W-hairs of the black holes in three-dimensional spacetime*

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Abstract: In a previous publication, we claimed that a black hole can be considered as a topological insulator. A direct consequence of this claim is that their symmetries should be related. In this paper, we give a representation of the near-horizon symmetry algebra of the BTZ black hole using the $W_{1+\infty}$ symmetry algebra of the topological insulator in three-dimensional spacetime. Based on the $W_{1+\infty}$ algebra, we count the number of the microstates of the BTZ black holes and obtain the Bekenstein-Hawking entropy.

Keywords: BTZ black hole, near-horizon symmetry, $W_{1+\infty}$ symmetry, W-hairs

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1 Introduction

The information paradox [1] is still a challenging problem in theoretical physics. It says that the evaporation of the black hole breaks unitarity. There have been many proposals for its resolution, see Refs. [2, 3] for some reviews. In 2016, Hawking, Perry and Strominger [4] suggested to use "soft hairs" to solve this paradox. They proposed that the black hole microstates could be related to the soft hairs, that is, the zero energy excitations on the horizon. Since then, there have been a number of works along this direction, see Refs. [5, 6] and references therein.

A related concept, named "horizon fluff", was presented in Refs. [7–9]. Based on a new near-horizon boundary condition [10, 11], a novel near-horizon symmetry, which is an infinite copy of the Heisenberg algebra, was obtained. The horizon fluff forms a finite subset of the related "soft Heisenberg hairs". Using this algebra one can generate descendants of the physical states which are interpreted as black hole microstates in three-dimensional spacetime. The number of these microstates determines the entropy of the black holes.

In previous publications [12, 13], we claimed that the black hole can be considered as a quantum spin Hall state in three-dimensional spacetime. The quantum Hall states have a dynamical infinite dimensional symmetry group – the $W_{1+\infty}$ group , which is the quantum version of the area-preserving diffeomorphism group. On the other hand, for black holes, it also has an infinite dimensional symmetry group – the near-horizon symmetry group. In this paper, we derive a representation of the latter group from the former one. Based on the $W_{1+\infty}$ algebra, we give the "W-hairs" [14] of the BTZ black hole. Actually, the $W_{1+\infty}$ algebra was used to retain the information in two-dimensional stringy black holes [15–18]. This algebra also appears in the spectrum of the Hawking radiation [19–24].

The paper is organized as follows. In Section 2, the representation of $W_{1+\infty}$ for black holes is outlined. The embedding of the near-horizon symmetry algebra into the $W_{1+\infty}$ algebra is obtained. In Section 3, following the horizon fluff proposal, the W-hairs of the BTZ black hole are proposed, and the Bekenstein-Hawking entropy is obtained. Section 4 contains the conclusion.

2 Near-horizon symmetry algebra from $W_{1+\infty}$ symmetry algebra

The generators V^i_n of the $W_{1+\infty}$ algebra are characterized by a mode index $n\in Z$ and a conformal spin $h=i+1$, and satisfy the algebra [25]

$$[V^i_n,V^j_m]=(jn-im)V^{i+j-1}_{n+m}+q(i,j,n,m)V^{i+j-3}_{n+m}+\cdots+c^i(n)\delta^{ij}\delta_{m+n,0},\tag{1}$$

where $q(i,j,n,m)$ are the pertinent polynomials, and $c^i(n)$ represents the relativistic quantum anomaly. The dots stand for a series of terms involving the operators $V^{i+j-1-2k}_{n+m}$.

In the simplest case, the generators V^0_n and V^1_n form a

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sub-algebra of the $W_{1+\infty}$ algebra:

$$\begin{aligned}\left[V_n^0, V_m^0\right] &= nc\delta_{n+m,0},\\ \left[V_n^1, V_m^0\right] &= -mV_{n+m}^0,\\ \left[V_n^1, V_m^1\right] &= (n-m)V_{n+m}^1 + \frac{c}{12}n\left(n^2-1\right)\delta_{n+m,0},\end{aligned} \tag{2}$$

with the central charge $c=1$. It contains the abelian Kac-Moody algebra and the $c=1$ Virasoro algebra.

All unitary, irreducible, highest-weight representations were found by Kac and Radul [26, 27]. This result was applied to the incompressible quantum Hall fluid by Cappelli et al. [28, 29]. These representations exist only for positive integer central charges $c=m=1,2,\cdots$. If $c=1$, the representations are equivalent to those of the abelian Kac-Moody algebra $\widehat{U(1)}$ of $W_{1+\infty}$, corresponding to the edge excitations of the single abelian Chern-Simons theory. For $c=m=2,3,\cdots$, there are two kinds of representations, generic and degenerate, depending on the weight. The generic representations are equivalent to the corresponding representations of the multi-component abelian algebra $\widehat{U(1)}^m$, which corresponds to the edge excitations of the multiple abelian Chern-Simons theory. On the other hand, the degenerate representations are contained in the $\widehat{U(1)}^m$ representations.

Any unitary, irreducible representation contains a bottom state – the highest-weight state, and an infinite tow (descendants) above it. The highest-weight state $|\Omega\rangle$ is defined by the condition

$$V_n^i|\Omega\rangle = 0, \quad \forall n>0, i\geqslant 0. \tag{3}$$

Using the polynomials of $V_n^i(n<0)$ in $|\Omega\rangle$ gives the other excitations.

It was claimed that the black hole can be considered as a quantum spin Hall state in three-dimensional space-time [12, 13]. A quantum spin Hall state can be realized as a bilayer integer quantum Hall system with opposite T−symmetry. Hence, the symmetry algebra for a quantum spin Hall state is $W_{1+\infty}\otimes\bar{W}_{1+\infty}$, which has opposite chirality. For the integer quantum Hall fluid $c=1$, the representation is the same as of the $\widehat{U(1)}$ algebra. For black holes, the corresponding algebra is $W=\widehat{U(1)}\otimes\widehat{\bar{U}(1)}$, which has opposite chirality. This result can also be obtained from the Chern-Simons theory [30].

We consider now the representation of the algebra $W=\widehat{U(1)}\otimes\widehat{\bar{U}(1)}$ [31]. First, let us consider the chiral part $\widehat{U(1)}$. The generators α_n^+ satisfy

$$[\alpha_n^+,\alpha_m^+] = n\delta_{n+m,0}. \tag{4}$$

All V_n^i can be written as polynomials of the current modes α_n^+.

All unitary, irreducible representations can be built on top of the highest-weight state $|r_1\rangle, r_1\in R$, which satisfies

$$\alpha_n^+|r_1\rangle = 0 \quad (n>0), \quad \alpha_0^+|r_1\rangle = r_1|r_1\rangle. \tag{5}$$

A general descendant can be written as

$$|\{n_1,n_2,\cdots,n_s\}\rangle = \alpha_{-n_1}^+\alpha_{-n_2}^+\cdots\alpha_{-n_s}^+|r_1\rangle,\ \ n_1\geqslant n_2\geqslant\cdots\geqslant n_s>0. \tag{6}$$

Note that the operator α_0^+ commutes with all other generators, which means that the eigenvalues of α_0^+ are the same for all descendants in a given representation.

The Virasoro generator L_n^+ can be obtained using the Sugawara construction

$$L_n^+ = \frac{1}{2}\sum_{l\in Z} :\alpha_{n-l}^+\alpha_l^+:, \tag{7}$$

where :: means normal ordering. Acting on the highest-weight state, this gives

$$L_n^+|r_1\rangle = 0 \quad (n>0), \quad L_0^+|r_1\rangle = \frac{r_1^2}{2}|r_1\rangle. \tag{8}$$

which satisfies the Virasoro algebra in (2) with $c=1$.

Second, we consider the anti-chiral part. The generators $\bar{\alpha}_n^+$ satisfy

$$[\bar{\alpha}_n^+,\bar{\alpha}_m^+] = -n\delta_{n+m,0}. \tag{9}$$

The highest-weight state $|r_2\rangle, r_2\in R$ is defined by

$$\bar{\alpha}_n^+|r_2\rangle = 0 \quad (n<0), \quad \bar{\alpha}_0^+|r_2\rangle = r_2|r_2\rangle. \tag{10}$$

The generators $\bar{L}_n^+$ can also be obtained using the Sugawara construction (7), but unfortunately they do not satisfy the standard Virasoro algebra (2). However, it is possible to define new operators

$$\alpha_n^- \equiv \bar{\alpha}_{-n}^+, \quad L_n^- \equiv -\bar{L}_{-n}^+, \tag{11}$$

which indeed satisfy the standard algebras (2) and (7), and conditions (5) and (8).

Finally, we get two copies of the $\widehat{U(1)}$ algebra,

$$[\alpha_n^\pm,\alpha_m^\pm] = n\delta_{n+m,0}, \tag{12}$$

which is the same as the algebra in Ref. [7] , except for the irrelevant factor $1/2$.

With these algebras, one can construct the near-horizon symmetry algebra. Let us define

$$T_n = \alpha_n^+ + \alpha_{-n}^-, \quad Y_n = L_n^+ - L_{-n}^-. \tag{13}$$

It is easy to show that these operators satisfy the near-horizon symmetry algebra [32]

$$\begin{aligned}[T_m,T_n] &= 0,\\ [Y_m,T_n] &= -nT_{m+n},\\ [Y_m,Y_n] &= (m-n)Y_{m+n},\end{aligned} \tag{14}$$

where T_n generates a super-translation, and Y_n generates a super-rotation.

3 W-hairs of the BTZ black holes

In this section we discuss the representations of the algebra (12). According to the rules of the conformal field theory [33], these representations should be closed under the "fusion algebra". For $\widehat{U(1)}$, this just means addition of r. For $W=\widehat{U(1)}\otimes\widehat{\bar{U}(1)}$, the highest-weight states can be written as $|r_1,r_2\rangle, r_1,r_2\in R$. The operators (13) acting on this state give

$$T_0|r_1,r_2\rangle = (r_1+r_2)|r_1,r_2\rangle,\quad Y_0|r_1,r_2\rangle = \frac{r_1^2-r_2^2}{2}|r_1,r_2\rangle. \quad (15)$$

A general descendant can be written as

$$|\{n_i^\pm\}\rangle = \prod_{n_i^\pm}(\alpha^+_{-n_i^+}\alpha^-_{-n_i^-})|r_1,r_2\rangle,\quad n_1^\pm \geqslant n_2^\pm \geqslant \cdots \geqslant n_s^\pm > 0. \quad (16)$$

The operators acting on these states give

$$\begin{aligned} T_0|\{n_i^\pm\}\rangle &= (r_1+r_2)|\{n_i^\pm\}\rangle,\\ Y_0|\{n_i^\pm\}\rangle &= \left(\frac{r_1^2-r_2^2}{2}+\sum n_i^+ - \sum n_i^-\right)|\{n_i^\pm\}\rangle. \end{aligned} \quad (17)$$

The key problem is to choose which representations correspond to the BTZ black hole microstates. The metric of the BTZ black hole can be written as [34]

$$\mathrm{d}s^2 = -N^2\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2(\mathrm{d}\varphi + N^\varphi \mathrm{d}v)^2, \quad (18)$$

where $N^2 = -8GM + \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2}, N^\varphi = -\frac{4GJ}{r^2}$. M,J are the mass and the angular momentum of the BTZ black hole, respectively.

Following the horizon fluff proposal, we make the following assumption: the BTZ black hole microstates correspond to the descendants of the absolute vacuum state $|(r_1=0,r_2=0)\rangle$. Thus, the BTZ black hole microstates can be written as [7]

$$|B\{n_i^\pm\}\rangle = N\{n_i^\pm\}\prod_{n_i^\pm}(\alpha^+_{-n_i^+}\alpha^-_{-n_i^-})|0,0\rangle,\quad n_1^\pm \geqslant n_2^\pm \geqslant \cdots \geqslant n_s^\pm > 0, \quad (19)$$

where $N\{n_i^\pm\}$ is the normalization factor.

It is useful to compare the above results with the quantum Hall fluid. For the quantum Hall fluid, T_0 represents the electric charge and Y_0 the angular momentum of the quasi-particles. For black holes, the meaning of T_0 is unclear, but Y_0 still represents the angular momentum. Let us define another operator, $H = L_0^+ + L_0^-$, which is the dimensionless Hamiltonian. Then we can identify the BTZ black hole microstates with parameters (M,J) with the descendants $|B\{n_i^\pm\}\rangle$, which satisfy

$$\begin{aligned} \langle B'\{n_i^{'\pm}\}|Y_0|B\{n_i^\pm\}\rangle &= cJ\delta_{B',B},\\ \langle B'\{n_i^{'\pm}\}|H|B\{n_i^\pm\}\rangle &= cMl\delta_{B',B}, \end{aligned} \quad (20)$$

where $c = 3l/2G$ is the central charge [9]. Substituting (19) into (20) gives

$$\sum n_i^+ - \sum n_i^- = cJ,\quad \sum n_i^+ + \sum n_i^- = cMl. \quad (21)$$

The solution is very simple,

$$\sum n_i^+ = c\frac{Ml+J}{2},\quad \sum n_i^- = c\frac{Ml-J}{2}. \quad (22)$$

Different $\{n_i^\pm\}$ correspond to different microstates of the BTZ black hole with the same (M,J). The total number of microstates for the BTZ black hole with parameters (M,J) is given by the famous Hardy-Ramanujan formula [35],

$$p(N) \simeq \frac{1}{4N\sqrt{3}}\exp\left(2\pi\sqrt{\frac{N}{6}}\right). \quad (23)$$

The entropy of the BTZ black hole is given by the logarithm of the number of microstates $|B\{n_i^\pm\}\rangle$,

$$S = \ln p\left(c\frac{Ml+J}{2}\right) + \ln p\left(c\frac{Ml-J}{2}\right) + \cdots = \frac{2\pi r_+}{4G} + \cdots, \quad (24)$$

which is just the Bekenstein-Hawking entropy with low order corrections.

The next question is what do the other highest-weight states $|r_1,r_2\rangle, r_1,r_2\in R$ mean. Let us turn back to the quantum Hall fluid. In the quantum Hall fluid, there are two kinds of excitations: the neutral excitations and the charged excitations, which correspond to quasi-holes and quasi-particles in the bulk of the fluid. For the integer quantum Hall effect, the highest-weight state is the vacuum state $|0\rangle$. For the fractional quantum Hall effect, the other highest-weight states $|Q\rangle$ appear, which have fractional charges and statistics. In the corresponding case of black holes, the pure black hole could be associated with the absolute vacuum state $|0\rangle$, and the black holes interacting with matter could correspond to the other highest-weight states.

4 Conclusion

In this paper, we considered the infinite dimensional symmetry algebras in the quantum Hall fluid and in the BTZ black holes. For the quantum Hall fluid, this is the $W_{1+\infty}$ algebra. Different quantum Hall fluids have different central charges c , which correspond to different representations of this algebra [28, 29]. For the BTZ black hole, which can be considered as a special quantum spin Hall fluid with central charge $c=1$, the corresponding algebra is $W = \widehat{U(1)}\otimes\widehat{\bar{U}(1)}$. From this algebra, one can get easily the near-horizon symmetry algebra (14). The explicit form is given in (12). The near-horizon symmetry algebra is a sub-algebra of the full algebra W. Note that the near-horizon symmetry algebra depends on the choice of the boundary conditions, so that one could maybe find a weak boundary condition to get the full algebra $W = \widehat{U(1)}\otimes\widehat{\bar{U}(1)}$.

The infinite set of W-charges provides an infinite set of discrete gauge hairs (W-hairs) [14], which were used to maintain the quantum coherence of the two-dimensional stringy black hole. In this paper, we associated these W-hairs with the microstates of black holes, following the sprit of the horizon fluff proposal. The BTZ black hole microstates can be considered as the descendants of the absolute vacuum state, Eq. (19). For the BTZ black hole with parameters (M,J) , we counted the number of these microstates and obtained the Bekenstein-Hawking entropy. The essential difference with respect to the horizon fluff proposal is that we used the $W_{1+\infty}$ algebra instead of the Heisenberg algebra, even though their representations for black holes are very similar.

The near-horizon symmetry algebra is related to the fluid symmetry algebra in Ref. [36]. In this paper, we used an explicit fluid, the quantum Hall fluid. The microscopic structure of this fluid is fairly well understood. The relation between these two infinite dimensional algebras also provides further evidence for our claim that "a black hole can be considered as a kind of topological insulator". This claim relates the black hole physics with the condensed matter physics. It is also a starting point for relating gravity with non-trivial condensed matter systems [37–40].

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