boldsymbol{p_4},\boldsymbol{k})|^2\\&\times\{f(\boldsymbol{p}_1)f(\boldsymbol{p}_2)[1+f(\boldsymbol{p}_3)][1+f(\boldsymbol{p}_4)][1+f(\boldsymbol{k})]\\&-[1+f(\boldsymbol{p}_1)][1+f(\boldsymbol{p}_2)]f(\boldsymbol{p}_3)f(\boldsymbol{p}_4)f(\boldsymbol{k})\}\\&\times(2\pi)^4\delta^{(4)}(P_1+P_2-P_3-P_4-K)\\&+\frac{1}{12|\boldsymbol{p}_1|N_g}\int_{\boldsymbol{p_2},\boldsymbol{p_3},\boldsymbol{p_4},\boldsymbol{k}}|\mathcal{M}^{ggg}_{gg}(\boldsymbol{p}_1,\boldsymbol{p_2},\boldsymbol{k};\boldsymbol{p_3},\boldsymbol{p_4})|^2\\&\times\{f(\boldsymbol{p}_1)f(\boldsymbol{p}_2)f(\boldsymbol{k})[1+f(\boldsymbol{p}_3)][1+f(\boldsymbol{p}_4)]\\&-[1+f(\boldsymbol{p}_1)][1+f(\boldsymbol{p}_2)][1+f(\boldsymbol{k})]f(\boldsymbol{p}_3)f(\boldsymbol{p}_4)\}\\&\times(2\pi)^4\delta^{(4)}(P_1+P_2+K-P_3-P_4).\end{aligned} \tag{7}$$

Here, $N_g=16$ is the helicity and color degeneracy of the gluons, and the Lorentz invariant momentum integration is $\int_{\boldsymbol{p}}\equiv\int\frac{\mathrm{d}^3\boldsymbol{p}}{2p^0(2\pi)^3}$.

The distribution function corresponding to a small departure from equilibrium state can be written as the sum of a local equilibrium term plus the departure term:

$$f(\boldsymbol{p})=f_0(\boldsymbol{p})+f_0(\boldsymbol{p})[1+f_0(\boldsymbol{p})]f_1(\boldsymbol{p}). \tag{8}$$

Here $f_0(\boldsymbol{p})$ is the gluon distribution function in the local equilibrium state, which satisfies the Bose distribution:

$$f_0(\boldsymbol{p})=\frac{1}{e^{\beta|\boldsymbol{p}|}-1}. \tag{9}$$

On the basis of distribution function expansion, we can expand the collision term, and get the linearized collision term

$$C[f]=C[f_0]+\mathcal{C}[f_1]+O(f_1^2). \tag{10}$$

The local equilibrium part has no contribution, $C[f_0]=0$. $\mathcal{C}[f_1]$ is the linearized collision term, $\mathcal{C}[f_1]=\mathcal{C}^{2\leftrightarrow2}[f_1]+\mathcal{C}^{2\leftrightarrow3}[f_1]$, where,

$$\begin{aligned}\mathcal{C}^{2\leftrightarrow2}[f_1]\ =\ &\frac{1}{4|\boldsymbol{p}_1|N_g}\int_{\boldsymbol{p_2},\boldsymbol{p_3},\boldsymbol{p_4}}|\mathcal{M}^{gg}_{gg}(\boldsymbol{p}_1,\boldsymbol{p_2};\boldsymbol{p_3},\boldsymbol{p_4})|^2\\&\times f_0(\boldsymbol{p}_1)f_0(\boldsymbol{p}_2)[1+f_0(\boldsymbol{p}_3)][1+f_0(\boldsymbol{p}_4)]\\&\times[f_1(\boldsymbol{p}_1)+f_1(\boldsymbol{p}_2)-f_1(\boldsymbol{p}_3)-f_1(\boldsymbol{p}_4)]\\&\times(2\pi)^4\delta^{(4)}(P_1+P_2-P_3-P_4),\end{aligned} \tag{11}$$

and

$$\begin{aligned}\mathcal{C}^{2\leftrightarrow3}[f_1]\ =\ &\frac{1}{8|\boldsymbol{p}_1|N_g}\int_{\boldsymbol{p_2},\boldsymbol{p_3},\boldsymbol{p_4},\boldsymbol{k}}|\mathcal{M}^{gg}_{ggg}(\boldsymbol{p}_1,\boldsymbol{p_2};\boldsymbol{p_3},\boldsymbol{p_4},\boldsymbol{k})|^2\\&\times f_0(\boldsymbol{p}_1)f_0(\boldsymbol{p}_2)[1+f_0(\boldsymbol{p}_3)][1+f_0(\boldsymbol{p}_4)][1+f_0(\boldsymbol{k})]\\&\times[f_1(\boldsymbol{p}_1)+f_1(\boldsymbol{p}_2)-f_1(\boldsymbol{p}_3)-f_1(\boldsymbol{p}_4)-f_1(\boldsymbol{k})]\\&\times(2\pi)^4\delta^{(4)}(P_1+P_2-P_3-P_4-K)\\&+\frac{1}{12|\boldsymbol{p}_1|N_g}\int_{\boldsymbol{p_2},\boldsymbol{p_3},\boldsymbol{p_4},\boldsymbol{k}}|\mathcal{M}^{ggg}_{gg}(\boldsymbol{p}_1,\boldsymbol{p_2},\boldsymbol{k};\boldsymbol{p_3},\boldsymbol{p_4})|^2\\&\times f_0(\boldsymbol{p}_1)f_0(\boldsymbol{p}_2)f_0(\boldsymbol{k})[1+f_0(\boldsymbol{p}_3)][1+f_0(\boldsymbol{p}_4)]\\&\times[f_1(\boldsymbol{p}_1)+f_1(\boldsymbol{p}_2)+f_1(\boldsymbol{k})-f_1(\boldsymbol{p}_3)-f_1(\boldsymbol{p}_4)]\\&\times(2\pi)^4\delta^{(4)}(P_1+P_2+K-P_3-P_4).\end{aligned} \tag{12}$$

On the left-hand side of the Boltzmann equation, the gradients acting on f_1 give a higher order of the departure from equilibrium, so that in the linear approximation, only the gradients acting on f_0 should be considered. The Boltzmann equation becomes a linear integral equation

$$\left[\frac{\partial}{\partial_t}+\boldsymbol{v_p}\cdot\frac{\partial}{\partial_{\boldsymbol{x}}}\right]f_0(\boldsymbol{p},x,t)=-\mathcal{C}f_1(\boldsymbol{p},x,t) \tag{13}$$

The left-hand side of the linearized Boltzmann equation has the form [8, 10],

$$LHS=\beta f_0(\boldsymbol{p})[1+f_0(\boldsymbol{p})]q_\kappa(\boldsymbol{p})I_{ij}(\hat{\boldsymbol{p}})X_{ij}(x). \tag{14}$$

where the spatial tenor $X_{ij}(x)$ denotes the driving force, which is defined as:

$$X_{ij}(x)=\begin{cases}\nabla_i u_j+\nabla_j u_i-\dfrac{2}{3}\delta_{ij}\nabla\cdot\boldsymbol{u}, & \text{(shear viscosity)}\\ \delta_{ij}\nabla\cdot\mathbf{u}, & \text{(bulk viscosity)}\end{cases} \quad (15)$$

and

$$I_{ij}(\hat{\boldsymbol{p}})=\begin{cases}\dfrac{1}{2}\left(\hat{p}_i\hat{p}_i-\dfrac{1}{3}\delta_{ij}\right), & \text{(shear viscosity)}\\ \dfrac{1}{3}\delta_{ij}, & \text{(bulk viscosity)}\end{cases} \quad (16)$$

with $q_\kappa(\boldsymbol{p})$

$$\begin{cases}q_\eta(\boldsymbol{p})=|\boldsymbol{p}|. & \text{(shear viscosity)}\\ q_\zeta(\boldsymbol{p})=\dfrac{\boldsymbol{p}\cdot\boldsymbol{v_p}}{3}-v_s^2\dfrac{\partial(\beta E_p)}{\partial\beta}. & \text{(bulk viscosity)}\end{cases} \quad (17)$$

AMY and ADM give the $q_\kappa(\boldsymbol{p})$ of Eq. (17) in Ref. [10, 14]. In this paper we do not consider the mass of the gluon and the thermal masses. So, the parameter $q_\zeta(\boldsymbol{p})$ can be given as

$$q_\zeta(\boldsymbol{p})=\left(\frac{1}{3}-v_s^2\right)|\boldsymbol{p}|. \quad (18)$$

Here we have determined the speed of sound v_s in the gluon plasma. In a conformally invariant system, it can easily be verified that $v_s^2=\frac{1}{3}$ and $q_\zeta(\boldsymbol{p})=0$. Then in the conformally non-invariant system, we can determine the speed of sound with the temperature dependence of the pressure. According to Ref. [14], the speed of sound v_s is written as follows in a high temperature and weak coupling gluon-plasma system,

$$v_s^2=\frac{1}{3}-\frac{55\alpha_s^2}{24\pi^2}, \quad (19)$$

where the strong coupling coefficient is $\alpha_s\equiv\frac{g^2}{4\pi}$.

Consequently, giving the left-hand side of Eq.(14), the departure distribution function $f_1(\boldsymbol{p})$ on the right-hand side, which will solve the linearized Boltzmann equation, must have the corresponding form

$$f_1(\boldsymbol{p},x)=\beta^2X_{ij}(x)\chi_{ij}(\boldsymbol{p}). \quad (20)$$

Here, $\chi_{ij}(p)=I_{ij}(\hat{\boldsymbol{p}})\chi(p)$. From Eq. (13), Eq. (14)and Eq. (16), we can write the linearized Boltzmann equation in the concise form [10, 14],

$$\mathcal{S}_{ij}(\boldsymbol{p})=\mathcal{C}\chi_{ij}(\boldsymbol{p}), \quad (21)$$

with

$$\mathcal{S}_{ij}(\boldsymbol{p})\equiv-\frac{T}{N_g}f_0(\boldsymbol{p})[1+f_0(\boldsymbol{p})]q_\kappa(\boldsymbol{p}))I_{ij}(\hat{\boldsymbol{p}}). \quad (22)$$

To solve Eq. (21), we use the variational approach proposed in Ref. [10]. First, we introduce an inner product as

$$(f,g)\equiv\beta^3N_g\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}f(\boldsymbol{p})g(\boldsymbol{p}), \quad (23)$$

with

$$(\chi_{ij},\mathcal{S}_{ij})=-\beta^2\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}f_0(\boldsymbol{p})[1+f_0(\boldsymbol{p})]q_\kappa(\boldsymbol{p})\chi(p), \quad (24)$$

$$\begin{aligned}(\chi_{ij},\mathcal{C}^{2\leftrightarrow2}\chi_{ij}) = \frac{\beta^3}{8}\int_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3,\boldsymbol{p}_4}&|\mathcal{M}^{gg}_{gg}(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_3,\boldsymbol{p}_4)|^2\\ &\times f_0(\boldsymbol{p}_1)f_0(\boldsymbol{p}_2)[1+f_0(\boldsymbol{p}_3)][1+f_0(\boldsymbol{p}_4)]\\ &\times[\chi_{ij}(p_1)+\chi_{ij}(p_2)-\chi_{ij}(p_3)-\chi_{ij}(p_4)]^2\\ &\times(2\pi)^4\delta^{(4)}(P_1+P_2-P_3-P_4),\end{aligned} \quad (25)$$

and

$$\begin{aligned}&(\chi_{ij},\mathcal{C}^{2\leftrightarrow3}\chi_{ij})\\ =&\frac{\beta^3}{12}\int_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3,\boldsymbol{p}_4,\boldsymbol{k}}|\mathcal{M}^{gg}_{ggg}(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_3,\boldsymbol{p}_4,\boldsymbol{k})|^2\\ &\times f_0(\boldsymbol{p}_1)f_0(\boldsymbol{p}_2)[1+f_0(\boldsymbol{p}_3)][1+f_0(\boldsymbol{p}_4)][1+f_0(\boldsymbol{k})]\\ &\times[\chi_{ij}(\boldsymbol{p}_1)+\chi_{ij}(\boldsymbol{p}_2)-\chi_{ij}(\boldsymbol{p}_3)-\chi_{ij}(\boldsymbol{p}_4)-\chi_{ij}(\boldsymbol{k})]^2\\ &\times(2\pi)^4\delta^{(4)}(P_1+P_2-P_3-P_4-K).\end{aligned} \quad (26)$$

Solving the linearized Boltzmann equation, we define the functional of the variational function $\chi(\boldsymbol{p})$,

$$\begin{aligned}&Q[\chi]\equiv(\chi_{ij},\mathcal{S}_{ij})-1/2(\chi_{ij},\mathcal{C}\chi_{ij})\\ &\mathcal{C}\chi=\mathcal{C}^{2\leftrightarrow2}\chi+\mathcal{C}^{2\leftrightarrow3}\chi.\end{aligned} \quad (27)$$

If $\chi_{ij}(\boldsymbol{p})$ satisfies Eq.(21), $Q[\chi_{ij}]$ takes the maximum. Then the shear viscosity η and the bulk viscosity ζ satisfy the following form:

$$\begin{cases}\eta=2/15Q[\chi]_{\max}|_{q_\eta(\boldsymbol{p})},\\ \zeta=2Q[\chi]_{\max}|_{q_\zeta(\boldsymbol{p})}.\end{cases} \quad (28)$$

3 Collision integrals

3.1 $gg\leftrightarrow gg$ collision integrals

Before calculating the collision term of the $gg\leftrightarrow gg$ scattering process, some approximations should be made first:

(1) The high temperature approximation: the temperature T of the gluon-plasma is extremely high, so that the theory is weakly coupled with the scale of the temperature, $g(T)\ll1$.

(2) The forward scattering approximation: the momentum transfer between the incident particles $q\sim gT$ is rather small, so that the the momentum difference between incoming and outgoing particles on the same interaction vertex can be ignored in the distribution functions.

The integrand of Eq. (25) is composed of three parts: the matrix element, the distribution functions, and the χ

term of the variational functions. With the $q\to 0$ Feynman rule, it is trivial to evaluate the matrix element for the $gg\leftrightarrow gg$ scattering processes

$$|\mathcal{M}_{gg}^{gg}|^2=16g^4 d_A C_A^2\left(3-\frac{su}{t^2}-\frac{st}{u^2}-\frac{tu}{s^2}\right), \tag{29}$$

with $d_A=8$ and $C_A=3$.

The $gg\leftrightarrow gg$ collision integrals in Refs. [10, 29] have been calculated:

$$(\chi,\mathcal{C}^{gg\leftrightarrow gg}\chi) = \frac{48\alpha_s^2\ln\alpha_s^{-1}}{\pi}\int_0^\infty p_1^2 \mathrm{d}p_1 f_0(p_1)[1+f_0(p_1)] \times([\chi(p_1)']^2+\frac{6}{p_1^2}[\chi(p_1)]^2). \tag{30}$$

3.2 $gg\leftrightarrow ggg$ collision integrals

For the treatment of the $gg\leftrightarrow ggg$ process, we consider the case in Fig. 1.

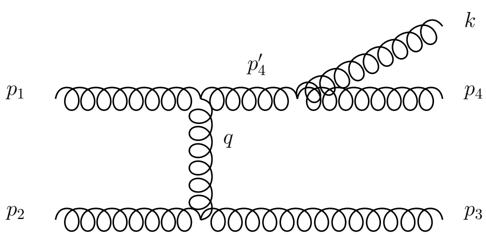

Fig. 1. Soft gluon bremsstrahlung in gluon plasma.

Here, we consider the contribution of the soft gluon bremsstrahlung $gg\leftrightarrow ggg$ process to shear or bulk viscosity. We use the Gunion-Bertsch (GB) formula [25–27] for the $gg\leftrightarrow ggg$ matrix element which is valid for the soft gluon bremsstrahlung. The GB formula is given in Ref. [25]:

$$\begin{aligned}|\mathcal{M}_{ggg}^{gg}|^2 &= 12g^2\frac{9}{2}g^4\frac{s^2}{t^2}\frac{1}{k_\perp^2}(1+\frac{t^2}{s^2})\\ &= 54g^6\frac{1}{k_\perp^2}(1+\frac{s^2}{t^2}).\end{aligned} \tag{31}$$

Starting calculation of the collision terms of $gg\leftrightarrow ggg$ soft gluon bremsstrahlung, some approximations should be considered:

(1) The momentum of soft gluon $\boldsymbol{k}\to 0$ in bremsstrahlung. Thus, we consider collinear approximation, and the angle between $\boldsymbol{p}_4$ and $\boldsymbol{k}$ is small, so that $\boldsymbol{p}'_4=\boldsymbol{p}_4+\boldsymbol{k}$ in Fig. 1.

(2) The forward scattering approximation is considered in the $gg\leftrightarrow gg$ scattering process.

(3) Soft gluon bremsstrahlung does not impact on the $gg\leftrightarrow gg$ forward scattering process.

The momentum of the soft gluon $\boldsymbol{k}\to 0$, so we have $\boldsymbol{p}'_4=\boldsymbol{p}_4+\boldsymbol{k}\simeq\boldsymbol{p}_4$. In Eq. (26), $\boldsymbol{p}'_4-\boldsymbol{p}_1=\boldsymbol{q}$. Thus, we can approximately get $\mathrm{d}^3\boldsymbol{p}_4=\mathrm{d}^3\boldsymbol{q}$. We integrate over $\mathrm{d}^3\boldsymbol{p}_3$ with the help of $(2\pi)^3\delta^3(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}_3-\boldsymbol{p}_4-\boldsymbol{k})$. Then, we consider the momentum of the soft gluon $\boldsymbol{k}\to 0$ with the collinear approximation. We may write the angular integrals in spherical coordinates with $\boldsymbol{q}$ on the z axis and $\boldsymbol{p}$ lying in the $x-z$ plane. One may transform the equation into:

$$\begin{aligned}&(\chi_{ij},\mathcal{C}^{gg\leftrightarrow ggg}\chi_{ij})\\ =& \frac{\beta^3}{3\cdot 2^{13}\cdot\pi^8}\int_0^\infty \mathrm{d}q\int_{-q}^{q}\mathrm{d}\omega\int_0^\infty \mathrm{d}p_1\\ &\times\int_0^\infty \mathrm{d}p_2\int_0^\infty k\mathrm{d}k\int_0^{2\pi}\mathrm{d}\phi|\mathcal{M}_{ggg}^{gg}(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_3,\boldsymbol{p}_4,\boldsymbol{k})|^2\\ &\times f_0(\boldsymbol{p}_1)f_0(\boldsymbol{p}_2)[1+f_0(\boldsymbol{p}_3)][1+f_0(\boldsymbol{p}_4)][1+f_0(\boldsymbol{k})]\\ &\times[\chi_{ij}(\boldsymbol{p}_1)+\chi_{ij}(\boldsymbol{p}_2)-\chi_{ij}(\boldsymbol{p}_3)-\chi_{ij}(\boldsymbol{p}_4)-\chi_{ij}(\boldsymbol{k})]^2,\end{aligned} \tag{32}$$

where, $p_4\equiv|\boldsymbol{p}_1+\boldsymbol{q}|\simeq p_1+\omega$, $p_3\equiv|\boldsymbol{p}_2-\boldsymbol{q}|=p_2-\omega$. To get Eq. (32), we use the following approximation,

$$\delta(p_1+p_2-p_3-p_4-k)\simeq\delta(p_1+p_2-p_3-p_4), \tag{33}$$

then, following Baym et al. [28], eliminate $\delta(p_1+p_2-p_3-p_4)$ and introduce a dummy integration variable ω.

Because the momentum of the soft gluon $\boldsymbol{k}\to 0$, we do not consider the impact of soft gluon bremsstrahlung on the $gg\leftrightarrow gg$ forward scattering process. In the variational method, considering the collinear approximation of soft gluon bremsstrahlung and the momentum of soft gluon $\boldsymbol{k}\to 0$, we specify the χ term as

$$\chi_{ij}(\boldsymbol{p}'_4)=\chi_{ij}(\boldsymbol{p}_4)+\chi_{ij}(\boldsymbol{k}). \tag{34}$$

Next, we must simplify the matrix element of the $gg\leftrightarrow ggg$ soft gluon bremsstrahlung process. Following Eq. (30), we should use the relationship of Mandelstam variables in Ref. [25],

$$k_\perp^2=\frac{(s+t+u)^2}{s};\qquad (s+t+u)=2\boldsymbol{p}_1\cdot\boldsymbol{k} \tag{35}$$

with the approximation of the Debye screening mass $m_D=gT\sim 0$, $s\geqslant 4m_D^2$ and $s=\langle s\rangle=18T^2$. We do not consider the direction vector of soft gluon momentum. Using the forward scattering approximation

$$-\frac{s}{t}\simeq\frac{u}{t}\simeq\frac{2p_1p_2}{q^2}(1-\cos\phi) \tag{36}$$

we can simplify the matrix element of the soft gluon bremsstrahlung $gg\leftrightarrow ggg$ process as

$$|\mathcal{M}^{gg\leftrightarrow ggg}|^2\approx 124416\pi^3\alpha_s^3T^2\frac{p_2^2}{q^4k^2}(1-\cos\phi)^2. \tag{37}$$

We have used Eq. (37) to simplify Eq. (32), to eliminate $d\omega$, $d\phi$. So the $gg\leftrightarrow ggg$ collision integrals can be written as

$$(\chi_{ij},\mathcal{C}^{gg\leftrightarrow ggg}\chi_{ij}) = \frac{162\alpha_s^3\beta}{\pi^4}\int_{\alpha_s T}^{T}\frac{1}{q}\mathrm{d}q\int_0^\infty \mathrm{d}p_1\int_0^\infty p_2^2\mathrm{d}p_2 \times\int_0^\infty\frac{1}{k}\mathrm{d}kf_0(p_1)f_0(p_2) \times[1+f_0(p_1)][1+f_0(p_2)][1+f_0(k)] \times([\chi(p_1)']^2+\frac{6}{p_1^2}[\chi(p_1)]^2). \quad (38)$$

In Eq. (38), a factor 4 is considered due to the symmetry between particle 1 and 2 as well as the symmetry between momenta p_1 and p_2 . The χ term contributes a small q^2 in the forwarding scattering approximation [10]. It is specified as

$$[\chi(\boldsymbol{p}_4')-\chi(\boldsymbol{p}_1)]^2\simeq[\chi(\boldsymbol{p}_3)-\chi(\boldsymbol{p}_2)]^2 \simeq\omega^2[\chi(p_1)']^2+3\frac{q^2-\omega^2}{p_1^2}[\chi(p_1)]^2. \quad (39)$$

In the dk integration, we should consider infrared divergence. The upper cutoff is $q\sim T$, and the lower cutoff occurs $q\sim\alpha_s T$, because we have considered the high temperature approximation with $\alpha_s\ll 1$.

$$\int_0^\infty \mathrm{d}k\frac{1}{k}[1+f_0(\boldsymbol{k})]\simeq\int_{\alpha_s T}^{T}\mathrm{d}k\left(\frac{T}{k^2}+\frac{1}{2k}\right) = \frac{1}{\alpha_s}-1+\frac{1}{2}\ln\alpha_s^{-1}. \quad (40)$$

Here, we use $[1+f_0(k)]\simeq(\frac{T}{k}+\frac{1}{2})$ with the approximation $\boldsymbol{k}\to 0$ in the soft gluon bremsstrahlung $gg\leftrightarrow ggg$ process.

Then, the integration over $\mathrm{d}p_2$ can be carried out,

$$\int_0^\infty p_2^2\mathrm{d}p_2 f_0(p_2)[1+f_0(p_2)]=\frac{\pi^2}{3}T^3. \quad (41)$$

With the above integration over $\mathrm{d}k$ and $\mathrm{d}p_2$, the $gg\leftrightarrow gg$ collision integrals can be simplified as

$$(\chi_{ij},\mathcal{C}^{gg\leftrightarrow ggg}\chi_{ij}) = \frac{54\alpha_s^3\ln\alpha_s^{-1}T^2}{\pi^2}\left(\frac{1}{\alpha_s}-1+\frac{1}{2}\ln\alpha_s^{-1}\right) \times\int_0^\infty \mathrm{d}p_1 f_0(p_1)[1\pm f_0(p_1)]\times([\chi(p_1)']^2+\frac{6}{p_1^2}[\chi(p_1)]^2). \quad (42)$$

4 Results of variational method

For calculating Eq. (21), we can determine the variational function $\chi(p)$. In this subsection, we will expand the variational function $\chi(p)$ by a finite set of bases, and maximize the functional $Q[\chi]$.

We know that in the case of viscosities, the variational function χ only has one component in gluon-plasma. Expanding the $\chi(p)$ on a set of bases

$$\chi(p)=\sum_n a_n\phi_n(p), \quad (43)$$

one can get the basis set components of $\widetilde{S}$ and $\widetilde{C}$,

$$(\chi,\mathcal{S})=\sum_n a_n\widetilde{S}_n,\quad (\chi,\mathcal{C}\chi)=\sum_{m,n}a_m\widetilde{C}_{mn}a_n. \quad (44)$$

Here,

$$\widetilde{S}_n^a\equiv N_g\int_{\boldsymbol{p}}\phi_n(p)\mathcal{S}_n^a(p)=(\phi(p),\mathcal{S}(p)), \quad (45)$$

$$\widetilde{C}_{mn}^{ab}\equiv N_g\int_{\boldsymbol{p}}\phi_m(p)\mathcal{C}^{ab}\phi_n(p)=(\phi(p),\mathcal{C}\phi(p)). \quad (46)$$

When the Boltzmann equation is satisfied, the coefficients a_n in front of the bases will be expressed as $a=\widetilde{C}^{-1}\widetilde{S}$.

With the natural function ansatz $\phi_1(p)=\frac{p^2}{T}$, one can evaluate the integral of $\widetilde{S}$ and $\widetilde{C}$.

4.1 Shear viscosity

Accordingly, the shear viscosity becomes

$$\eta=\frac{2}{15}\widetilde{Q}_\eta|_{\max}=\frac{1}{15}\widetilde{S}_\eta^\top\widetilde{C}^{-1}\widetilde{S}_\eta. \quad (47)$$

Here, $\widetilde{S}_\eta$ includes $q_\eta(\boldsymbol{p})=|\boldsymbol{p}|$.

In the section on effective kinetic theory, we wrote the linearized Boltzmann equation as Eq. (21). In gluon-plasma the left-hand side can be written directly as:

$$(\chi,\mathcal{S}_\eta)=-N_g\beta^2\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}q_\eta(\boldsymbol{p})f_0(p)[1+f_0(p)]\chi(p). \quad (48)$$

From Eq. (45), we substitute $q_\eta(\boldsymbol{p})$ in the $\widetilde{S}_\eta$.

So

$$\widetilde{S}_\eta=-100.86T^3. \quad (49)$$

From Eq. (46), combining Eq. (30)and Eq. (42), we can fulfil the collision integrals of the $gg\leftrightarrow gg$ process and $gg\leftrightarrow ggg$ process:

$$\widetilde{C}^{gg\leftrightarrow gg}=3968.8\alpha_s^2\ln\alpha_s^{-1}T^3, \quad (50)$$

and

$$\widetilde{C}^{gg\leftrightarrow ggg}=180\alpha_s^2T^3\left[\ln\alpha_s^{-1}-\alpha_s\ln\alpha_s^{-1}+\frac{\alpha_s}{2}(\ln\alpha_s^{-1})^2\right]. \quad (51)$$

On the basis of the above, the shear viscosity can be obtained from Eq. (47)

$$\eta_{22}=\frac{0.171T^3}{\alpha_s^2\ln\alpha_s^{-1}}, \quad (52)$$

$$\eta\approx\frac{0.163T^3}{\alpha_s^2\ln\alpha_s^{-1}}+\frac{0.0071T^3}{\alpha_s\ln\alpha_s^{-1}}-\frac{0.0035T^3}{\alpha_s}. \quad (53)$$

Here, η_{22} is the leading-log order shear viscosity which only considers the contribution of the $gg\leftrightarrow gg$ forward scattering process, which agrees with that of Refs. [10, 13, 29], and η is the full shear viscosity including both the $gg\leftrightarrow gg$ forward scattering process and the soft gluon bremsstrahlung $gg\leftrightarrow ggg$ process. In Eq. (53), the first term is the leading-log formula. It is smaller than η_{22}, because the correction of the soft gluon bremsstrahlung $gg\leftrightarrow ggg$ process depresses the leading-log shear viscosity to about $\sim$5% level. The last two terms are the higher order corrections with shear viscosity from the $gg\leftrightarrow ggg$ process, which are far smaller than the leading-log term with $\alpha_s\ll 1$ in the the high-temperature system.

Figure 2 shows the ratio η/η_{22}, and we can obtain the correction at the $\sim$5% level. Our results agree with AMY and ADM [11, 30], who found that $g\leftrightarrow gg$ only contributes at $\sim$10% level for the three flavor quark diffusion constant for $\alpha_s<0.3$, and with Q. Wang [13] for vary small α_s [9, 13].

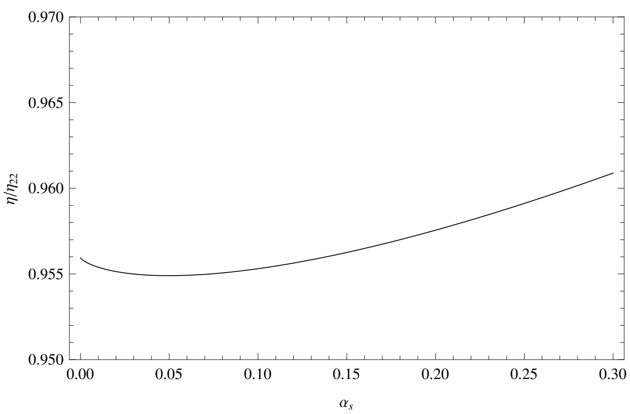

Fig. 2. Ratio of the full shear viscosity η to leading-log order shear viscosity η_{22}.

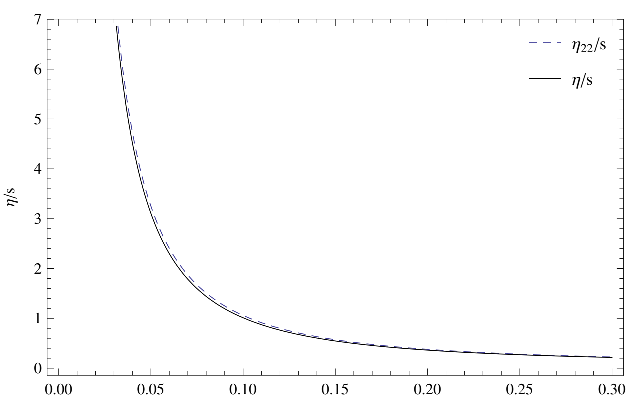

Fig. 3. The solid line is the shear viscosity η to entropy density s ratio η/s; the dashed line is η_{22}/s.

We use the entropy density for non-interacting gluons, $s=N_g\frac{2\pi^2}{45}T^3$. In Fig. 3, we give the function η/s of α_s for two cases, η_{22}/s and η/s, both of which decrease rapidly with increasing α_s. We calculate the collision integrals of the $gg\leftrightarrow gg$ process $\widetilde{C}^{gg\leftrightarrow gg}$ in Eq. (30), and the $gg\leftrightarrow ggg$ process $\widetilde{C}^{gg\leftrightarrow ggg}$ in Eq. (42). $\widetilde{C}^{gg\leftrightarrow ggg}$ is just a small correction to $\widetilde{C}^{gg\leftrightarrow gg}$.

Figure 4 shows the relationship between shear viscosity η/s and temperature. To isolate the temperature dependence, we use a simple form of the strong coupling constant in a high temperature system [31],

$$\alpha_s=\frac{2\pi}{11}\left(\ln\frac{4T}{1.5T_c}\right)^{-1} \tag{54}$$

Here T_c is the critical temperature of the QCD matter phase transition. We use T_c=175 MeV.

We use an analytical method to solve the linearized Boltzmann equation. The dominant contribution of shear viscosity η (the first term in Eq. (53)) considering the correction of the soft gluon bremsstrahlung $gg\leftrightarrow ggg$ process, and the leading-log η_{22} only counting $gg\leftrightarrow gg$ scattering process, have the same form $\sim\frac{T^3}{\alpha_s^2\ln\alpha_s^{-1}}$. In Fig. 1, there are three vertexes, providing g^6 (or α_s^3) in the matrix element. But, when calculating the phase space integrals dk of soft gluons, considering infrared divergence, we do infrared cutoff and get an enhancement factor α_s^{-1}, such as the first term in Eq. (40). Therfore, α_s^2 appears in the denominator of shear viscosity.

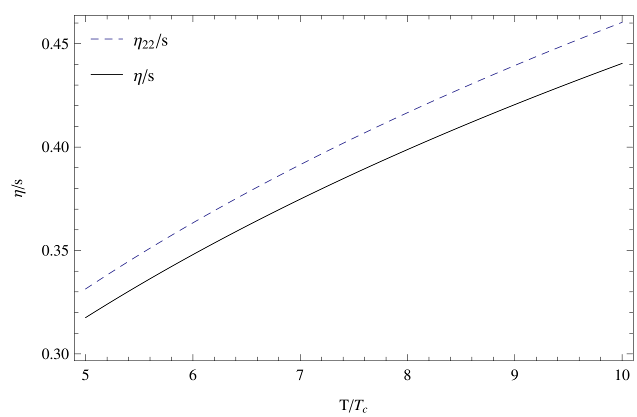

Fig. 4. The shear viscosity η to entropy density s ratio, as a function of the ratio of temperature to the critical temperature T_c=175 MeV. The solid line is the full shear viscosity η to entropy density s ratio η/s; the dashed line is η_{22}/s.

4.2 Bulk viscosity

The bulk viscosity is:

$$\zeta=2\widetilde{Q}_\zeta|_{\max}=\widetilde{S}_\zeta^\top\widetilde{C}^{-1}\widetilde{S}_\zeta, \tag{55}$$

and,

$$\widetilde{S}_\zeta=(\chi,\mathcal{S}_\zeta)=-N_g\beta^2\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}q_\zeta(\boldsymbol{p})f_0(p)[1+f_0(p)]\chi(p), \tag{56}$$

with $q_\zeta(\boldsymbol{p})=(\frac{1}{3}-v_s^2)|\boldsymbol{p}|$ included in $\widetilde{S}_\zeta$.

Then we substitute $q_\zeta(\boldsymbol{p})$ in $\widetilde{S}_\zeta$:

$$\widetilde{S}_\zeta = -23.4192\alpha_s^2 T^3. \tag{57}$$

Combining the results of the collision integrals of the $gg \leftrightarrow gg$ process (Eq. (50)) and the $gg \leftrightarrow ggg$ process (Eq. (51)), we can get the bulk viscosity of the gluon-plasma:

$$\zeta = \frac{0.132\alpha_s^2 T^3}{\ln\alpha_s^{-1}} + \frac{0.0057\alpha_s^3 T^3}{\ln\alpha_s^{-1}} - 0.0029\alpha_s^3 T^3. \tag{58}$$

In Eq. (58), we give the form of bulk viscosity in high temperature gluon plasma: $\zeta \sim \frac{\alpha_s^2 T^3}{\ln\alpha_s^{-1}}$. Comparing to the shear viscosity Eq. (53),

$$\zeta/\eta = 15(\frac{1}{3} - v_s^2)^2 = 0.81\alpha_s^4. \tag{59}$$

This result is dependent on $q_\kappa(\boldsymbol{p})$ (Eq. (17)) and shear and bulk viscosity with the variational function (28), which is the same as the emission of light quanta by the medium [32], $\zeta \ll \eta$, because $\mathcal{S}_\zeta(\boldsymbol{p})$ contains $(\frac{1}{3} - v_s^2) \sim \alpha_s^2$ in the variational method.

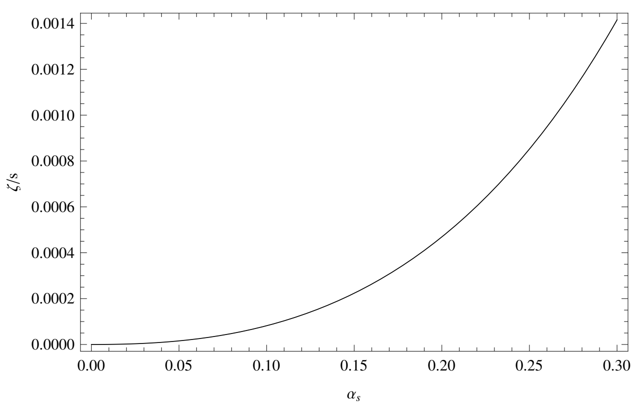

Fig. 5. Bulk viscosity ζ/s as a function of strong coupling coefficient α_s in high temperature gluon-plasma system.

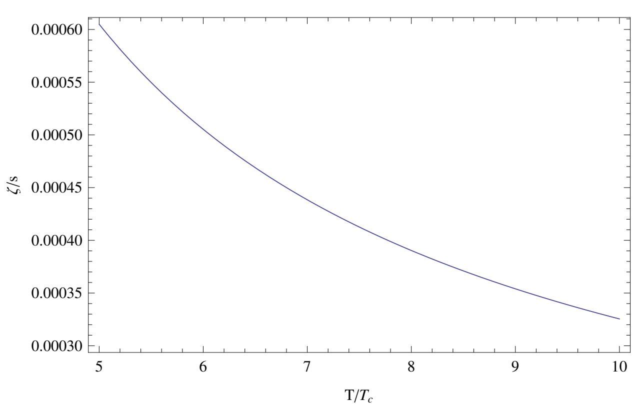

Fig. 6. The bulk viscosity ζ to entropy density s ratio as a function of the temperature ratio T/T_c.

In a high temperature gluon-plasma system, the bulk viscosity is much smaller than shear viscosity. In Fig. 5, we give the growth of ζ/s with α_s. The shear viscosity to entropy density η/s ratio as a function of α_s, however, decreases in Fig. 3. Moreover, contrary to the growth trend of η/s in Fig. 4, ζ/s decreases as T increases in Fig. 6.

In Fig. 7, we show that η/s decreases quickly with the increase of α_s, but ζ/s increases slowly with α_s. ζ/s is very small; when $\alpha_s = 0.2$, $\zeta/s \simeq 4.6\times10^{(-4)} \ll 1$. Also, $\eta/s \gg \zeta/s$; when $\alpha_s = 0.187$, η/s is 1000 times larger than ζ/s.

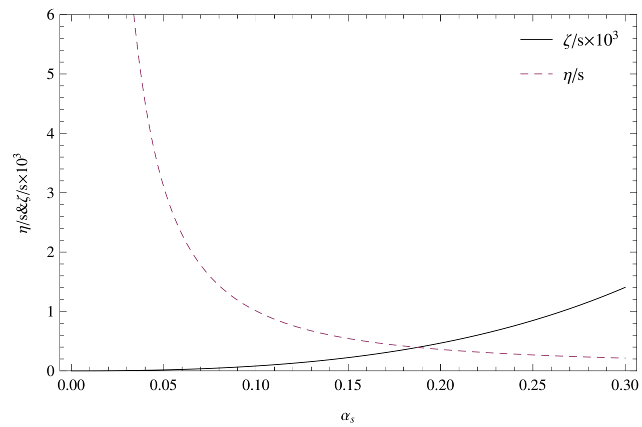

Fig. 7. Shear versus bulk viscosity: η/s and ζ/s as a function of α_s. The solid line is $\zeta/s\times10^3$, and the dashed line is η/s.

5 Conclusion

In this paper, we use the variational method to solve the linearized Boltzmann equation, getting the shear and bulk viscosity of gluon-plasma in a high-temperature system, including both the elastic $gg \leftrightarrow gg$ forward scattering and the inelastic soft gluon bremsstrahlung $gg \leftrightarrow ggg$ processes. On the basis of leading-log order result of shear viscosity, we calculate the correction of the $gg \leftrightarrow ggg$ soft gluon bremsstrahlung process. Calculating bulk viscosity, we consider the $gg \leftrightarrow ggg$ soft gluon bremsstrahlung process as the process for particle number violation and give our result. We will consider the correction of the Debye mass m_D [9, 11, 13], LPM corrections [8, 13, 33] and all $2 \leftrightarrow 3$ number changing processes in QCD. In this paper, we only use the basis function $\phi_1(p)$ in the variational method. If more basis functions are considered, a more accurate result may be obtained.

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