Generalized uncertainty principle and black hole thermodynamics *

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Abstract: Banerjee-Ghosh's work shows that the singularity problem can be naturally avoided by the fact that black hole evaporation stops when the remnant mass is greater than the critical mass when including the generalized uncertainty principle (GUP) effects with first- and second-order corrections. In this paper, we first follow their steps to reexamine Banerjee-Ghosh's work, but we find an interesting result: the remnant mass is always equal to the critical mass at the final stage of black hole evaporation with the inclusion of the GUP effects. Then, we use Hossenfelder's GUP, i.e., another GUP model with higher-order corrections, to restudy the final evolution behavior of the black hole evaporation, and we confirm the intrinsic self-consistency between the black hole remnant and critical masses once more. In both cases, we also find that the thermodynamic quantities are not singular at the final stage of black hole evaporation.

Keywords: black hole thermodynamics, generalized uncertainty principle, remnant mass, critical mass

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1 Introduction

The generalized uncertainty principle (GUP), which modifies the uncertainty principle to include a minimal length, has received increasing amounts of attention in the past decade [1-4]. On the one hand, one may predict the GUP effects through various experiments, such as the hydrogen Lamb shift [5-7], electron tunneling [5, 6], mechanical oscillators [8, 9], gravitational bar detectors [10], ultra-cold atom experiments [11, 12], gravitational wave experiments [13, 14], sub-kilogram acoustic resonators [15], and large molecular wave-packets [16]. On the other hand, one can also apply the GUP to study the effects of quantum gravity on small- or large-scale physical systems. For example, the GUP effects have been studied with respect to the early Universe [17-21], compact stellar objects [22-24], the Newtonian law of gravity [25], the equivalence principle [26-28], the entropic nature of gravitational force [29-33], the Casimir effect [34, 35], the Dirac δ-function potential [36], and post-Newtonian potential [37].

As far as we are concerned, the GUP affects the well-known semi-classical laws of black hole thermodynamics [38-68]. For example, the black hole entropy is no longer proportional to the horizon area [51-59]; the black hole does not evaporate completely, but leaves a remnant mass at the final stage of evaporation [57-65]; the remnant with the Planck scale can store information, which gives a possible solution to the singularity problem [57-65]; and there is a metastable remnant that asymptotes to zero mass when considering the negative GUP correction [66]. In [68], Banerjee and Ghosh intriguingly constructed a GUP that contains the term predicted by string theory and a series of higher-order correction terms, and they studied the GUP effects on black hole thermodynamics. Their results show that, when considering the first- and second-order quantum corrections to black hole thermodynamics, black hole evaporation always stops when the remnant mass is greater than the critical mass, and the singularity problem in the semi-classical ap-

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proach is bypassed at the final stage of black hole evaporation.

However, Banerjee-Ghosh's results in [68] lack credibility because some necessary terms have been omitted in their treatment. For example, when dealing with the first-order correction, all terms regarding a'_1 should be included in the corrected temperature-mass relation. However, the term $a'^2_1(k_BT/M_pc^2)^2$ has been omitted in their treatment, perhaps because they consider this term to be negligible. When dealing with the second-order correction, Banerjee and Ghosh have omitted the necessary terms $2a'_1a'_2(k_BT/M_pc^2)^4$ and $a'^2_2(k_BT/M_pc^2)^6$. If these omitted terms are recovered, the final evolution behavior of black hole evaporation may be different. Anyway, Banerjee-Ghosh's work cannot truly demonstrate the final evolution behavior of a black hole system with the inclusion of the GUP effects.

In this paper, we reexamine Banerjee-Ghosh's work in Sec. 2, and we restudy the final evolution behavior of black hole evaporation when including the GUP effects with first- and second-order corrections. In Sec. 3, we review the GUP proposed by Hossenfelder *et al.* in [69], i.e., another GUP model with higher-order corrections. In Sec. 4, we use Hossenfelder's GUP to precisely study first- and second-order quantum corrections to black hole thermodynamics, and we aim to discover the intrinsic self-consistency between the black hole remnant and critical masses when including the effects of quantum gravity. Finally, Sec. 5 provides some conclusions.

2 Reexamination of Banerjee-Ghosh's work

In [68], Banerjee and Ghosh have assumed that the function relation between the wave vector k and the momentum p satisfies certain properties: 1) the function has to be an odd function to preserve parity; 2) the function should be chosen to satisfy $p=\hbar k$ at small energy; and 3) the wave vector k should have an upper bound of $2\pi/L_p$. Thus, Banerjee and Ghosh have assumed an infinite-order polynomial to satisfy these properties of the function, which is expressed as

$$k=f(p)=\frac{1}{L_p}\sum_{i=0}^{\infty}a_i(-1)^i\left(\frac{L_pp}{\hbar}\right)^{2i+1}. \tag{1}$$

Here, only odd powers of the momentum p appear in the polynomial because the function $f(p)$ is an odd function to preserve parity. The coefficients $\{a_i\}$ are all positive, and $a_0=1$ to recover $p=\hbar k$ at small energy. The factor $(-1)^i$ ensures property 3), and we have a constraint for $p\to\infty$, $k\to\frac{2\pi}{L_p}$, i.e.

$$\sum_{i=0}^{\infty}a_i(-1)^i\left(\frac{L_pp}{\hbar}\right)^{2i+1}\to 2\pi. \tag{2}$$

From (1), we can obtain

$$\frac{\partial p}{\partial k}=\hbar\sum_{i=0}^{\infty}a'_i\left(\frac{L_pp}{\hbar}\right)^{2i}, \tag{3}$$

where the new coefficients of expansions $\{a'_i\}$ are functions of $\{a_i\}$, and $a'_0=1$. Hence, the form of the GUP proposed by Banerjee and Ghosh is given by

$$\Delta x\Delta p\geqslant\frac{\hbar}{2}\sum_{i=0}^{\infty}a'_i\left(\frac{L_p\Delta p}{\hbar}\right)^{2i}, \tag{4}$$

where the coefficients $\{a'_i\}$ are all positive.

Subsequently, we use the GUP (4) to study the quantum-corrected thermodynamic entities of a Schwarzschild black hole and attempt to find relationships among them. In [68], by comparison with the standard semi-classical Hawking temperature, the mass-temperature relationship of a Schwarzschild black hole is given by

$$M=\frac{M_p}{8\pi}\sum_{i=0}^{\infty}a'_i\left(\frac{k_BT}{M_pc^2}\right)^{2i-1}. \tag{5}$$

According to the definition of the heat capacity of a black hole $C=c^2\dfrac{\mathrm{d}M}{\mathrm{d}T}$, we have

$$C=\frac{k_B}{8\pi}\sum_{i=0}^{\infty}a'_i(2i-1)\left(\frac{k_BT}{M_pc^2}\right)^{2i-2}. \tag{6}$$

The entropy of a black hole is given by

$$S=\int\frac{C\mathrm{d}T}{T}=\frac{k_B}{16\pi}\left[\left(\frac{M_pc^2}{k_BT}\right)^2+a'_1\ln\left(\frac{k_BT}{M_pc^2}\right)^2+\sum_{i=2}^{\infty}a'_i\frac{(2i-1)}{(i-1)}\left(\frac{k_BT}{M_pc^2}\right)^{2(i-1)}\right]. \tag{7}$$

In [68], because the heat capacity and entropy are expressed in terms of the mass, the expression for T^2 in terms of M is given by

$$\left(\frac{8\pi M}{M_p}\right)^2=\left(\frac{M_pc^2}{k_BT}\right)^2+2a'_1+(a'^2_1+2a'_2)\left(\frac{k_BT}{M_pc^2}\right)^2+2(a'_1a'_2+a'_3)\left(\frac{k_BT}{M_pc^2}\right)^4+(2a'_1a'_3+a'^2_2)\left(\frac{k_BT}{M_pc^2}\right)^6+\cdots. \tag{8}$$

In Banerjee-Ghosh's treatment, when dealing with the first-order correction, they have obtained the corrected mass-temperature relation as

$$\left(\frac{8\pi M}{M_p}\right)^2=\left(\frac{M_pc^2}{k_BT}\right)^2+2a'_1. \tag{9}$$

In fact, all terms concerning a'_1 should be included in the first-order correction, so the corrected Mass-Temperature relation should be written as

$$\left(\frac{8\pi M}{M_p}\right)^2=\left(\frac{M_pc^2}{k_BT}\right)^2+2a_1'+a_1'^2\left(\frac{k_BT}{M_pc^2}\right)^2. \tag{10}$$

Therefore, the term $a_1'^2\left(\frac{k_BT}{M_pc^2}\right)^2$ has been omitted in Banerjee-Ghosh's treatment. Based on Eq. (9), Banerjee and Ghosh further obtained the result that the remnant mass is greater than the critical mass when considering the first-order correction.

Based on Eq. (10), where the omitted term is recovered in the first-order correction, we can obtain

$$\left(\frac{k_BT}{M_pc^2}\right)^2=\frac{2}{\left(\frac{8\pi M}{M_p}\right)^2-2a_1'+\left(\frac{8\pi M}{M_p}\right)\sqrt{\left(\frac{8\pi M}{M_p}\right)^2-4a_1'}}. \tag{11}$$

Obviously, as a thermodynamic system, the black hole has a *critical* mass below (at) which the thermodynamic entities become complex (ill-defined) [54, 68], and which is given by

$$M_{cr}=\frac{\sqrt{a_1'}}{4\pi}M_p. \tag{12}$$

From (6) and (11), the heat capacity with the first-order correction is given by

$$C=\frac{k_B}{8\pi}\left[-\frac{\left(\frac{8\pi M}{M_p}\right)^2-2a_1'+\left(\frac{8\pi M}{M_p}\right)\sqrt{\left(\frac{8\pi M}{M_p}\right)^2-4a_1'}}{2}+a_1'\right]. \tag{13}$$

When the heat capacity becomes zero at the final stage of black hole evaporation [58, 65, 66], the *remnant* mass is obtained as follows

$$M_{\rm rem}=\frac{\sqrt{a_1'}}{4\pi}M_p. \tag{14}$$

Thus, the remnant mass is equal to the critical mass when including the necessary term $a_1'^2\left(\frac{k_BT}{M_pc^2}\right)^2$ for the first-order correction term.

Next, let us focus on the effects of the second-order correction. The mass-temperature relation should be given here, according to (8), as

$$\left(\frac{8\pi M}{M_p}\right)^2=\left(\frac{M_pc^2}{k_BT}\right)^2+2a_1'+(a_1'^2+2a_2')\left(\frac{k_BT}{M_pc^2}\right)^2+2a_1'a_2'\left(\frac{k_BT}{M_pc^2}\right)^4+a_2'^2\left(\frac{k_BT}{M_pc^2}\right)^6. \tag{15}$$

However, in Banerjee-Ghosh's treatment, the contributions of the correction terms $2a_1'a_2'\left(\frac{k_BT}{M_pc^2}\right)^4$ and $a_2'^2\left(\frac{k_BT}{M_pc^2}\right)^6$ were both omitted [68]. When the omitted terms are recovered in the second-order correction, we have

$$\frac{k_BT}{M_pc^2}=\frac{1}{2}\sqrt{-\frac{2a_1'}{3a_2'}+B+D}-\frac{1}{2}\sqrt{-\frac{4a_1'}{3a_2'}-B-D+\frac{2\times(8\pi M/M_p)}{a_2'\sqrt{-2a_1'/3a_2'+B+D}}}, \tag{16}$$

where

$$B=\frac{2^{1/3}(a_1'^2+12a_2')}{3a_2'\left(F+\sqrt{F^2-G}\right)^{1/3}}, \tag{17}$$

$$D=\frac{\left(F+\sqrt{F^2-G}\right)^{1/3}}{3\times2^{1/3}a_2'}, \tag{18}$$

$$F=2a_1'^3-72a_1'a_2'+27a_2'(8\pi M/M_p)^2, \tag{19}$$

$$G=4(a_1'^2+12a_2')^3, \tag{20}$$

From Eq. (16), the critical mass below (at) which the thermodynamic entities become complex (ill-defined) can be determined by $F^2-G\geqslant0$, that is

$$\left(\frac{8\pi M_{cr}}{M_p}\right)=\frac{1}{3}\sqrt{\frac{2}{3}}\sqrt{36a_1'-\frac{a_1'^3}{a_2'}+\sqrt{(a_1'^2+12a_2')^3}}. \tag{21}$$

From (6), the heat capacity with the second-order correction can now be written as

$$C=\frac{k_B}{8\pi}\left[-\left(\frac{M_pc^2}{k_BT}\right)^2+a_1'+3a_2'\left(\frac{k_BT}{M_pc^2}\right)^2\right]. \tag{22}$$

Then, at the final stage of black hole evaporation, the *remnant* mass with the second-order correction is given by

$$\left(\frac{8\pi M_{\rm rem}}{M_p}\right)=\frac{1}{3}\sqrt{\frac{2}{3}}\sqrt{\frac{-a_1'+\sqrt{(a_1'^2+12a_2')^3}}{a_2'}}\times\left(2a_1'+\sqrt{(a_1'^2+12a_2')^3}\right)=\frac{1}{3}\sqrt{\frac{2}{3}}\sqrt{36a_1'-\frac{a_1'^3}{a_2'}+\sqrt{(a_1'^2+12a_2')^3}}. \tag{23}$$

It is clear that when the omitted terms $2a_1'a_2'\left(\frac{k_BT}{M_pc^2}\right)^4$ and $a_2'^2\left(\frac{k_BT}{M_pc^2}\right)^6$ are recovered in the second-order correction, the remnant mass is also equal to the critical mass.

In this section, it is found that, in Banerjee-Ghosh's work, some necessary terms, e.g., the term $a_1'^2\left(\frac{k_BT}{M_pc^2}\right)^2$ for the first-order correction and the terms $2a_1'a_2'\left(\frac{k_BT}{M_pc^2}\right)^4$

and $a_2'^2\left(\frac{k_BT}{M_pc^2}\right)^6$ for the second-order correction, have been omitted when considering the GUPeffects on the final evolution behavior of black hole evaporation. In fact, these omitted terms become necessary because without them, the final evolution behavior of the black hole system cannot truly emerge with the inclusion of the GUP effects, as discussed above. When these omitted terms are recovered in the first- and second-order quantum corrections, the black hole always stops evaporation when the remnant mass is equal to the critical mass. In the following section, we use another GUP model with higher-order corrections to restudy the final evolution behavior of black hole evaporation, and we attempt to confirm the intrinsic self-consistency between the black hole remnant and critical mass once again.

3 New generalized uncertainty principle proposed by Hossenfelder *et al.*

To implement the notion of a minimal length L_f, Hossenfelder *et al.* have assumed that particles cannot possess arbitrarily small Compton wavelengths ($\lambda = 2\pi/k$); then, the vector k has an upper bound [69]. This effect would show up when p approaches a certain scale M_f. To incorporate this behavior, they have assumed that the relation $k(p)$ between p and k is an uneven function (because of parity) and asymptotically approaches $1/L_f$. Thus, Hossenfelder *et al.* have assumed the function behavior of $k(p)$ is [69]

$$L_f k(p) = \tanh^{1/\gamma}\left[\left(\frac{p}{M_f c}\right)^{\gamma}\right], \tag{24}$$

where γ is a positive constant and L_f and M_f satisfy the relation $L_f M_f c = \hbar$. For simplicity, we set $\gamma = 1$. Expanding the modified relation (24), there are two cases: (a) the regime of expanding $\tanh(x)$ for small arguments (i.e., $|x| < \frac{\pi}{2}$); and (b) the high-energy limit $p \gg M_f$.

For case (a), its expanding expression is given by

$$k(p) = \frac{1}{L_f}\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}\left(\frac{L_f p}{\hbar}\right)^{2n-1}, \tag{25}$$

where B is the Bernoulli number. The expanding expression (25) can also be found in Banerjee-Ghosh's relation (1). That is to say, Hossenfelder's relation (24) between the wave vector and the momentum exhibits much more physics because its expanding expression not only contains the regime of Banerjee-Ghosh's relation, but also includes the high-energy limit $p \gg M_f$.

According to Eq. (25), we have

$$\frac{1}{\hbar}\frac{\partial p}{\partial k} = 1 + \left(\frac{p}{M_f c}\right)^2 + \frac{1}{3}\left(\frac{p}{M_f c}\right)^4 + \frac{2}{45}\left(\frac{p}{M_f c}\right)^6 + \cdots. \tag{26}$$

According to the well-known commutation relation

$$[\hat{x}, \hat{p}(\hat{k})] = i\frac{\partial p}{\partial k}, \tag{27}$$

the uncertainty relation is given by

$$\Delta x \Delta p \geqslant \frac{1}{2}\left|\left\langle\frac{\partial p}{\partial k}\right\rangle\right|. \tag{28}$$

From (26) and (28), we can obtain

$$\Delta x \Delta p \geqslant \frac{\hbar}{2}\left(1 + \frac{\langle\hat{p}^2\rangle}{M_f^2 c^2} + \frac{1}{3}\frac{\langle\hat{p}^4\rangle}{M_f^4 c^4} + \frac{2}{45}\frac{\langle\hat{p}^6\rangle}{M_f^6 c^6} + \cdots\right). \tag{29}$$

Here, $\langle p^{2i}\rangle \geqslant \langle p^2\rangle^i$ has been used. For a minimal position uncertainty, we have $\langle p\rangle = 0$, so Hossenfelder's GUP is given by

$$\Delta x \Delta p \geqslant \frac{\hbar}{2}\left[1 + f^2\left(\frac{\Delta p}{M_p c}\right)^2 + \frac{f^4}{3}\left(\frac{\Delta p}{M_p c}\right)^4 + \frac{2f^6}{45}\left(\frac{\Delta p}{M_p c}\right)^6 + \cdots\right]. \tag{30}$$

Here, $M_f = M_p/f$, and M_p is the Planck mass. It is noteworthy that Hossenfelder's GUP contains not only the term described in string theory [70, 71], but also higher-order quantum corrections. However, the GUP derivations are phenomenological and normally take only the first-order or second-order correction as the subdominant one; there is nothing new about higher orders here. In the next section, we will use Hossenfelder's GUP to precisely reexamine the first- and second-order corrections to black hole thermodynamics.

4 Black hole thermodynamics with corrections

Near the horizon of a Schwarzschild black hole, when the production of a particle-antiparticle pair occurs as a result of quantum fluctuation in a vacuum, the particle with negative energy falls into the horizon, and that with positive energy escapes to outside the horizon and is detected by an observer at infinity. For simplicity, we consider the emitted particle to be massless and its spectrum to be thermal. Therefore, we have [58]

$$k_B T = \triangle p c, \tag{31}$$

where k_B is the Boltzmann constant, and the momentum of the emitted particle p is on the order of its momentum uncertainty $\triangle p$. For thermodynamic equilibrium, the temperature of the emitted particle is identified with the temperature of the black hole itself. Moreover, near the horizon of a Schwarzschild black hole, the position uncertainty of a particle will be on the order of the Schwarzschild radius for the black hole [58, 59]. Consequently,

$$\Delta x = r_s = \frac{2GM}{c^2}, \tag{32}$$

where r_s is the Schwarzschild radius, G is Newton's gravitational constant, c is the speed of light, and M is the mass of a Schwarzschild black hole.

We can now relate the temperature T with the mass M of the black hole by recasting Hossenfelder's GUP (30) in terms of T and M. Hossenfelder's GUP (30) with first-order and second-order corrections is given by

$$\Delta x\Delta p = \epsilon\frac{\hbar}{2}\left[1+f^2\left(\frac{\Delta p}{M_pc}\right)^2+\frac{f^4}{3}\left(\frac{\Delta p}{M_pc}\right)^4\right], \tag{33}$$

where the parameter ϵ is a scale factor saturating the uncertainty relation. It should be noted that we take only the first-order or second-order correction as the subdominant one because there is nothing new about higher orders here. Substituting Eqs. (31) and (32) into Eq. (33), Hossenfelder's GUP in terms of T and M is recast as

$$M = \epsilon\frac{M_p}{4}\left[\left(\frac{M_pc^2}{k_BT}\right)+f^2\left(\frac{k_BT}{M_pc^2}\right)+\frac{f^4}{3}\left(\frac{k_BT}{M_pc^2}\right)^3\right], \tag{34}$$

where $M_p = L_pc^2/G$ and $c\hbar/L_p = M_pc^2$ have been used. If Hossenfelder's GUP effects are not considered (i.e., $f = 0$ in (34)), the semi-classical mass-temperature relation is reproduced by $T = M_p^2c^2/(8\pi k_BM)$ [72]. We thus fix the calibration factor $\epsilon = 1/2\pi$. Therefore, the corrected mass-temperature relation with the first- and second-order corrections is given by

$$M = \frac{M_p}{8\pi}\left[\left(\frac{M_pc^2}{k_BT}\right)+f^2\left(\frac{k_BT}{M_pc^2}\right)+\frac{f^4}{3}\left(\frac{k_BT}{M_pc^2}\right)^3\right]. \tag{35}$$

The heat capacity of a black hole is defined as $C = c^2\frac{\mathrm{d}M}{\mathrm{d}T}$, so the heat capacity with the first- and second-order corrections is obtained as

$$C = \frac{k_B}{8\pi}\left[-\left(\frac{M_pc^2}{k_BT}\right)^2+f^2+f^4\left(\frac{k_BT}{M_pc^2}\right)^2\right]. \tag{36}$$

Obviously, in the semi-classical case (i.e., $f = 0$ in (36)), a Schwarzschild black hole always possesses a negative heat capacity, which means that a black hole is an unstable system that loses its mass with an increase in its temperature during the evaporation process. When including the effects of quantum gravity, the corrected heat capacity (36) has some positive corrections that cause the heat capacity to monotonically increase as the black hole temperature gradually increases during the evaporation process.

According to the first law of black hole thermodynamics $S = \int\frac{c^2\mathrm{d}M}{T}$, the black hole entropy with the first- and second-order corrections is given by

$$S = \frac{k_B}{16\pi}\left[\left(\frac{M_pc^2}{k_BT}\right)^2+f^2\ln\left(\frac{k_BT}{M_pc^2}\right)^2+f^4\left(\frac{k_BT}{M_pc^2}\right)^2\right]. \tag{37}$$

Interestingly, we find that the corrected entropy (37) of the black hole is no longer proportional to the horizon area when considering the effects of quantum gravity [55-59], and the leading-order correction has a logarithmic form similar to that obtained by other methods, such as field theory [73], quantum geometry [74], string theory [75], and loop quantum gravity [76, 77].

Using Hossenfelder's GUP, we derive the corrected black hole thermodynamics with the first- and second-order corrections, which leads to some interesting properties that may solve certain puzzles in the semi-classical case. In the following section, we will focus on the first- and second-order corrections to black hole thermodynamics without loss of generality to precisely reexamine the final evolution behavior of the black hole system from different physical perspectives.

4.1 First-order correction

From (35), we can obtain the mass-temperature relation with the first-order correction, given by

$$\mathcal{M} = \frac{1}{\mathcal{T}}+f^2\mathcal{T}, \tag{38}$$

where we introduce the notations $\mathcal{M} = 8\pi M/M_p$ and $\mathcal{T} = k_BT/(M_pc^2)$ for convenience. Thus, the temperature $\mathcal{T}$ in terms of the mass $\mathcal{M}$ can be written as

$$\mathcal{T} = \frac{2}{\mathcal{M}\pm\sqrt{\mathcal{M}^2-4f^2}}. \tag{39}$$

Here, only the (+) sign is acceptable, and the (−) sign is physically problematic because it cannot recover the classical result if $f = 0$. Obviously, as a thermodynamic system, the black hole has a lower limit for the mass to guarantee a meaningful range of the thermodynamic temperature with the first-order correction. During the evaporation process, if the black hole mass exceeds the lower limit, the thermodynamic temperature becomes complex, and the thermodynamic system cannot be well described here. Therefore, the lower limit of the black hole mass is usually called the *critical* mass [54, 58, 68], which is given, from (39), by

$$M_{cr} = \frac{f}{4\pi}M_p. \tag{40}$$

Next, we start from the corrected thermodynamic entropy to reexamine the lower limit of the black hole mass. By substituting (39) into (37), the thermodynamic entropy with the first-order correction is given by

$$\frac{S}{k_B} = \frac{\mathcal{A}}{4L_p^2}-\frac{f^2}{16\pi}\ln\left[\frac{\mathcal{A}}{4L_p^2}\right]-\frac{f^2}{16\pi}\ln[16\pi], \tag{41}$$

where $A = 16\pi G^2M^2c^{-4}$ is the semi-classical area of the black hole horizon, and

$$\mathcal{A} = \left[\frac{\sqrt{A}}{2}+\sqrt{\frac{A}{4}-\frac{f^2}{4\pi}L_p^2}\right]^2, \tag{42}$$

is the reduced area, which is introduced to produce the area theorem in tractable form, and the semi-classical area A is reproduced for $f=0$ in (42). Obviously, the entropy is explicitly expressed as a function of the reduced area $\mathcal{A}$ rather than the actual area A. From (42), we can also see that there is a lower limit of the black hole mass below which the reduced area becomes a complex quantity, which is given by

$$M_{cr}=\frac{f}{4\pi}M_p. \tag{43}$$

The *critical* mass is correctly reproduced again by analyzing the black hole thermodynamic entropy and reduced area.

By substituting (39) into (36), the heat capacity in terms of the mass $\mathcal{M}$ is then given by

$$C=\frac{k_B}{8\pi}\left[-\frac{\mathcal{M}^2+\mathcal{M}\sqrt{\mathcal{M}^2-4f^2}-2f^2}{2}+f^2\right]. \tag{44}$$

Clearly, there is a positive correction to the semi-classical heat capacity when including the first-order quantum correction. In the semi-classical case, the black hole with a negative heat capacity could evaporate completely in late evolution. In the presence of the first-order correction, the positive correction emerges to prevent further evaporation at zero heat capacity ($C=c^2\frac{\mathrm{d}M}{\mathrm{d}T}=0$) when the black hole mass no longer changes with the black hole temperature [58, 65, 66, 68]. This result means that black hole evaporation stops at a finite mass when including the first-order correction, which is called the *remnant* mass, given by

$$M_{\rm rem}=\frac{f}{4\pi}M_p. \tag{45}$$

The *remnant* mass can also be obtained by minimizing the entropy (41), $\frac{\mathrm{d}S}{\mathrm{d}M}=0$, and looking at the second derivative ($\frac{\mathrm{d}^2S}{\mathrm{d}M^2}>0$). Obviously, the black hole stops evaporation when the *remnant* mass is equal to the *critical* mass. This reveals the intrinsic self-consistency between the black hole remnant and critical masses when including the first-order correction. In addition, we can easily find that the thermodynamic temperature (39) and reduced area (42) are always positive, even at the final stage of black hole evaporation, and there is therefore no singularity in the thermodynamic temperature (39) and entropy (41) when including the first-order quantum correction.

4.2 Second-order correction

Next, we will continue this issue by considering the effects of the second-order correction. From (35), the mass-temperature relation with the second-order correction is given by

$$\mathcal{M}=\frac{1}{\mathcal{T}}+f^2\mathcal{T}+\frac{f^4}{3}\mathcal{T}^3. \tag{46}$$

Then, through complicated calculation, the temperature in terms of the mass can be expressed as

$$\frac{1}{\mathcal{T}}=\frac{\mathcal{M}}{4}+\frac{1}{2}\sqrt{\frac{\mathcal{M}^2}{4}+H}+\frac{1}{2}\left[\frac{\mathcal{M}^2}{2}-H-2f^2+\frac{1}{4}\left(\frac{\mathcal{M}^2}{4}+H\right)^{-\frac{1}{2}}(\mathcal{M}^3-4f^2\mathcal{M})\right]^{\frac{1}{2}}, \tag{47}$$

where

$$H=\frac{5}{3}f^4K^{-1}+\frac{1}{3}K-\frac{2}{3}f^2, \tag{48}$$

$$K=2^{-\frac{1}{3}}\left(9f^4\mathcal{M}^2+F-22f^6\right)^{\frac{1}{3}}, \tag{49}$$

$$F=\sqrt{81f^8\mathcal{M}^4-396f^{10}\mathcal{M}^2-16f^{12}}. \tag{50}$$

The semi-classical Hawking temperature can be well reproduced by the quantum-corrected temperature (47) at $f=0$ when $F=K=H=0$. From (47), there is a lower limit of the black hole mass below which the quantum-corrected temperature becomes a complex quantity, which is determined by $81f^8\mathcal{M}^4-396f^{10}\mathcal{M}^2-16f^{12}\geqslant 0$. Therefore, the lower limit of the black hole mass (i.e., the *critical* mass) is given by

$$M_{cr}=\frac{\left(22+10\sqrt{5}\right)^{\frac{1}{2}}}{24\pi}fM_p. \tag{51}$$

Substituting (47) into (37), the entropy with the second-order correction is given by

$$\frac{S}{k_B}=\frac{\mathcal{A}}{4L_p^2}-\frac{f^2}{16\pi}\ln\left[\frac{\mathcal{A}}{4L_p^2}\right]-\frac{f^2}{16\pi}\ln[16\pi]+\frac{f^4L_p^2}{64\pi^2\mathcal{A}}, \tag{52}$$

where $\mathcal{A}$ is the reduced area, given by

$$\mathcal{A}=\left\{\frac{\sqrt{A}}{4}+\frac{1}{2}\sqrt{\frac{A}{4}+\frac{HL_p^2}{4\pi}}+\frac{1}{2}\left[\frac{A}{2}-\frac{HL_p^2}{4\pi}-\frac{f^2L_p^2}{2\pi}+\sqrt{A}\left(\frac{A}{4}+\frac{HL_p^2}{4\pi}\right)^{-\frac{1}{2}}\left(\frac{A}{4}-\frac{f^2L_p^2}{4\pi}\right)\right]^{\frac{1}{2}}\right\}^2. \tag{53}$$

which is introduced to produce the area theorem in tractable form. A is the usual area of the black hole horizon, and

$$H=\frac{5}{3}f^4K^{-1}+\frac{1}{3}K-\frac{2}{3}f^2, \tag{54}$$

$$K=2^{-\frac{1}{3}}\left(36\pi f^4L_p^{-2}A+F-22f^6\right)^{\frac{1}{3}}, \tag{55}$$

$$F=\sqrt{1296\pi^2f^8L_p^{-4}A^2-1584\pi f^{10}L_p^{-2}A-16f^{12}}. \tag{56}$$

To guarantee an effective range of the reduced area (53), the lower limit of the black hole mass is given by $1296\pi^2f^8\ L_p^{-4}A^2-1584\pi f^{10}L_p^{-2}A-16f^{12}=0$. In this case, the *critical* mass with the second-order correction is given by

$$M_{cr} = \frac{\left(22+10\sqrt{5}\right)^{\frac{1}{2}}}{24\pi} f M_p. \tag{57}$$

The *critical* mass is consistently obtained by guaranteeing the effective ranges of the black hole thermodynamic temperature (47) and reduced area (53).

From (36), the heat capacity with the second-order correction can be written as

$$C = \frac{k_B}{8\pi}\left(-\frac{1}{\mathcal{T}^2} + f^2 + f^4\mathcal{T}^2\right). \tag{58}$$

There are two positive corrections in the corrected heat capacity (58), enabling the black hole evaporation to stop at zero heat capacity easier than that with the first-order correction, where there is only one positive correction. At the final stage of black hole evaporation, when the black hole mass no longer changes with the black hole temperature (i.e., $C = c^2\frac{\mathrm{d}M}{\mathrm{d}T} = 0$), the *remnant* mass with the second-order correction is obtained as

$$M_{\rm rem} = \frac{\left(22+10\sqrt{5}\right)^{\frac{1}{2}}}{24\pi} f M_p. \tag{59}$$

Obviously, the black hole always stops evaporation when the *remnant* mass is equal to the *critical* mass. This result reveals again the intrinsic self-consistency between the black hole remnant and critical masses when including the second-order correction. In addition, the thermodynamic temperature (47) and reduced area (53) are always positive, even during the final evolution of the black hole system, so there is no singularity in the thermodynamic temperature (47) and entropy (52) when including the second-order correction.

The *critical* mass is the lower limit of the black hole mass below which the thermodynamic temperature, entropy, and the reduced area go beyond its effective range [54, 58, 69], and the *remnant* mass is determined by the zero heat capacity $C = 0$ or by minimizing the entropy [58, 66, 67, 69]. Although the *remnant* and *critical* masses are respectively determined from different physical perspectives with respect to the final evolution behavior of the black hole system, they are equal to each other when including the first- and second-order corrections. In addition, we can easily find that the thermodynamic quantities, e.g., the thermodynamic temperature and entropy, are not singular at the final stage of black hole evaporation with the inclusion of the first- and second-order corrections.

5 Conclusions

In this paper, we reveal the intrinsic self-consistency between the remnant and critical masses during the final stage of black hole evaporation when including the effects of quantum gravity. When including all the necessary terms in the first- and second-order corrections, we first reexamine Banerjee-Ghosh's work and find that the black hole stops evaporation when the remnant mass is equal to the critical mass with the inclusion of the GUP effects. Then, we use another GUP model with higher-order corrections proposed by Hossenfelder *et al.* to restudy the final evolution behavior of the black hole evaporation, and we again reveal the intrinsic self-consistency between the black hole remnant and critical masses. In addition, we can easily find that the thermodynamic quantities, e.g., the thermodynamic temperature and entropy, are not singular at the final stage of black hole evaporation with the inclusion of the first- and second-order corrections; therefore, the singularity problem in the semi-classical approach can be naturally bypassed here.

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