Operators of quantum theory of Dirac's free field

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Abstract: The Pryce (e) spin and position operators of the quantum theory of Dirac's free field were re-defined and studied recently with the help of a new spin symmetry and suitable spectral representations [Eur. Phys. J. C 82, 1073 (2022)]. This approach is generalized here, associating a pair of integral operators acting directly on particle and antiparticle wave spinors in momentum representation to any integral operator in configuration representation, acting on mode spinors. This framework allows an effective quantization procedure, giving a large set of one-particle operators with physical meaning as the spin and orbital parts of the isometry generators, the Pauli-Lubanski and position operators, or other spin-type operators proposed to date. Special attention is paid to the operators that mix the particle and antiparticle sectors whose off-diagonal associated operators have oscillating terms producing Zitterbewegung. The principal operators of this type, including the usual coordinate operator, are derived here for the first time. As an application, it is shown that an apparatus measuring these new observables may prepare and detect one-particle wave packets moving uniformly without Zitterbewegung or spin dynamics, spreading in time normally as any other relativistic or even non-relativistic wave packet.

Keywords: Dirac theory, integral operators, Pryce's operators, integral representations, canonical quantization, propagation

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I. INTRODUCTION

In the relativistic quantum mechanics (RQM) of Dirac's field, one traditionally considers the usual coordinate operator, which is affected by Zitterbewegung [1−3], and the Pauli-Dirac spin operator, whose components generate the rotations of the Dirac representation of the $SL(2,\mathbb{C})$ group [1] but are not conserved. For this reason, many researchers have struggled to find a suitable *conserved* spin operator [4−9], giving rise to a rich literature (e.g., see Refs. [10−13] and the references therein). As this problem remains of interest [14−17], we attempted to reach the next step to quantization [18]. As a result, we observed that the required spin operator has been known for a long time and was proposed by Pryce in momentum representation (MR) according to his hypothesis (e) [5]. In fact, Pryce studied the relativistic mass-center operator, analyzing many possible definitions; among them, versions (c), (d), and (e) are of interest regarding Dirac's theory. Each version gives its own specific angular momentum related to a convenient spin operator, ensuring the conservation of the total angular momentum. Pryce's hypothesis (e) is a unique version with correct physical meaning, giving a would-be mass-center vector-operator with commuting components related to a conserved spin operator whose components generate an $su(2)$ algebra.

Foldy and Wouthuysen later showed that their famous transformation [6] leads to the Newton-Wigner representation [19] in which the Dirac Hamiltonian is diagonal, while the Pryce (e) spin and position operators become the aforementioned usual ones. Besides the Pauli-Dirac and Pryce (e) spin operators, other versions have been proposed by Frenkel [4], Pryce (c) and Czochor [5, 9], Fradkin and Good [7], and Chakrabarti [8]. Among them, only the components of Pauli-Dirac and Chakrabarti spin operators generate $su(2)$ algebras, but these operators are not conserved. In contrast, the operators proposed by Frenkel, Pryce (c)-Czochor, and Fradkin-Good are conserved, but their components do not close $su(2)$ algebras. For this reason, we say that these are spin-type operators.

We can understand the role of the Pryce (e) spin operator by studying the symmetry of Pauli polarization spinors, which define the fermion polarization. These spinors enter in the structure of the plane wave solutions of the Dirac equation that form the basis of mode (or fun-

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damental) spinors. Technically, the fermion polarization depends on the direction of spin projection, which can be chosen arbitrarily. When this direction depends on momentum, as in the case of the largely used momentum-helicity basis, we say that the polarization is *peculiar*. Otherwise, we have a *common* polarization independent of momentum, such as in the momentum-spin basis defined in Ref. [20]. In both these cases, the polarization spinors offer us the degrees of freedom of the new $SU(2)$ *spin* symmetry that we require to construct a spin operator conserved via Noether's theorem [18]. However, this symmetry has been neglected so far because of the difficulties in finding suitable operators in configuration (or coordinate) representation (CR) able to transform only the polarization spinors in MR without affecting other quantities. Fortunately, we have found a spectral representation of a class of integral operators allowing us to define the action of the little group $SU(2)$ upon the polarization spinors [18], showing that the generators of these transformations are the components of the conserved spin operator whose Fourier transform is just the operator of the Pryce (e) version (see the third of Eqs. (6), (7) in Ref. [5]). In this new framework, we defined the operator of fermion polarization and studied how the principal operators of Dirac's theory depend on polarization through new momentum-dependent Pauli-type matrices and covariant momentum derivatives [18].

This was a crucial step towards quantization, allowing us to derive the principal one-particle operators of quantum field theory (QFT). After quantization, the would-be mass-center operator of the Pryce (e) version becomes the time-dependent *dipole* one-particle operator whose velocity is the conserved part of the Dirac current (unaffected by Zitterbewegung), often called the classical current [21, 22] and referred to here as the *conserved* current. Quantifying, in addition, the spin, and polarization operators as well the isometry generators for any polarization we outlined a coherent version of Dirac's QFT [18].

In this paper, we would like to continue and complete this study by improving the general formalism to eliminate the difficulties that impeded the aforementioned results for more than seven decades. In our opinion, the principal impediment was the manner in which the action of the integral operators of RQM was considered so far. The Dirac free fields in CR can be expanded in terms of particle and antiparticle Pauli wave spinors in MR in a basis of Dirac's mode spinors. The matrix, differential, and integral operators act directly on the mode spinors. Difficulties arise because of some integral operators with complicated actions that cannot be manipulated or interpreted, as in the case of all Pryce's operators. The solution is to associate a pair of integral operators acting in MR on the particle and antiparticle wave spinors to each integral operator in CR, acting on mode spinors. In this manner, the kernels of the integral operators in CR can be related to those of the associated operators in MR through spectral representations, which are generalized here to a large class of integral operators. We thus obtain a friendly approach by which we may study and interpret the principal integral operators of RQM, taking a decisive step to towards quantization.

In view of the above arguments, this paper presents an extended review of the operators of Dirac's theory, following three major objectives. The first is to improve the entire formalism, focusing on the theory of integral operators acting on the wave spinors. The second objective is to develop and complete the quantum theory outlined in Ref. [18], studying the entire collection of operators with physical meaning of Dirac's QFT derived from the operators of RQM proposed to date, including the operators with oscillating terms producing Zitterbewegung. Finally, for the first time, we present an example of Dirac's wave packet prepared and detected by an apparatus able to measure the new Pryce's spin and position operators, presenting the image of a natural smooth propagation without Zitterbewegung or spin dynamics.

In the next section, we start with the Dirac theory in CR and MR, presenting our framework and explicitly defining the new spin and orbital symmetries in CR before considering the solutions in MR where the mode spinors are constructed according to Wigner's method, allowing us to demonstrate the role of the polarization spinors. Then, we present the equal-time and Fourier integral operators acting on the mode spinors through their kernels. We pay special attention to the operators proposed by Pryce but without neglecting the other historical proposals of spin or spin-type operators [4, 6–9].

Section III is devoted to our principal technical improvement of the operator theory, namely, the method of associated operators, relating the operators acting on fields to pairs of operators acting directly on the Pauli wave spinors in MR, which we call associated operators. The operators that do not mix particle and antiparticle wave functions are called reducible; otherwise, they are irreducible. We show that the irreducible operators have associated operators whose off-diagonal kernels, between particle and antiparticle wave functions, oscillate with high frequency. Fortunately, the principal operators we require are reducible, without oscillating terms. We derive and study the operators associated with the spin, position, polarization and Pauli-Lubanski ones, paying special attention to the isometry generators of the covariant representation of the Dirac field in CR, which is equivalent to a pair of associated Wigner-induced representations in MR [23, 25, 26]. Remarkably, our approach can show that the spin part of the rotation generators of the covariant representation are just the components of Pryce's spin operator in CR associated with the spin parts of the rotation generators of Wigner's representations defined in MR. In addition, we study the conserved spin-type operators proposed by Frankel, Pryce (c)-Czogor, and Fradkin-Good, analyzing their algebraic properties. Section IV generalizes the spectral representations defined in Ref. [18], expressing the kernels of the integral operators acting in CR in terms of kernels of associated operators defined in MR. This method allows us to particularly focus on the principal non-Fourier operators whose kernels in MR are momentum derivatives of Dirac's δ-distributions of complicated arguments.

The previous results prepare the quantization presented in Sec. V, where we apply the Bogolyubov quantization method [27], transforming the expectation values of RQM in operators of QFT. We find that, after quantization, the reducible operators of RQM become one-particle operators, which we divide into even and odd operators according to the relative sign between the particle and antiparticle terms (*i.e.*, charge parity). We define the operators of unitary transformations under isometries with general calculation rules, and we study the algebra of principal observables generated by the reducible operators of RQM. The last subsection is devoted to the quantization of the irreducible operators with oscillating terms. The new results presented here are the operators of QFT corresponding to the traditional Pauli-Dirac spin and coordinate operators of RQM, which can be related to the vector or axial currents, and other interesting operators, such as the Chakrabarti spin operator and the generators of the Foldy-Wouthuysen transformations.

Turning back to RQM but now as a one-particle restriction of QFT, in Sec. VI, we consider wave packets prepared and detected by two different observers. We first present the general theory, assuming that the detector filters momenta oriented along the direction source-detector such that this measures a one-dimensional wave packet governed by radial observables. An isotropic wave packet example is presented, showing that it has an inertial motion spreading in time just as other scalar or even non-relativistic wave packets do, without Zitterbewegung or spin dynamics [28].

Concluding remarks are presented in Sec. VII. The four Appendices successively present the Dirac representation of the $SL(2,\mathbb{C})$ group, the commutation relations of the algebra of associated operators in MR, the Pryce (c) and (d) position operators, and examples of known peculiar and common fermion polarizations.

II. DIRAC'S FREE FIELD

In special relativity, the covariant free fields [20, 29] are defined in Minkowski's space-time M with the metric $\eta = \mathrm{diag}(1,-1,-1,-1)$ and Cartesian coordinates x^μ labeled by Greek indices ($\alpha,\beta,...\mu,\nu... = 0,1,2,3$). These fields transform covariantly under Poincaré isometries, $(\Lambda,a): x \to x' = \Lambda x + a$, which form the group $P_+^\uparrow = T(4)\circledS L_+^\uparrow$ [30] constituted by the transformations $\Lambda \in L_+^\uparrow$ of the orthochronous proper Lorentz group, preserving the metric η, and the four-dimensional translations $a \in \mathbb{R}^4$ of the invariant subgroup $T(4)$. For the fields with half-integer spins, in addition, the universal covering group of the Poincaré one, $\tilde{P}_+^\uparrow = T(4)\circledS SL(2,\mathbb{C})$, formed by the mentioned translations and transformations $\lambda \in SL(2,\mathbb{C})$ is related to those of the group $L_+^\uparrow$ through the canonical homomorphism $\lambda \to \Lambda(\lambda) \in L_+^\uparrow$ [30] obeying the condition (A.2). In this framework, the covariant fields with spin can be defined on M with values in vector spaces carrying reducible finite-dimensional representations of the $SL(2,\mathbb{C})$ group where invariant Hermitian forms can be defined.

A. Lagrangian theory and its symmetries

The Dirac field $\psi: M \to \mathcal{V}_D$ takes values in the space of Dirac spinors $\mathcal{V}_D = \mathcal{V}_P \oplus \mathcal{V}_P$, which is the orthogonal sum of two spaces of Pauli spinors, $\mathcal{V}_P$, carrying the irreducible representations $(1/2,0)$ and $(0,1/2)$ of the $SL(2,\mathbb{C})$ group. These form the Dirac representation $\rho_D = (1/2,0)\oplus(0,1/2)$, where one may define the Dirac γ-matrices and invariant Hermitian form $\overline{\psi}\psi$ with the help of the Dirac adjoint $\overline{\psi} = \psi^+\gamma^0$ of ψ (see Appendix A for details). The fields ψ and $\overline{\psi}$ are the canonical variables of the action

$$\mathcal{S}[\psi,\overline{\psi}] = \int \mathrm{d}^4x \mathcal{L}_D(\psi,\overline{\psi}), \tag{1}$$

defined by the Lagrangian density

$$\mathcal{L}_D(\psi,\overline{\psi}) = \frac{\mathrm{i}}{2}[\overline{\psi}\gamma^\alpha\partial_\alpha\psi - (\overline{\partial_\alpha\psi})\gamma^\alpha\psi] - m\overline{\psi}\psi, \tag{2}$$

depending on the mass $m \neq 0$ of the Dirac field. This action gives rise to the Dirac equation $E_D\psi = (\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0$, which can be put in Hamiltonian form as

$$\mathrm{i}\partial_t\psi(x) = H_D\psi(x), \quad H_D = -\mathrm{i}\gamma^0\gamma^i\partial_i + m\gamma^0. \tag{3}$$

In other respects, the conservation of the electric charge via Noether's theorem [20, 29] suggests the form of the Dirac relativistic scalar product

$$\langle\psi,\psi'\rangle_D = \int \mathrm{d}^3x \overline{\psi}(x)\gamma^0\psi'(x) = \int \mathrm{d}^3x \psi^+(x)\psi'(x). \tag{4}$$

We denote by $\mathcal{F} = \{\psi \mid E_D\psi = 0\}$ the space of *free fields* that can be organized as a rigged Hilbert space by using the Dirac scalar product.

The action (1) is invariant under the transformations of the well-known symmetries, namely, the Poincaré isometries and $U(1)_{\rm em}$ transformations of the electromagnetic gauge. The Dirac field transforms under isometries according to the *covariant* representation $T:(\lambda,a)\to T_{\lambda,a}\in{\rm Aut}(\mathcal{F})$ of the group $\tilde{P}^{\uparrow}_{+}$ as [30]

$$(T_{\lambda,a}\psi)(x)=\lambda\psi\left[\Lambda(\lambda)^{-1}(x-a)\right], \tag{5}$$

generated by the basis generators of the corresponding representation of the Lie algebra ${\rm Lie}(T)$ that reads

$$P_\mu=-\,{\rm i}\frac{\partial T_{1,a}}{\partial a^\mu}\bigg|_{a=0},\quad J_{\mu\nu}={\rm i}\frac{\partial T_{\lambda(\omega),0}}{\partial\omega^{\mu\nu}}\bigg|_{\omega=0}. \tag{6}$$

To demonstrate the physical meaning of these generators, one separates the momentum components, $P^i=-{\rm i}\partial_i$, and the energy operator, $H=P_0={\rm i}\partial_t$, denoting the $SL(2,\mathbb{C})$ generators as

$$J_i=\frac{1}{2}\varepsilon_{ijk}J_{jk}=-{\rm i}\varepsilon_{ijk}\underline{x}^j\partial_k+s_i, \tag{7}$$

$$K_i=J_{0i}={\rm i}(\underline{x}^i\partial_t+t\partial_i)+s_{0i}, \tag{8}$$

where $\underline{x}^i$ are the components of the *coordinate* vector-operator $\vec{\underline{x}}$ acting as $(\underline{x}^i\psi)(x)=x^i\psi(x)$. The reducible matrices s_i and s_{0i} are given by Eqs. (A6) and (A8), respectively. The operators $\{H,P^i,J_i,K_i\}$ form the usual basis of the Lie algebra ${\rm Lie}(T)$ of the representation (5) [30].

The scalar product (4) helps us to simply write the quantities conserved via Noether's theorem as expectation values, $\langle\psi,X\psi\rangle_D$, of the generators of the symmetry transformations $\psi\to T\psi=\psi-{\rm i}\xi X\psi+...$, which leave invariant the action (1) and implicitly the scalar product, $\langle T\psi,T\psi'\rangle_D=\langle\psi,\psi'\rangle_D$. Hereby, we deduce that the generators X are self-adjoint, obeying

$$\langle\psi,X^+\psi'\rangle_D=\langle X\psi,\psi'\rangle_D=\langle\psi,X\psi'\rangle_D. \tag{9}$$

Therefore, we may conclude that the covariant representation (5) is *unitary* with respect to the relativistic scalar product (4).

The above operators may freely generate new ones, such as the Pauli-Lubanski pseudo-vector [30]

$$W^\mu=-\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}P_\nu J_{\alpha\beta}, \tag{10}$$

with components

$$W^0=J_iP^i=s_iP^i,\quad W^i=HJ_i+\varepsilon_{ijk}P^jK_k, \tag{11}$$

where $\varepsilon^{0123}=-\varepsilon_{0123}=-1$. This operator is considered by many authors as the covariant four-dimensional spin operator as long as W_0 is just the helicity operator [31]. Moreover, this gives rise to the second Casimir operator of the pair [1]

$$C_1=P_\mu P^\mu\sim m^2, \tag{12}$$

$$C_2=W^\mu W_\mu\sim -m^2s(s+1),\quad s=\frac{1}{2}, \tag{13}$$

whose eigenvalues depend on the invariants (m,s) determining the representation T.

Here, the subgroup $SU(2)\subset SL(2,\mathbb{C})$ will play a special role in studying the spin operator. For this reason, we consider the restriction of the covariant representation T to this subgroup, $T^r\equiv T|_{SU(2)}$, such that $T_{r,0}=T^r_{\hat{r}}$ for any $\hat{r}\in SU(2)$ or $r={\rm diag}(\hat{r},\hat{r})\in\rho_D$. The basis generators of the representation T^r are the components of the total angular momentum operator $\vec{J}=\vec{\underline{x}}\wedge\vec{P}+\vec{s}$, defined by Eq. (7), which is formed by the orbital term $\vec{\underline{x}}\wedge\vec{P}$ and Pauli-Dirac spin matrix $\vec{s}$. However, as mentioned before, these operators are not conserved separately; thus, we must look for a new conserved spin operator $\vec{S}$ related to a suitable new position operator, $\vec{X}=\vec{\underline{x}}+\delta\vec{X}$, allowing the new splitting

$$\vec{J}=\vec{\underline{x}}\wedge\vec{P}+\vec{s}=\vec{L}+\vec{S},\qquad \vec{L}=\vec{X}\wedge\vec{P}, \tag{14}$$

which imposes the correction $\delta\vec{X}$ to satisfy $\delta\vec{X}\wedge\vec{P}=\vec{s}-\vec{S}$. This new splitting gives rise to a pair of new $su(2)\sim so(3)$ symmetries, namely, the *orbital* symmetry generated by $\{L_1,L_2,L_3\}$ and the *spin* one generated by $\{S_1,S_2,S_3\}$. Moreover, we have shown that the Fourier transforms of the operators $\vec{S}$ and $\delta\vec{X}$ are just the Pryce (e) operators [18].

To write the plane wave solutions of the Dirac equation, it is known that we must choose the same orthonormal basis of polarization spinors $\xi=\{\xi_\sigma|\sigma=\pm\frac{1}{2}\}$ in both the spaces $\mathcal{V}_P$ of Pauli spinors carrying the irreducible representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ of ρ_D. Because the polarization spinors are free parameters, we may consider the Dirac field as $\psi:M\times\mathcal{V}_P\to\mathcal{V}_D$, denoting it explicitly by ψ_ξ instead of ψ. The basis of polarization spinors can be changed at any time, $\xi\to\hat{r}\xi$, by applying a rotation $\hat{r}\in SU(2)$, which changes the form of the Dirac spinor, giving rise to the new representation $T^s:\hat{r}\to T^s_{\hat{r}}$ of the group $SU(2)$, which encapsulates the spin symmetry. The operators of this representation have the action

$$\left(T^s_{\hat{r}(\theta)}\psi_\xi\right)(x)=\psi_{\hat{r}(\theta)\xi}(x), \tag{15}$$

where $\hat{r}(\theta)$ are the rotations (A7) with Cayley-Klein parameters. The components of the spin operator can now be defined as the generators of this representation [18],

$$S_i = \mathrm{i}\frac{\partial T^s_{\hat{r}(\theta)}}{\partial \theta^i}\bigg|_{\theta^i=0} \quad \Rightarrow \quad S_i\psi_\xi = \psi_{\hat{s}_i\xi}\,, \tag{16}$$

whose action is obvious. For the first time, we similarly define the *orbital* representation $T^o : \hat{r} \to T^o_{\hat{r}}$ as

$$\left(T^o_{\hat{r}(\theta)}\psi_\xi\right)(t,\vec{x}) = r(\theta)\psi_{\hat{r}(\theta)^{-1}\xi}\left(t, R[\hat{r}(\theta)]^{-1}\vec{x}\right) \tag{17}$$

to accomplish the factorization $T^r = T^o \otimes T^s$. The basis generators of the orbital representation

$$L_i = \mathrm{i}\frac{\partial T^o_{\hat{r}(\theta)}}{\partial \theta^i}\bigg|_{\theta^i=0} \tag{18}$$

are the components of the new conserved orbital angular momentum operator $\vec{L}$. In the following, we pay special attention to the new operators $\vec{S}$, $\vec{L}$, and $\vec{X}$.

B. Momentum representation

In MR, all quantities are defined on orbits in momentum space, $\Omega_{\mathring{p}} = \{\vec{p}\,|\,\vec{p} = \Lambda\mathring{p}, \Lambda \in L^\uparrow_+\}$, which can be built by applying Lorentz transformations on a *representative* momentum $\mathring{p}$ [23−25]. In the case of massive particles, the representative momentum is just the rest frame momentum, $\mathring{p} = (m,0,0,0)$. The rotations that leave $\mathring{p}$ invariant, $\Lambda(r)\mathring{p} = \mathring{p}$, form the *stable* group $SO(3) \subset L^\uparrow_+$ of $\mathring{p}$, whose universal covering group $SU(2)$ is called the *little* group associated with the representative momentum $\mathring{p}$.

The momenta $\vec{p} \in \Omega_{\mathring{p}}$ may be obtained as $\vec{p} = \Lambda_{\vec{p}}\mathring{p}$ by applying transformations $\Lambda_{\vec{p}} = L_{\vec{p}}R(r(\vec{p}))$ formed by genuine Lorentz boosts and arbitrary rotations $R(r(\vec{p})) = \Lambda(r(\vec{p}))$ that do not change the representative momentum. The corresponding transformations $\lambda_{\vec{p}} \in \rho_D$, which satisfy $\Lambda(\lambda_{\vec{p}}) = \Lambda_{\vec{p}}$ and $\lambda_{\vec{p}=0} = 1 \in \rho_D$, have the form

$$\lambda_{\vec{p}} = l_{\vec{p}}\, r(\vec{p})\,, \tag{19}$$

where the transformations $l_{\vec{p}}$ given by Eq. (A11) are related to the genuine Lorentz boosts $L_{\vec{p}} = \Lambda(l_{\vec{p}})$ with the matrix elements from (A12). The invariant measure on the massive orbits [30]

$$\mu(\vec{p}) = \mu(\Lambda\vec{p}) = \frac{\mathrm{d}^3 p}{E(p)}\,, \quad \forall\Lambda \in L^\uparrow_+ \tag{20}$$

is the last tool required for relating CR and MR.

The general solutions of the free Dirac equation, $\psi \in \mathcal{F}$, may be expanded in terms of mode spinors spinors, $U_{\vec{p},\sigma}$ and $V_{\vec{p},\sigma} = CU^*_{\vec{p},\sigma}$, of positive and negative frequencies, related through the charge conjugation defined by the matrix $C = C^{-1} = \mathrm{i}\gamma^2$. The mode spinors are particular solutions of the Dirac equation that satisfy the eigenvalues problems

$$HU_{\vec{p},\sigma} = E(p)U_{\vec{p},\sigma}\,, \qquad HV_{\vec{p},\sigma} = -E(p)V_{\vec{p},\sigma}\,, \tag{21}$$

$$P^i U_{\vec{p},\sigma} = p^i\, U_{\vec{p},\sigma}\,, \qquad P^i V_{\vec{p},\sigma} = -p^i\, V_{\vec{p},\sigma}\,, \tag{22}$$

depending explicitly on the polarization spinors, which will be specified later. Therefore, the general solutions of the Dirac equation are free fields that can be expanded as [20, 29]

$$\begin{aligned}\psi(x) &= \psi^+(x) + \psi^-(x)\\ &= \int \mathrm{d}^3p \sum_\sigma \left[U_{\vec{p},\sigma}(x)\alpha_\sigma(\vec{p}) + V_{\vec{p},\sigma}(x)\beta^*_\sigma(\vec{p})\right]\,,\end{aligned} \tag{23}$$

in terms of spinors-functions $\alpha : \Omega_{\mathring{p}} \to \mathcal{V}_P$ and $\beta : \Omega_{\mathring{p}} \to \mathcal{V}_P$ representing the particle and antiparticle *wave spinors*, respectively. Thus, the space of free fields $\mathcal{F}$ can be split into two subspaces of positive and negative frequencies, $\mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^-$, which are orthogonal with respect to the scalar product (4).

The mode spinors prepared at the initial time $t_0 = 0$ by an observer staying at rest in origin have the general form

$$U_{\vec{p},\sigma}(x) = u_\sigma(\vec{p})\frac{1}{(2\pi)^{3/2}}\,\mathrm{e}^{-\mathrm{i}E(p)t + \mathrm{i}\vec{p}\cdot\vec{x}}\,, \tag{24}$$

$$V_{\vec{p},\sigma}(x) = v_\sigma(\vec{p})\frac{1}{(2\pi)^{3/2}}\,\mathrm{e}^{\mathrm{i}E(p)t - \mathrm{i}\vec{p}\cdot\vec{x}}\,, \tag{25}$$

where $v_\sigma(\vec{p}) = Cu^*_{s\sigma}(\vec{p})$. According to Wigner's general method [1, 23, 24], we use the transformations of (19) and (A.11) to represent the spinors

$$\begin{aligned}u_\sigma(\vec{p}) &= n(p)\lambda_{\vec{p}}\,\mathring{u}_\sigma = n(p)l_{\vec{p}}r(\vec{p})\,\mathring{u}_\sigma\\ &= n(p)l_{\vec{p}}\,\mathring{u}_\sigma(\vec{p})\,,\end{aligned} \tag{26}$$

$$\begin{aligned}v_\sigma(\vec{p}) &= Cu^*_\sigma(\vec{p}) = n(p)\lambda_{\vec{p}}\mathring{v}_\sigma = n(p)l_{\vec{p}}r(\vec{p})\mathring{v}_\sigma\\ &= n(p)l_{\vec{p}}\mathring{v}_\sigma(\vec{p})\,,\end{aligned} \tag{27}$$

depending on a normalization factor satisfying $n(0) = 1$. The rest frame spinors $\mathring{u}_\sigma = u_\sigma(0)$ and $\mathring{v}_\sigma = v_\sigma(0) = C\mathring{u}^*_\sigma$ are solutions of the Dirac equation in the rest frame obeying $\gamma^0\mathring{u}_\sigma=\mathring{u}_\sigma$ and $\gamma^0\mathring{v}_\sigma=-\mathring{v}_\sigma$. If these equations are satisfied, then the spinors (24) and (25) are solutions of the Dirac equation in MR,

$$(\gamma p-m)u_\sigma(\vec{p})=0\,,\quad (\gamma p+m)v_\sigma(\vec{p})=0\,, \tag{28}$$

because $\gamma p=E(p)\gamma^0-\gamma^i p^i=ml_{\vec{p}}\gamma^0 l_{\vec{p}}^{-1}$.

Considering that the rotations $r(\vec{p})$ are arbitrary, we separate the quantities

$$\mathring{u}_\sigma(\vec{p})=r(\vec{p})\mathring{u}_\sigma=\frac{1}{\sqrt{2}}\begin{pmatrix}\xi_\sigma(\vec{p})\\ \xi_\sigma(\vec{p})\end{pmatrix}, \tag{29}$$

$$\mathring{v}_\sigma(\vec{p})=r(\vec{p})\mathring{v}_\sigma=\frac{1}{\sqrt{2}}\begin{pmatrix}\eta_\sigma(\vec{p})\\ -\eta_\sigma(\vec{p})\end{pmatrix}, \tag{30}$$

which are eigenspinors of the matrix γ^0 corresponding to the eigenvalues 1 and -1, respectively, as $r(\vec{p})$ commutes with γ^0. These Dirac spinors depend on the related Pauli spinors $\xi_\sigma(\vec{p})$ and $\eta_\sigma(\vec{p})=\mathrm{i}\sigma_2\xi^*_\sigma(\vec{p})$, which we call the polarization spinors, observing that only the spinors $\xi_\sigma(\vec{p})$ remain arbitrary. The orthogonality and completeness properties of these spinors (presented in Appendix C) ensure the normalization of the spinors (29) and (30), which give rise to the complete orthogonal system of projection matrices

$$\sum_\sigma \mathring{u}_\sigma(\vec{p})\mathring{u}^+_\sigma(\vec{p})=\sum_\sigma \mathring{u}_\sigma\mathring{u}^+_\sigma=\frac{1+\gamma^0}{2}\,, \tag{31}$$

$$\sum_\sigma \mathring{v}_\sigma(\vec{p})\mathring{v}^+_\sigma(\vec{p})=\sum_\sigma \mathring{v}_\sigma\mathring{v}^+_\sigma=\frac{1-\gamma^0}{2}\,, \tag{32}$$

on the proper subspaces of the matrix γ^0.

Finally, by setting the normalization factor in accordance with Eq. (A16),

$$n(p)=\sqrt{\frac{m}{E(p)}}\,, \tag{33}$$

we obtain the orthonormalization,

$$\langle U_{\vec{p},\sigma},U_{\vec{p}\,',\sigma'}\rangle_D=\langle V_{\vec{p},\sigma},V_{\vec{p}\,',\sigma'}\rangle_D=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}\,')\,, \tag{34}$$

$$\langle U_{\vec{p},\sigma},V_{\vec{p}\,',\sigma'}\rangle_D=\langle V_{\vec{p},\sigma},U_{\vec{p}\,',\sigma'}\rangle_D=0\,, \tag{35}$$

and completeness,

$$\int \mathrm{d}^3p\sum_\sigma\left[U_{\vec{p},\sigma}(t,\vec{x})U^+_{\vec{p},\sigma}(t,\vec{x}\,')+V_{\vec{p},\sigma}(t,\vec{x})V^+_{\vec{p},\sigma}(t,\vec{x}\,')\right]=\delta^3(\vec{x}-\vec{x}\,')\,, \tag{36}$$

of the basis of mode spinors.

Equation (23) can now be seen as the expansion of the free field ψ in the basis of mode spinors whose "coefficients" are just the wave spinors

$$\alpha=\begin{pmatrix}\alpha_{\frac12}\\ \alpha_{-\frac12}\end{pmatrix}\in\tilde{\mathcal{F}}^+,\quad \beta=\begin{pmatrix}\beta_{\frac12}\\ \beta_{-\frac12}\end{pmatrix}\in\tilde{\mathcal{F}}^-, \tag{37}$$

which encapsulate the physical meaning of ψ. When the field ψ is known, then the wave spinors can be derived by applying the inversion formulas

$$\alpha_\sigma(\vec{p})=\langle U_{\vec{p},\sigma},\psi\rangle_D\,,\quad \beta_\sigma(\vec{p})=\langle\psi,V_{\vec{p},\sigma}\rangle_D\,, \tag{38}$$

resulting from Eqs. (34) and (35). We assume now that the spaces $\tilde{\mathcal{F}}^+\sim\tilde{\mathcal{F}}^-$ are rigged Hilbert spaces, including Hilbert spaces $\mathcal{L}^2(\Omega_{\mathring{p}},d^3p,\mathcal{V}_P)$, equipped with the same scalar product,

$$\langle\alpha,\alpha'\rangle=\int \mathrm{d}^3p\,\alpha^+(\vec{p})\alpha'(\vec{p})=\int \mathrm{d}^3p\sum_\sigma\alpha^*_\sigma(\vec{p})\alpha'_\sigma(\vec{p})\,, \tag{39}$$

and similarly for the spinors β. Then, after using Eqs. (34) and (35), we obtain the important identity

$$\langle\psi,\psi'\rangle_D=\langle\alpha,\alpha'\rangle+\langle\beta,\beta'\rangle\,, \tag{40}$$

expressing the Dirac scalar product in terms of wave spinors. We remind the reader that when $\langle\psi,\psi\rangle_D=1$, the quantities $|\alpha_\sigma(\vec{p})|^2$ and $|\beta_\sigma(\vec{p})|^2$ are the densities of probability in momentum space of a particle and antiparticle of polarization σ, respectively.

III. OPERATORS OF DIRAC'S THEORY

The observables of Dirac's RQM are linear operators acting on the space of free fields, $A,B,...\in\mathrm{Aut}(\mathcal{F})$, which must be self-adjoint with respect to the scalar product (4). Apart from the familiar multiplicative and differential operators, there are integral operators that deserve special attention.

A. From differential to integral operators

The differential operators are 4×4 matrices depending on derivatives $f(\mathrm{i}\partial_\mu)\in\rho_D$, whose action on the mode spinors,

$$\left[f(\mathrm{i}\partial_\mu)\psi\right](x) = \int \mathrm{d}^3p \sum_\sigma \left[f(p^\mu)U_{\vec{p},\sigma}(x)\alpha_\sigma(\vec{p}) + f(-p^\mu)V_{\vec{p},\sigma}(x)\beta^*_\sigma(\vec{p})\right], \tag{41}$$

is given by the momentum-dependent matrices $f(p^\mu)$. The principal differential operators are the translation generators $P_\mu = \mathrm{i}\partial_\mu$, the operator of the Dirac equation, and implicitly the Dirac Hamiltonian (3). However, there are important operators, such as those proposed by Pryce, that are integral operators and cannot be reduced to differential ones.

In general, the integral operators, $Z : \mathcal{F} \to \mathcal{F}$, have the action

$$(Z\psi)(x) = \int \mathrm{d}^4x' \mathfrak{Z}(x,x')\psi(x'), \tag{42}$$

defined by their kernels $\mathfrak{Z} : M \times M \to \rho_D$, denoted here by the corresponding Fraktur symbol, *e.g.*, $Z \to \mathfrak{Z}$. These operators are linear, forming an algebra in which the multiplication, $Z = Z_1Z_2$, is defined by the composition rule of the corresponding kernels,

$$\mathfrak{Z}(x,x') = \int \mathrm{d}^4x'' \mathfrak{Z}_1(x,x'')\mathfrak{Z}_2(x'',x'). \tag{43}$$

The identity operator I of this algebra acting as $(I\psi)(x) = \psi(x)$ has the kernel $\mathfrak{I}(x,x') = \delta^4(x-x')$. For any integral operator Z, we may write the Dirac bracket at the given time t as

$$\langle\psi, Z\psi'\rangle_D|_t = \int \mathrm{d}^3x\,\mathrm{d}^4x' \psi^+(t,\vec{x})\mathfrak{Z}(t,\vec{x},x')\psi(x'), \tag{44}$$

integrating only over the space coordinates $\vec{x}$. The multiplicative or differential operators are particular cases of integral ones. For example, the derivatives ∂_μ can be seen as integral operators with the kernels $\partial_\mu\delta^4(x)$. In general, the operators with kernels depending on t and t' or only on t-t' play the role of *propagators*.

For describing usual observables, it is sufficient to consider *equal-time operators*, A, whose kernels of the form

$$\mathfrak{A}(x,x') = \delta(t-t')\mathfrak{A}(t,\vec{x},\vec{x}') \tag{45}$$

define the operator action

$$(A\psi)(t,\vec{x}) = \int \mathrm{d}^3x' \mathfrak{A}(t,\vec{x},\vec{x}')\psi(t,\vec{x}'), \tag{46}$$

preserving the time. The operator multiplication takes over this property:

$$A = A_1A_2 \Rightarrow \mathfrak{A}(t,\vec{x},\vec{x}') = \int \mathrm{d}^3x'' \mathfrak{A}_1(t,\vec{x},\vec{x}'')\mathfrak{A}_2(t,\vec{x}'',\vec{x}'), \tag{47}$$

which means that the set of equal-time operators forms an algebra, $E[t] \subset \mathrm{Aut}(\mathcal{F})$, at any fixed time t. The expectation values of these operators at a given time t,

$$\langle\psi, A\psi'\rangle_D|_t = \int \mathrm{d}^3x\,\mathrm{d}^3x' \psi^+(t,\vec{x})\mathfrak{A}(t,\vec{x},\vec{x}')\psi'(t,\vec{x}'), \tag{48}$$

are dynamic quantities evolving in time as

$$\partial_t\langle\psi, A\psi'\rangle_D|_t = \langle\psi, \mathrm{d}A\psi'\rangle_D|_t \qquad dA = \partial A + \mathrm{i}[H_D, A], \tag{49}$$

where dA plays the role of total time derivative assuming that the new operator ∂A has the action

$$(\partial A\psi)(t,\vec{x}) = \int \mathrm{d}^3x' \partial_t\mathfrak{A}(t,\vec{x},\vec{x}')\psi(t,\vec{x}'). \tag{50}$$

As mentioned before, we say that an operator is conserved if its expectation value is independent of time. This means that an equal-time operator A is conserved if and only if it satisfies dA=0. Thus, we have a tool allowing us to identify the conserved operators without resorting to Noether's theorem.

A special subalgebra, $F[t] \subset E[t]$, is formed by Fourier operators with local kernels, $\mathfrak{A}(t,\vec{x},\vec{x}') = \mathfrak{A}(t,\vec{x}-\vec{x}')$, allowing three-dimensional Fourier representations,

$$\mathfrak{A}(t,\vec{x}) = \int \mathrm{d}^3p \frac{\mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}}}{(2\pi)^3}\hat{A}(t,\vec{p}), \tag{51}$$

depending on the matrices $\hat{A}(t,\vec{p}) \in \rho_D$, which we call the Fourier transforms of the operators A. Then, the action (46) on a field (23) can be written as

$$\begin{aligned}(A\psi)(t,\vec{x}) &= \int \mathrm{d}^3x' \mathfrak{A}(t,\vec{x}-\vec{x}')\psi(t,\vec{x}') \\ &= \int \mathrm{d}^3p \sum_\sigma \left[\hat{A}(t,\vec{p})U_{\vec{p},\sigma}(t,\vec{x})\alpha_\sigma(\vec{p}) + \hat{A}(t,-\vec{p})V_{\vec{p},\sigma}(t,\vec{x})\beta^*_\sigma(\vec{p})\right].\end{aligned} \tag{52}$$

One can verify that a Fourier operator A is self-adjoint with respect to the scalar product (4) if its Fourier transform is a Hermitian matrix, $\hat{A}(t,\vec{p}) = \hat{A}(t,\vec{p})^+$.

In the $F[t]$ algebra, the operator multiplication,

$A = A_1 A_2$, is given by the convolution of the corresponding kernels, $\mathfrak{A} = \mathfrak{A}_1 * \mathfrak{A}_2$, defined as

$$\mathfrak{A}(t,\vec{x}-\vec{x}\,') = \int d^3x''\mathfrak{A}_1(t,\vec{x}-\vec{x}\,'')\mathfrak{A}_2(t,\vec{x}\,''-\vec{x}\,'), \tag{53}$$

which leads to the multiplication, $\hat{A}(t,\vec{p}) = \hat{A}_1(t,\vec{p})\hat{A}_2(t,\vec{p})$, of the Fourier transforms. Thus, one obtains the new algebra $\hat{F}[t]$ in MR, formed by the Fourier transforms of the Fourier operators, in which the identity is the matrix $\hat{I}(\vec{p}) = 1 \in \rho_D$. Obviously, the operator $A \in F[t]$ is invertible if its Fourier transform is invertible in $\hat{F}[t]$.

As there are many equal-time or Fourier operators whose kernels are independent of time, we denote their algebras by $F[0] \subset E[0]$, observing that the time-independent Fourier transforms of the operators of the $F[0]$ algebra constitute the algebra $\hat{F}[0]$. An example is the Dirac Hamiltonian (3), whose Fourier transform

$$\hat{H}_D(\vec{p}) = m\gamma^0 + \gamma^0\vec{\gamma}\cdot\vec{p} \in \hat{F}[0], \tag{54}$$

acts as

$$\hat{H}_D(\vec{p})U_{\vec{p},\sigma}(x) = E(p)U_{\vec{p},\sigma}(x), \tag{55}$$

$$\hat{H}_D(-\vec{p})V_{\vec{p},\sigma}(x) = -E(p)V_{\vec{p},\sigma}(x). \tag{56}$$

Other elementary examples are the momentum-independent matrices of ρ_D, γ^μ, $s_{\mu\nu}$, *etc*. which can be seen as Fourier operators whose Fourier transforms are just the matrices themselves.

During the last century, many authors have preferred to work in the $\hat{F}[0]$ algebra, exclusively manipulating the time-independent Fourier transforms of the operators under consideration. In this manner, Pryce proposed his versions (c), (d), and (e) of related spin and position operators and a complete set of orthogonal projection operators, defining their Fourier transforms [5]. In the same pape, r Pryce proposed a transformation that differs only through a parity from the famous Foldy-Wouthuysen transformation proposed two years later [6], whose action remains exclusively at the level of the $\hat{F}[0]$ algebra.

B. Diagonal and oscillating terms

The Pryce projection operators, $\Pi_\pm \in F[0]$, are defined by their Fourier transforms from $\hat{F}[0]$ that read

$$\hat{\Pi}_+(\vec{p}) = \frac{m}{E(p)} l_{\vec{p}} \frac{1+\gamma^0}{2} l_{\vec{p}} = \frac{1}{2}\left(1 + \frac{\hat{H}_D(\vec{p})}{E(p)}\right), \tag{57}$$

$$\hat{\Pi}_-(\vec{p}) = \frac{m}{E(p)} l_{\vec{p}}^{-1} \frac{1-\gamma^0}{2} l_{\vec{p}}^{-1} = \frac{1}{2}\left(1 - \frac{\hat{H}_D(\vec{p})}{E(p)}\right), \tag{58}$$

where $\hat{H}_D(\vec{p})$, defined by Eq. (54), can now be written in the form

$$\hat{H}_D(\vec{p}) = E(p)\left[\hat{\Pi}_+(\vec{p}) - \hat{\Pi}_-(\vec{p})\right]. \tag{59}$$

Moreover, according to Eq. (56), we verify that

$$\begin{aligned}
(\Pi_+ U_{\vec{p},\sigma})(x) &= \hat{\Pi}_+(\vec{p})U_{\vec{p},\sigma}(x) = U_{\vec{p},\sigma}(x),\\
(\Pi_- U_{\vec{p},\sigma})(x) &= \hat{\Pi}_-(\vec{p})U_{\vec{p},\sigma}(x) = 0,\\
(\Pi_+ V_{\vec{p},\sigma})(x) &= \hat{\Pi}_+(-\vec{p})V_{\vec{p},\sigma}(x) = 0,\\
(\Pi_- V_{\vec{p},\sigma})(x) &= \hat{\Pi}_-(-\vec{p})V_{\vec{p},\sigma}(x) = V_{\vec{p},\sigma}(x),
\end{aligned}$$

concluding that the operators $\Pi_+ = \Pi_+^2$ and $\Pi_- = \Pi_-^2$ satisfy $\Pi_+\Pi_- = \Pi_-\Pi_+ = 0$ and $\Pi_+ + \Pi_- = I$, thus forming a complete system of orthogonal projection operators. With their help, one may separate the subspaces of positive and negative frequencies, $\Pi_+\mathcal{F} = \mathcal{F}^+$ and $\Pi_-\mathcal{F} = \mathcal{F}^-$ [5]. These projection operators allow us to define the new operator $N \in F[0]$ with Fourier transform

$$\begin{aligned}
\hat{N}(\vec{p}) = \hat{\Pi}_+(\vec{p}) - \hat{\Pi}_-(\vec{p}) = \frac{\hat{H}_D(\vec{p})}{E(p)},\\
\Rightarrow \quad \hat{N}^2(\vec{p}) = 1 \in \rho_D \Rightarrow N^2 = I.
\end{aligned} \tag{60}$$

We postpone its interpretation as it is discussed later.

The Pryce projection operators help us to study how an operator $A \in E[t]$ acts on the orthogonal subspaces of $\mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^-$, resorting to the expansion

$$\begin{aligned}
A &= A^{(+)} + A^{(-)} + A^{(\pm)} + A^{(\mp)}\\
&= \Pi_+ A \Pi_+ + \Pi_- A \Pi_- + \Pi_+ A \Pi_- + \Pi_- A \Pi_+
\end{aligned} \tag{61}$$

suggested by Pryce [5] and written here in a self-explanatory notation. When A is a Hermitian operator, we have

$$\left[A^{(+)}\right]^+ = A^{(+)}, \quad \left[A^{(-)}\right]^+ = A^{(-)}, \quad \left[A^{(\pm)}\right]^+ = A^{(\mp)}. \tag{62}$$

The first two terms form the *diagonal* part of A, denoted by $A_{\rm diag} = A^{(+)} + A^{(-)}$, which does not mix the subspaces $\mathcal{F}^+$ and $\mathcal{F}^-$ among themselves. The off-diagonal terms, $A^{(\pm)}$ and $A^{(\mp)}$, are nilpotent operators changing the sign of frequency. Under such circumstances, we adopt the following definition: an equal-time operator $A \in E[t]$ is said to be *reducible* if $A = A_{\rm diag}$ as $A^{(\pm)} = A^{(\mp)} = 0$. Otherwise, the operator is irreducible with off-diagonal terms.

In the case of time-dependent Fourier operators $A\in F[t]$, the expansion (61) gives the equivalent expansion of the Fourier transforms in $\hat{F}[t]$ algebra that reads

$$\begin{aligned}\hat{A}(t,\vec{p}) &= \hat{A}^{(+)}(t,\vec{p})+\hat{A}^{(-)}(t,\vec{p})+\hat{A}^{(\pm)}(t,\vec{p})+\hat{A}^{(\mp)}(t,\vec{p})\\ &= \hat{\Pi}_+(\vec{p})\hat{A}(t,\vec{p})\hat{\Pi}_+(\vec{p})+\hat{\Pi}_-(\vec{p})\hat{A}(t,\vec{p})\hat{\Pi}_-(\vec{p})\\ &\quad+\hat{\Pi}_+(\vec{p})\hat{A}(t,\vec{p})\hat{\Pi}_-(\vec{p})+\hat{\Pi}_-(\vec{p})\hat{A}(t,\vec{p})\hat{\Pi}_+(\vec{p}). \end{aligned} \tag{63}$$

In addition, we observe that the total time derivative (49) acts on the Fourier transforms of the operator A as

$$\mathrm{d}\hat{A}(t,\vec{p}) = \partial_t\hat{A}(t,\vec{p})+\mathrm{i}\left[\hat{H}_D(\vec{p}),\hat{A}(t,\vec{p})\right]. \tag{64}$$

Considering that the operator (59) depends on Pryce's projection operators, we can calculate the following commutators:

$$\left[\hat{H}_D(\vec{p}),\hat{A}^{(+)}(t,\vec{p})\right] = \left[\hat{H}_D(\vec{p}),\hat{A}^{(-)}(t,\vec{p})\right] = 0, \tag{65}$$

$$\left[\hat{H}_D(\vec{p}),\hat{A}^{(\pm)}(t,\vec{p})\right] = 2E(p)\hat{A}^{(\pm)}(t,\vec{p}), \tag{66}$$

$$\left[\hat{H}_D(\vec{p}),\hat{A}^{(\mp)}(t,\vec{p})\right] = -2E(p)\hat{A}^{(\mp)}(t,\vec{p}), \tag{67}$$

concluding that a Fourier operator A is conserved (obeying dA=0) only if this is reducible and independent of time, $A=A_{\rm diag}\in F[0]$. In fact, all the diagonal parts of the Fourier operators of the algebra $F[0]$ are conserved. In contrast, the off-diagonal terms are oscillating in time with frequency $2E(p)$, resulting from Eqs. (66) and (67). These terms form the *oscillating* part $A_{\rm osc}=A^{(\pm)}+A^{(\mp)}$ of the operator A. A well-known example is the operator of Dirac's current density, whose oscillating terms give rise to Zitterbewegung [2, 3, 21, 22].

We must stress that the criteria for selecting conserved Fourier operators cannot be extended to any equal-time operators, even those satisfying a similar condition $A=A_{\rm diag}\in E[0]$. An example is the position operator, which satisfies this condition but evolves linearly in time, as we shall show in Sec. IV.B.

C. Pryce (e) spin and related operators

Pryce's principal proposal is his version (e) defining the Fourier transforms of a conserved spin operator $\vec{S}_{\rm Pr(e)}$ related to a suitable correction to the coordinate operator, $\delta\vec{X}_{\rm Pr(e)}$. These Fourier transforms

$$\hat{\vec{S}}_{\rm Pr(e)}(\vec{p}) = \frac{m}{E(p)}\vec{s}+\frac{\vec{p}(\vec{s}\cdot\vec{p})}{E(p)(E(p)+m)}+\frac{\mathrm{i}}{2E(p)}\vec{p}\wedge\vec{\gamma}, \tag{68}$$

$$\delta\vec{X}_{\rm Pr(e)}(\vec{p}) = \frac{\mathrm{i}\vec{\gamma}}{2E(p)}+\frac{\vec{p}\wedge\vec{s}}{E(p)(E(p)+m)}-\frac{\mathrm{i}\vec{p}(\vec{\gamma}\cdot\vec{p})}{2E(p)^2(E(p)+m)} \tag{69}$$

satisfy the identity $\delta\vec{X}_{\rm Pr(e)}(\vec{p})\wedge\vec{p}=\vec{s}-\hat{\vec{S}}_{\rm Pr(e)}(\vec{p})$ to ensure the conservation of the total angular momentum (14). The Pryce (e) spin operator was considered later by Foldy and Wouthuysen, who showed that their operator (A17) transforms the Pryce (e) spin operator into the Pauli-Dirac one in Eq. (A19). For this reason, many authors consider the Pryce (e) spin operator as the Foldy-Wouthuysena one, denoting it by $\vec{S}_{\rm FW}$ [11, 12]. In the following, we use the simpler notation of the spin operator $\vec{S}\equiv\vec{S}_{\rm Pr(e)}\equiv\vec{S}_{\rm FW}\in F[0]$, and similarly, for its Fourier transform, $\hat{\vec{S}}(\vec{p})\equiv\hat{\vec{S}}_{\rm Pr(e)}(\vec{p})\in\hat{F}[0]$, defined by Eq. (68).

In Ref. [18], we considered a spectral representation to show that $\vec{S}$ is just the operator defined by Eq. (16), whose components generate the spin symmetry. We found that its Fourier transform (68) can be put in the form [18]

$$\begin{aligned}\hat{\vec{S}}(\vec{p}) &= \frac{m}{E(p)}\left[l_{\vec{p}}\,\vec{s}\,\frac{1+\gamma^0}{2}l_{\vec{p}}+l_{\vec{p}}^{-1}\,\vec{s}\,\frac{1-\gamma^0}{2}l_{\vec{p}}^{-1}\right]\\ &= \vec{s}(\vec{p})\hat{\Pi}_+(\vec{p})+\vec{s}(-\vec{p})\hat{\Pi}_-(\vec{p}),\end{aligned} \tag{70}$$

laying out the operator

$$\hat{\vec{S}}_{\rm Ch}(\vec{p})\equiv\vec{s}(\vec{p})=l_{\vec{p}}\vec{s}l_{\vec{p}}^{-1}\in\hat{F}[0], \tag{71}$$

which was proposed by Chakrabarti [8] as the Fourier transform of an alternative spin operator, $\vec{S}_{\rm Ch}\in F[0]$. However, this operator is not conserved, having the same action as the Pryce (e) one but only in the particle sector, while in the antiparticle sector, there is a discrepancy generating oscillating terms, as we shall show in Sec. V.C. Nevertheless, the properties of the Chakrabarti operator,

$$\vec{s}(\vec{p})=\vec{s}^{\,+}(-\vec{p}),\quad \vec{s}(\pm\vec{p})\hat{\Pi}_\pm(\vec{p})=\hat{\Pi}_\pm(\vec{p})\vec{s}(\mp\vec{p}), \tag{72}$$

guarantee that $\vec{S}$ is a conserved Hermitian operator, and its Fourier transform obeys $\hat{\vec{S}}(\vec{p})=\hat{\vec{S}}^{+}(\vec{p})=\hat{\vec{S}}_{\rm diag}(\vec{p})\in\hat{F}[0]$. In addition, the components S_i are translation invariant, commuting with the momentum operator, having similar algebraic properties to the Pauli-Dirac operator,

$$\begin{aligned}&\left[\hat{S}_i(\vec{p}),\hat{S}_j(\vec{p})\right]=\mathrm{i}\epsilon_{ijk}\hat{S}_k(\vec{p})\ \Rightarrow\ \left[S_i,S_j\right]=\mathrm{i}\epsilon_{ijk}S_k,\\ &\left\{\hat{S}_i(\vec{p}),\hat{S}_j(\vec{p})\right\}=\frac{1}{2}\delta_{ij}\cdot 1\in\rho_D\ \Rightarrow\ \left\{S_i,S_j\right\}=\frac{1}{2}\delta_{ij}I,\\ &\hat{\vec{S}}^2(\vec{p})=\frac{3}{4}\cdot 1\in\rho_D\ \Rightarrow\ \vec{S}^2=\frac{3}{4}I,\end{aligned}$$

thus defining a spin half representation of the $SU(2)$ group. Furthermore, to explicitly write the action of this operator, we re-denote $\psi \to \psi_\xi$, $U_{\vec{p},\sigma} \to U_{\vec{p},\xi_\sigma}$, and $V_{\vec{p},\sigma} \to V_{\vec{p},\eta_\sigma}$. Then, by using the form of the spinors (26) and (27), we may write the actions

$$(S_i U_{\vec{p},\xi_\sigma})(x) = \hat{S}_i(\vec{p}) U_{\vec{p},\xi_\sigma}(x) = U_{\vec{p},\hat{s}_i\xi_\sigma}(x), \tag{73}$$

$$(S_i V_{\vec{p},\eta_\sigma})(x) = \hat{S}_i(-\vec{p}) V_{\vec{p},\eta_\sigma}(x) = V_{\vec{p},\hat{s}_i\eta_\sigma}(x), \tag{74}$$

concluding that $\vec{S}(\vec{p})$ is just the Fourier transform of the spin operator $\vec{S}$ defined by Eq. (16). The integral representation helping us to derive the identity (70) will be discussed and generalized in Sec. IV.D.

In applications, we may use the new auxiliary operators $\vec{S}^{(+)}$ and $\vec{S}^{(-)}$ whose components have the Fourier transforms

$$\hat{S}_i^{(+)}(\vec{p}) = \Theta_{ij}(\vec{p})\hat{S}_j(\vec{p}), \quad \hat{S}_i^{(-)}(\vec{p}) = \Theta^{-1}_{ij}(\vec{p})\hat{S}_j(\vec{p}), \tag{75}$$

where $\Theta(\vec{p})$ is the $SO(3)$ tensor defined in Eq. (A13) as the space part of the Lorentz boost $L_{\vec{p}}$ given by Eq. (A12). With these notations, the Fourier transform of the Pauli-Lubanski operator (11) can now be written as

$$\begin{aligned} &\hat{W}^\mu(\vec{p}) = m(L_{\vec{p}})^{\mu}_{\cdot i}\hat{S}_i(\vec{p}) \Rightarrow \\ &\hat{W}^0(\vec{p}) = \vec{p}\cdot\vec{\hat{S}}(\vec{p}) = \vec{p}\cdot\vec{s}, \quad \vec{\hat{W}}(\vec{p}) = m\vec{\hat{S}}^{(+)}(\vec{p}), \end{aligned} \tag{76}$$

satisfying $p^\mu \hat{W}_\mu(\vec{p}) = 0$ and $\hat{W}^\mu(\vec{p})\hat{W}_\mu(\vec{p}) = -m^2\frac{3}{4}\cdot 1 \in \rho_D$.

The form of the Pryce (e) spin operator allows us to define the operator of fermion polarization for any related polarization spinors, $\xi_\sigma(\vec{p})$ and $\eta_\sigma(\vec{p})$, satisfying the general eigenvalues problems

$$\hat{s}_i n^i(\vec{p})\xi_\sigma(\vec{p}) = \sigma\,\xi_\sigma(\vec{p}) \Rightarrow \hat{s}_i n^i(\vec{p})\eta_\sigma(\vec{p}) = -\sigma\,\eta_\sigma(\vec{p}), \tag{77}$$

where the unit vector $\vec{n}(\vec{p})$ gives the peculiar direction of spin projection. The corresponding polarization operator may be defined as the Fourier operator $W_s \in F[0]$, whose Fourier transform reads [18]

$$\hat{W}_s(\vec{p}) = w(\vec{p})\hat{\Pi}_+(\vec{p}) + w(-\vec{p})\hat{\Pi}_-(\vec{p}), \tag{78}$$

where $w(\vec{p}) = \vec{s}(\vec{p})\cdot\vec{n}(\vec{p})$. As in the case of the spin operator, we find that the operator of fermion polarization acts as

$$\begin{aligned} (W_s U_{\vec{p},\xi_\sigma(\vec{p})})(x) &= \hat{W}_s(\vec{p}) U_{\vec{p},\xi_\sigma(\vec{p})}(x) \\ &= U_{\vec{p},\hat{s}_i n^i(\vec{p})\xi_\sigma(\vec{p})}(x) = \sigma U_{\vec{p},\xi_\sigma(\vec{p})}(x), \end{aligned} \tag{79}$$

$$\begin{aligned} (W_s V_{\vec{p},\eta_\sigma(\vec{p})})(x) &= \hat{W}_s(-\vec{p}) V_{\vec{p},\eta_\sigma(\vec{p})}(x) \\ &= V_{\vec{p},\hat{s}_i n^i(\vec{p})\eta_\sigma(\vec{p})}(x) = -\sigma V_{\vec{p},\eta_\sigma(\vec{p})}(x). \end{aligned} \tag{80}$$

These eigenvalue problems demonstrate that W_s is the operator we need to complete the system of commuting operators $\{H, P^1, P^2, P^3, W_s\}$ defining the momentum bases of RQM.

Finally, we remind the reader that the conserved spin operator (70) is related to Pryce's position operator of version (e), whose correction $\delta\vec{X}$ has the Fourier transform (69), which can be written in the simpler form [18]

$$\delta\vec{\hat{X}}(\vec{p}) \equiv \delta\vec{\hat{X}}_{\rm Pr(e)}(\vec{p}) = \delta\vec{x}_+(\vec{p})\hat{\Pi}_+(\vec{p}) + \delta\vec{x}_-(\vec{p})\hat{\Pi}_-(\vec{p}), \tag{81}$$

where the components of $\delta\vec{x}_\pm(\vec{p})$ have the form

$$\delta x^i_\pm(\vec{p}) = -{\rm i}\frac{1}{n(p)}\left(\partial_{p^i} n(p) l_{\pm\vec{p}}\right) l_{\mp\vec{p}}, \tag{82}$$

depending on the normalization factor (33) and momentum derivatives. However, we cannot construct the whole position operator $\vec{X} = \underline{\vec{x}} + \delta\vec{X}$ with the tools we considered so far because of the coordinate operator $\underline{\vec{x}}$, which is no longer a Fourier one. For this reason, we shall study this operator in Sec. IV.B after constructing a convenient framework.

D. Other spin-type and position operators

Other conserved spin-type Fourier operators that cannot be integrated naturally in Dirac's theory have been proposed, as in the case of the Pryce (e) one, because their components do not satisfy $su(2)$ commutation relations. Nevertheless, these operators deserve to be briefly examined as they represent observables that could be measured in some dedicated experiments [11, 12].

The oldest proposal is the Frankel spin-type operator, which is a Fourier operator, $\vec{S}_{\rm Fr}$, with the Fourier transform [4]

$$\begin{aligned} \vec{\hat{S}}_{\rm Fr}(\vec{p}) &= \vec{s} + \frac{\rm i}{2m}\vec{p}\wedge\vec{\gamma} \\ &= \frac{E(p)}{m}\left(\vec{\hat{S}}(\vec{p}) - \frac{\vec{p}(\vec{p}\cdot\vec{\hat{S}}(\vec{p}))}{E(p)(E(p)+m)}\right) \\ &= \frac{E(p)}{m}\vec{\hat{S}}^{(-)}(\vec{p}), \end{aligned} \tag{83}$$

where the notation is the same as that for (75). The components of this operator are conserved and translation invariant, commuting with H_D and P^i, but these do not satisfy the $su(2)$ algebra such that the squared norm,

$$\hat{\vec{S}}^2_{\rm Fr}(\vec{p}) = \frac{1}{4}\left(1+2\frac{E(p)^2}{m^2}\right)\cdot 1 \in \rho_D\,, \tag{84}$$

is larger than 3/4. The Frankel spin-type operator may be generated as

$$\left[\hat{S}^{(+)}_i(\vec{p}),\hat{S}^{(+)}_j(\vec{p})\right] = {\rm i}\epsilon_{ijk}\hat{S}_{{\rm Fr}k}(\vec{p})$$
$$\Rightarrow \left[S^{(+)}_i,S^{(+)}_j\right] = {\rm i}\epsilon_{ijk}S_{{\rm Fr}k}\,, \tag{85}$$

with specific commutation rules

$$\left[\hat{S}_{{\rm Fr}i}(\vec{p}),\hat{S}_{{\rm Fr}j}(\vec{p})\right] = {\rm i}\epsilon_{ijk}\hat{C}_{{\rm Fr}k}(\vec{p})$$
$$\Rightarrow \left[S_{{\rm Fr}i},S_{{\rm Fr}j}\right] = {\rm i}\epsilon_{ijk}C_{{\rm Fr}k}\,, \tag{86}$$

which define the new Fourier operator $\vec{C}_{\rm Fr}$ whose Fourier transform reads

$$\hat{\vec{C}}_{\rm Fr}(\vec{p}) = \frac{E(p)}{m}\left(\hat{\vec{S}}(\vec{p})+\frac{\vec{p}(\vec{p}\cdot\hat{\vec{S}}(\vec{p}))}{m(E(p)+m)}\right) = \frac{E(p)}{m}\hat{\vec{S}}^{(+)}(\vec{p})\,. \tag{87}$$

A similar spin-type operator was considered initially by Pryce according to his hypothesis (c) [5] and then redefined and studied by Czochor [9] such that this is often caled the Czochor spin operator [11, 12]. Here, we speak about the Pryce (c)-Czochor (PC) operator defined as the diagonal part of the Pauli-Dirac one [9],

$$\vec{S}_{\rm PC} = \Pi_+\vec{s}\,\Pi_+ + \Pi_-\vec{s}\,\Pi_-\,. \tag{88}$$

This has the Fourier transform [9, 11, 12]

$$\hat{\vec{S}}_{\rm PC}(\vec{p}) = \hat{\Pi}_+(\vec{p})\vec{s}\,\hat{\Pi}_+(\vec{p})+\hat{\Pi}_-(\vec{p})\vec{s}\,\hat{\Pi}_-(\vec{p})$$
$$= \frac{m^2}{E(p)^2}\vec{s}+\frac{\vec{p}(\vec{p}\cdot\vec{s})}{E(p)^2}+\frac{{\rm i}m}{2E(p)^2}\vec{p}\wedge\vec{\gamma}$$
$$= \frac{m}{E(p)}\hat{\vec{S}}^{(+)}(\vec{p}), \tag{89}$$

whose squared norm,

$$\hat{\vec{S}}^2_{\rm PC}(\vec{p}) = \frac{1}{4}\left(1+2\frac{m^2}{E(p)^2}\right)\cdot 1 \in \rho_D\,, \tag{90}$$

takes values in the domain $\left(\frac{1}{4},\frac{3}{4}\right]$. The Pryce (c)-Czochor spin-type operator may be generated as

$$\left[\hat{S}^{(-)}_i(\vec{p}),\hat{S}^{(-)}_j(\vec{p})\right] = {\rm i}\epsilon_{ijk}\hat{S}_{{\rm PC}k}(\vec{p})$$
$$\Rightarrow \left[S^{(-)}_i,S^{(-)}_j\right] = {\rm i}\epsilon_{ijk}S_{{\rm PC}k}\,, \tag{91}$$

satisfying the commutation relations

$$\left[\hat{S}_{{\rm PC}i}(\vec{p}),\hat{S}_{{\rm PC}j}(\vec{p})\right] = {\rm i}\epsilon_{ijk}\hat{C}_{{\rm PC}k}(\vec{p})$$
$$\Rightarrow \left[S_{{\rm PC}i},S_{{\rm PC}j}\right] = {\rm i}\epsilon_{ijk}C_{{\rm PC}k}\,, \tag{92}$$

where the Fourier transform of the new operator $\vec{C}_{\rm PC}$ reads

$$\hat{\vec{C}}_{\rm PC}(\vec{p}) = \frac{m}{E(p)}\hat{\vec{S}}^{(-)}(\vec{p})\,. \tag{93}$$

We conclude that the Frankel and Pryce (c)-Czochor spin-type operators are elements of a larger algebraic structure depending only on the pair of operators $\vec{S}^{(+)}$ and $\vec{S}^{(-)}$. In other respects, all the Fourier transforms of conserved spin and spin-type operators discussed so far have the same projection along the momentum direction such that

$$\vec{p}\cdot\hat{\vec{S}}_{\rm Fr}(\vec{p}) = \vec{p}\cdot\hat{\vec{S}}_{\rm PC}(\vec{p}) = \vec{p}\cdot\hat{\vec{S}}(\vec{p}) = \vec{p}\cdot\vec{s}\in\hat{F}[0]\,. \tag{94}$$

This means that we can inverse Eqs. (83) and (89), relating the operators $\hat{\vec{S}}_{\rm Fr}(\vec{p})$ and $\hat{\vec{S}}_{\rm PC}(\vec{p})$ and implicitly their commutator operators, $\hat{\vec{C}}_{\rm Fr}(\vec{p})$ and $\hat{\vec{C}}_{\rm PC}(\vec{p})$, at any time .

Another conserved and translation-invariant operator was proposed by Fradkin and Good [7]. Its Fourier transform is defined as

$$\hat{\vec{S}}_{\rm FG}(\vec{p}) = \gamma^0\vec{s}+\frac{\vec{p}(\vec{p}\cdot\vec{s})}{p^2}\left(\frac{\hat{H}_D(\vec{p})}{E(p)}-\gamma^0\right)$$
$$= \hat{\vec{S}}(\vec{p})\hat{N}(\vec{p}) \;\Rightarrow\; \vec{S}_{\rm FG} = \vec{S}N\,, \tag{95}$$

where the operator N has the Fourier transform (60). As N commutes with the spin operator $\vec{S}$ and $N^2 = I$, we may write the commutators directly as

$$\left[S_{{\rm FG}i},S_{{\rm FG}j}\right] = {\rm i}\epsilon_{ijk}NS_{{\rm FG}k}\,,\;\Rightarrow\; \vec{S}^2_{\rm FG} = \vec{S}^2 = \frac{3}{4}I\,, \tag{96}$$

which guarantee a desired square norm but without defining a Lie algebra. The simple algebraic properties of the Fradkin-Good spin-type operator indicate that this is somewhat useless as it is equivalent with the Pryce (e) one. Other operators proposed recently [16, 17] could be related to the above spin and spin-type operators in further investigations.

The Pryce (c)-Czochor spin-type operator was constructed from the beginning according to Pryce's hypothesis (c). Moreover, it is not difficult to verify that the Frankel one complies with the hypothesis (d) such that both these operators are related to specific position operators, $\vec{X}_{\rm Pr(c)} = \underline{\vec{x}} + \delta\vec{X}_{\rm Pr(c)}$ and $\vec{X}_{\rm Pr(d)} = \underline{\vec{x}} + \delta\vec{X}_{\rm Pr(d)}$, respectively. Observing that the corrections are Fourier operators, it is convenient to use the artifice

$$\vec{X}_{\rm Pr(c)} = \vec{X} + \delta\vec{X}_{\rm Pr(c)} - \delta\vec{X}, \quad \vec{X}_{\rm Pr(d)} = \vec{X} + \delta\vec{X}_{\rm Pr(d)} - \delta\vec{X}, \tag{97}$$

providing us with simple Fourier transforms

$$\delta\vec{\tilde{X}}_{\rm Pr(c)}(\vec{p}) - \delta\vec{\tilde{X}}(\vec{p}) = \frac{\vec{p}\wedge\vec{\tilde{S}}(\vec{p})}{E(p)(E(p)+m)}, \tag{98}$$

$$\delta\vec{\tilde{X}}_{\rm Pr(d)}(\vec{p}) - \delta\vec{\tilde{X}}(\vec{p}) = -\frac{\vec{p}\wedge\vec{\tilde{S}}(\vec{p})}{m(E(p)+m)}, \tag{99}$$

resulting from the formulas of Ref. [5]. These position operators give alternative splittings of the total angular momentum,

$$\vec{J} = \vec{X}_{\rm Pr(c)} \wedge \vec{P} + \vec{S}_{\rm PC} = \vec{X}_{\rm Pr(d)} \wedge \vec{P} + \vec{S}_{\rm Fr},$$

but they are formal, without a precise physical meaning, as the components of the position operators do not commute among themselves, while those of the spin-type operators do not satisfy an $su(2)$ algebra. The only attribute of the above spin-type and related orbital angular momentum operators is that they are conserved.

We conclude that the study of various position operators reduces to the Pryce (e) one, which must be derived after passing beyond the technical difficulties of constructing another suitable effective framework.

IV. METHOD OF ASSOCIATED OPERATORS

The difficulties arising in Dirac's theory come from the fact that there are many equal-time integral operators with bi-local kernels that do not have Fourier transforms. To study such operators, we must resort to integral representations that can only be defined properly by relating the operators acting on the free fields to pairs of operators acting on the wave spinors (37); here, we call these *associated* operators. In other worlds, we transfer the action of a given operator from mode spinors to the wave spinors, thus obtaining a tool for systematically deriving expectation values in terms of the wave spinors we need for preparing the quantization.

A. Associated operators

We start by associating to each operator $A:\mathcal{F}\to\mathcal{F}$ in CR the pair of operators $\tilde{A}:\tilde{\mathcal{F}}^+\to\tilde{\mathcal{F}}$ and $\tilde{A}^c:\tilde{\mathcal{F}}^-\to\tilde{\mathcal{F}}$, obeying

$$\begin{aligned}(A\psi)(x) &= \int d^3p \sum_\sigma \left[(AU_{\vec{p},\sigma})(x)\alpha)_\sigma(\vec{p}) + (AV_{\vec{p},\sigma})(x)\beta^*_\sigma(\vec{p})\right]\\ &\equiv \int d^3p \sum_\sigma \left[U_{\vec{p},\sigma}(x)(\tilde{A}\alpha)_\sigma(\vec{p}) + V_{\vec{p},\sigma}(x)(\tilde{A}^c\beta)^*_\sigma(\vec{p})\right],\end{aligned} \tag{100}$$

such that the brackets of A for two different fields, ψ and ψ', can be calculated as

$$\langle\psi, A\psi'\rangle_D = \langle\alpha,\tilde{A}\alpha'\rangle + \langle\beta,\tilde{A}^{c+}\beta'\rangle. \tag{101}$$

Hereby, we deduce that if $A = A^+$ is Hermitian with respect to the Dirac scalar product (4), then the associated operators are Hermitian with respect to the scalar product (39), $\tilde{A} = \tilde{A}^+$ and $\tilde{A}^c = \tilde{A}^{c\,+}$. For simplicity, we denote the Hermitian conjugation of the operators acting on the spaces $\mathcal{F}$ and $\tilde{\mathcal{F}}$ with the same symbol but bearing in mind that the scalar products of these spaces are different.

In general, the operators $A\in E[t]$ and their associated operators $(\tilde{A},\tilde{A}^c)$ may depend on time such that we must be careful considering the entire algebra we manipulate as frozen at a fixed time t. The new operators $\tilde{A}$ and $\tilde{A}^c$ are well-defined at any time as their action can be derived by applying the inversion formulas (38) to Eq. (100) at a given instance t. Thus, we find that $\tilde{A}$ and $\tilde{A}^c$ are integral operators that may depend on time acting as

$$\begin{aligned}(\tilde{A}\alpha)_\sigma(\vec{p})\big|_t &= \int d^3p' \sum_{\sigma'} \langle U_{\vec{p},\sigma}, AU_{\vec{p}',\sigma'}\rangle_D\big|_t\, \alpha_{\sigma'}(\vec{p}')\\ &\quad + \int d^3p' \sum_{\sigma'} \langle U_{\vec{p},\sigma}, AV_{\vec{p}',\sigma'}\rangle_D\big|_t \beta^*_{\sigma'}(\vec{p}'),\end{aligned} \tag{102}$$

$$\begin{aligned}(\tilde{A}^c\beta)_\sigma(\vec{p})\big|_t &= \int d^3p' \sum_{\sigma'} \langle U_{\vec{p}',\sigma'}, AV_{\vec{p},\sigma}\rangle_D\big|_t\, \alpha^*_{\sigma'}(\vec{p}')\\ &\quad + \int d^3p' \sum_{\sigma'} \langle V_{\vec{p}',\sigma'}, AV_{\vec{p},\sigma}\rangle_D\big|_t \beta_{\sigma'}(\vec{p}'),\end{aligned} \tag{103}$$

through kernels that are the matrix elements of the operator A in the basis of mode spinors. Thus, we obtain the association $A \Leftrightarrow (\tilde{A},\tilde{A}^c)$ defined through Eq. (100), which is a bijective mapping between two isomorphic operator algebras, $E[t]\subset{\rm Aut}(\mathcal{F})$ and $\tilde{E}[t]\oplus\tilde{E}^c[t]\subset{\rm Aut}(\tilde{\mathcal{F}})$, preserving the linear and multiplication properties. Obviously, the identity operator of the algebras $\tilde{E}[t]$ and $\tilde{E}[t]^c$ is the matrix $1_{2\times2}$. To analyze the actions of these operators, we rewrite Eqs. (102) and (103) as

$$(\tilde{A}\alpha)_\sigma(\vec{p})\big|_t = (\tilde{A}^{(+)}\alpha)_\sigma(\vec{p})\big|_t + (\tilde{A}^{(\pm)}\beta^*)_\sigma(\vec{p})\big|_t\,, \tag{104}$$

$$(\tilde{A}^c\beta)_\sigma(\vec{p})\big|_t = (\tilde{A}^{(\mp)}\alpha^*)_\sigma(\vec{p})\big|_t + (\tilde{A}^{(-)}\beta)_\sigma(\vec{p})\big|_t \tag{105}$$

in terms of the new associated operators,

$$\begin{aligned}&\tilde{A}^{(+)} \in \mathrm{Aut}(\tilde{\mathcal{F}}^+), && \tilde{A}^{(-)} \in \mathrm{Aut}(\tilde{\mathcal{F}}^-)\\ &\tilde{A}^{(\pm)} \in \mathrm{Lin}(\tilde{\mathcal{F}}^+, \tilde{\mathcal{F}}^{-*}), && \tilde{A}^{(\mp)} \in \mathrm{Lin}(\tilde{\mathcal{F}}^-, \tilde{\mathcal{F}}^{+*}),\end{aligned}$$

which are integral operators in MR whose kernels are the matrix elements of the operators $A^{(+)}, A^{(-)}, A^{(\pm)}$, and $A^{(\mp)}$ defined by the expansion (61). Therefore, if $A \in E[t]$ is reducible, then we have

$$A^{(\pm)} = A^{(\mp)} = 0 \;\Rightarrow\; \tilde{A}^{(\pm)} = \tilde{A}^{(\mp)} = 0 \;\Rightarrow\; \begin{cases}\tilde{A} = \tilde{A}^{(+)},\\ \tilde{A}^c = \tilde{A}^{(-)}.\end{cases} \tag{106}$$

Anticipating this, we specify that all the Hermitian reducible operators $A \in E[t]$ we study here have associated operators related through *charge parity*, $\tilde{A}^c = \pm\tilde{A}$.

In the particular case of Fourier operators, $A \in F[t]$, having time-dependent Fourier transforms $\hat{A}(t,\vec{p})$, the matrix elements can be calculated easier as

$$\begin{aligned}\langle U_{\vec{p},\sigma}, AU_{\vec{p}\,',\sigma'}\rangle_D\big|_t &= \langle U_{\vec{p},\sigma}, \hat{A}(t,\vec{p}\,')U_{\vec{p}\,',\sigma'}\rangle_D\big|_t\\ &= \delta^3(\vec{p}-\vec{p}\,')\frac{m}{E(p)}\mathring{u}^+_\sigma(\vec{p})l_{\vec{p}}\hat{A}(t,\vec{p})l_{\vec{p}}\,\mathring{u}_{\sigma'}(\vec{p}),\end{aligned} \tag{107}$$

$$\begin{aligned}&\langle U_{\vec{p},\sigma}, AV_{\vec{p}\,',\sigma'}\rangle_D\big|_t = \langle U_{\vec{p},\sigma}, \hat{A}(t,-\vec{p}\,')V_{\vec{p}\,',\sigma'}\rangle_D\big|_t\\ &= \delta^3(\vec{p}+\vec{p}\,')\frac{m}{E(p)}\mathring{u}^+_\sigma(\vec{p})l_{\vec{p}}\hat{A}(t,\vec{p})l_{-\vec{p}}\,\mathring{v}_{\sigma'}(-\vec{p})\mathrm{e}^{2\mathrm{i}E(p)t},\end{aligned} \tag{108}$$

$$\begin{aligned}&\langle V_{\vec{p}\,',\sigma'}, AU_{\vec{p},\sigma}\rangle_D\big|_t = \langle V_{\vec{p}\,',\sigma'}, \hat{A}(t,\vec{p})U_{\vec{p},\sigma}\rangle_D\big|_t\\ &= \delta^3(\vec{p}+\vec{p}\,')\frac{m}{E(p)}\mathring{v}^+_{\sigma'}(-\vec{p})l_{-\vec{p}}\hat{A}(t,\vec{p})l_{\vec{p}}\,\mathring{u}_\sigma(\vec{p})\mathrm{e}^{-2iE(p)t},\end{aligned} \tag{109}$$

$$\begin{aligned}\langle V_{\vec{p},\sigma}, AV_{\vec{p}\,',\sigma'}\rangle_D\big|_t &= \langle V_{\vec{p},\sigma}, \hat{A}(t,-\vec{p}\,')V_{\vec{p}\,',\sigma'}\rangle_D\big|_t\\ &= \delta^3(\vec{p}-\vec{p}\,')\frac{m}{E(p)}\mathring{v}^+_\sigma(\vec{p})l_{\vec{p}}\hat{A}(t,-\vec{p})l_{\vec{p}}\,\mathring{v}_{\sigma'}(\vec{p}),\end{aligned} \tag{110}$$

observing that in this case, the associated operators are simple 2×2 matrix operators acting on the spaces $\tilde{\mathcal{F}}^+$ and $\tilde{\mathcal{F}}^-$. Hereby, we deduce the matrix elements of the associated diagonal operators

$$\tilde{A}^{(+)}_{\sigma\sigma'}(t,\vec{p}) = \frac{m}{E(p)}\,\mathring{u}^+_\sigma(\vec{p})l_{\vec{p}}\hat{A}(t,\vec{p})l_{\vec{p}}\,\mathring{u}_{\sigma'}(\vec{p}), \tag{111}$$

$$\tilde{A}^{(-)}_{\sigma\sigma'}(t,\vec{p}) = \frac{m}{E(p)}\,\mathring{u}^+_\sigma(\vec{p})l_{\vec{p}}\,C\hat{A}(t,-\vec{p})^T C l_{\vec{p}}\,\mathring{u}_{\sigma'}(\vec{p}), \tag{112}$$

and those of the off-diagonal ones

$$\tilde{A}^{(\pm)}_{\sigma\sigma'}(t,\vec{p}) = \frac{m}{E(p)}\,\mathring{u}^+_\sigma(\vec{p})l_{\vec{p}}\hat{A}(t,\vec{p})l_{-\vec{p}}\,\mathring{v}_{\sigma'}(-\vec{p})\mathrm{e}^{2\mathrm{i}E(p)t}, \tag{113}$$

$$\tilde{A}^{(\mp)}_{\sigma\sigma'}(t,\vec{p}) = \frac{m}{E(p)}\,\mathring{v}^+_{\sigma'}(-\vec{p})l_{-\vec{p}}\hat{A}(t,\vec{p})l_{\vec{p}}\,\mathring{u}_\sigma(\vec{p})\mathrm{e}^{-2\mathrm{i}E(p)t}, \tag{114}$$

which oscillate with frequency $2E(p)$.

B. Associated spin, polarization, and position operators

The simplest examples of reducible Fourier operators are the projection operators related to the operators $I, N \in F[0]$, for which we have to substitute the expressions (57) and (58) in Eqs. (111) and (112) using the identities (A.15) to obtain the associated operators,

$$\begin{aligned}&\Pi_+ \;\Rightarrow\; \tilde{\Pi}_+ = 1_{2\times2}, \quad \tilde{\Pi}^c_+ = 0,\\ &\Pi_- \;\Rightarrow\; \tilde{\Pi}_- = 0, \; \tilde{\Pi}^c_- = 1_{2\times2},\\ &I = \Pi_+ + \Pi_- \;\Rightarrow\; \tilde{I} = \tilde{I}^c = 1_{2\times2},\\ &N = \Pi_+ - \Pi_- \;\Rightarrow\; \tilde{N} = -\tilde{N}^c = 1_{2\times2},\end{aligned}$$

depending on the identity operator $1_{2\times2}$ of $\tilde{F}[0] \simeq \tilde{F}^c[0]$ algebras. More interesting are the operators associated to the new observables of our approach, namely, the spin, fermion polarization, and position operators, which we study in this section.

To derive the operators associated to the Pryce (e) spin $\vec{S}$, we substitute its Fourier transform (70) in Eqs. (111) and (112), taking into account that these operators are reducible, $\vec{S} = \vec{S}_{\rm diag}$. By again using the identity (A.15), we find that the associated operators of $\vec{S}$ have the components [18]

$$S_i \;\Rightarrow\; \tilde{S}_i = -\tilde{S}^c_i = \frac{1}{2}\Sigma_i(\vec{p}), \tag{115}$$

where the 2×2 matrices $\Sigma_i(\vec{p})$ have the matrix elements

$$\Sigma_{i\sigma\sigma'}(\vec{p}) = 2\mathring{u}^+_\sigma(\vec{p})s_i\mathring{u}_{\sigma'}(\vec{p}) = \xi^+_\sigma(\vec{p})\sigma_i\xi_{\sigma'}(\vec{p}), \tag{116}$$

depending on the polarization spinors and having the same algebraic properties as the Pauli matrices. Similar procedures give the operators

$$S_i^{(+)} \Rightarrow \tilde{S}_i^{(+)} = -\tilde{S}_i^{(+)c} = \frac{1}{2}\Theta_{ij}(\vec{p})\Sigma_j(\vec{p}), \tag{117}$$

$$S_i^{(-)} \Rightarrow \tilde{S}_i^{(-)} = -\tilde{S}_i^{(-)c} = \frac{1}{2}\Theta_{ij}^{-1}(\vec{p})\Sigma_j(\vec{p}), \tag{118}$$

associated to those defined by Eq. (75), as well as the simple associated operators of the polarization operator (78),

$$W_s \ \Rightarrow\ \tilde{W}_s = -\tilde{W}_s^c = \frac{1}{2}\sigma_3, \tag{119}$$

according to the definition of the polarization spinors (77).

The position operator, $\vec{X}$, is reducible but is no longer a Fourier operator even though the correction $\delta\vec{X}$ of the Pryce (e) version is of this type with the Fourier transform given by Eqs. (81) and (82). To extract the action of this operator, we apply the Green theorem after deriving the identities [18]

$$\begin{aligned}&\left(\delta X^i U_{\vec{p},\xi_\sigma}\right)(t,\vec{x}) = \delta\tilde{X}^i(\vec{p})U_{\vec{p},\xi_\sigma}(t,\vec{x})\\&= -i\partial_{p^i}U_{\vec{p},\xi_\sigma}(t,\vec{x}) - x^i U_{\vec{p},\xi_\sigma}(t,\vec{x}) + \frac{tp^i}{E(p)}U_{\vec{p},\xi_\sigma}(t,\vec{x})\\&\quad+\sum_{\sigma'}U_{\vec{p},\xi_{\sigma'}}(t,\vec{x})\Omega_{i\sigma'\sigma}(\vec{p}),\end{aligned} \tag{120}$$

$$\begin{aligned}&\left(\delta X^i V_{\vec{p},\eta_\sigma}\right)(t,\vec{x}) = \delta\tilde{X}^i(-\vec{p})V_{\vec{p},\eta_\sigma}(t,\vec{x})\\&= i\partial_{p^i}V_{\vec{p},\eta_\sigma}(t,\vec{x}) - x^i V_{\vec{p},\eta_\sigma}(t,\vec{x}) + \frac{tp^i}{E(p)}V_{\vec{p},\eta_\sigma}(t,\vec{x})\\&\quad-\sum_{\sigma'}V_{\vec{p},\eta_{\sigma'}}(t,\vec{x})\Omega^*_{i\sigma'\sigma}(\vec{p}).\end{aligned} \tag{121}$$

We find that this operator depends linearly on time, $\vec{X}(t)=\vec{X}+t\vec{V}$, and its components have simple and intuitive associated operators [18],

$$X^i \ \Rightarrow\ \tilde{X}^i = \tilde{X}^{ci} = \mathrm{i}\tilde{\partial}_i, \tag{122}$$

$$V^i \ \Rightarrow\ \tilde{V}^i = \tilde{V}^{ci} = \frac{p^i}{E(p)}, \tag{123}$$

where the *covariant* derivatives [18],

$$\tilde{\partial}_i = \partial_{p^i}1_{2\times2} + \Omega_i(\vec{p}), \tag{124}$$

are defined such that $\tilde{\partial}_i[\xi_\sigma(\vec{p})\alpha_\sigma(\vec{p})] = \xi_\sigma(\vec{p})\tilde{\partial}_i\alpha_\sigma(\vec{p})$. Therefore, the connections

$$\Omega_{i\sigma\sigma'}(\vec{p}) = \xi^+_\sigma(\vec{p})\left[\partial_{p^i}\xi_{\sigma'}(\vec{p})\right] = \left\{\eta^+_\sigma(\vec{p})\left[\partial_{p^i}\eta_{\sigma'}(\vec{p})\right]\right\}^* \tag{125}$$

are anti-Hermitian, $\Omega_{i\sigma\sigma'}(\vec{p}) = -\Omega^*_{i\sigma'\sigma}(\vec{p})$, which means that the operators $i\tilde{\partial}_i$ are Hermitian. We must stress that the principal property of the covariant derivatives is their commuting with the spin components, $[\tilde{\partial}_i,\tilde{S}_j]=0$. In the case of peculiar polarization, the connections $\Omega_i(\vec{p})$ guarantee this property, which becomes trivial in the case of common polarization when $\Omega_i=0$ and $\tilde{S}_i$ are independent of $\vec{p}$.

Initially, Pryce proposed the operator $\vec{X}$ as the relativistic mass-center operator of RQM. However, we showed in Ref. [18] that after quantization, this in fact becomes the operator of center of charges, or simply the dipole operator, while the velocity operator $\vec{V}$ becomes just the corresponding conserved vector current. For this reason, we defined another mass-center operator by changing the sign of the antiparticle term by hand. Now, we have the ability to use the operator N to define the mass-center operator from the beginning, at the level of RQM. We assume that this has the form $\vec{X}_{MC}(t)=\vec{X}_{MC}+t\vec{V}_{MC}$, where

$$\vec{X}_{MC}(t) = N\vec{X}(t) \ \Rightarrow X^i_{MC} = NX^i,\ V^i_{MC} = NV^i, \tag{126}$$

such that the associated operators $\tilde{X}^i_{MC} = -\tilde{X}^{ci}_{MC} = \tilde{X}^i$ and $\tilde{V}^i_{MC} = -\tilde{V}^{ci}_{MC} = \tilde{V}^i$ guarantee the desired sign of the antiparticle term after quantization.

Other position operators are the Pryce (c) and (d) ones depending on the principal position operator (e), as in Eqs. (97)−(99). As these operators are of marginal interest, we restrict ourselves to briefly present their associated operators and some algebraic properties in Appendix C.

C. Associated isometry generators

Let us now demonstrate how the Pryce (e) spin operator is related to the generators of the Poincaré isometries. In our approach, we may explicitly establish the equivalence between the covariant representation and a pair of Wigner's induced ones transforming the Pauli wave spinors. The covariant representation T defined by Eq. (5) may be associated to a pair of Wigner's representations whose operators $\tilde{T}\in\mathrm{Aut}(\tilde{\mathcal{F}}^+)$ and $\tilde{T}^c\in\mathrm{Aut}(\tilde{\mathcal{F}}^-)$ satisfy [1, 26, 30]

$$(T_{\lambda,a}\psi)(x) = \int \mathrm{d}^3p\sum_\sigma\left[U_{\vec{p},\sigma}(x)(\tilde{T}_{\lambda,a}\alpha)_\sigma(\vec{p})\right.$$

$$+V_{\vec{p},\sigma}(x)(\tilde{T}^c_{\lambda,a}\beta)^*_\sigma(\vec{p})\big] \,. \tag{127}$$

In other respects, by using the identity $(\Lambda x)\cdot p = x\cdot(\Lambda^{-1}p)$ and the invariant measure (20), we expand Eq. (5) by changing the integration variable as

$$\begin{aligned}(T_{\lambda,a}\psi)(x) &= \lambda\psi\left(\Lambda(\lambda)^{-1}(x-a)\right)\\ &= \int \mathrm{d}^3p \frac{E(p_\lambda)}{E(p)}\sum_\sigma \left[\lambda U'_{\vec{p},\sigma}(x)\alpha_\sigma(\vec{p}_\lambda)\mathrm{e}^{\mathrm{i}a\cdot p}\right.\\ &\quad \left.+\lambda V'_{\vec{p},\sigma}(x)\beta^*_\sigma(\vec{p}_\lambda)\mathrm{e}^{-\mathrm{i}a\cdot p}\right] ,\end{aligned} \tag{128}$$

where we denote $a\cdot p = a_\mu p^\mu = E(p)a^0 - \vec{p}\cdot\vec{a}$, while the new mode spinors,

$$U'_{\vec{p},\sigma}(x) = u_\sigma(\vec{p}_\lambda)\frac{1}{(2\pi)^{\frac{3}{2}}}\,\mathrm{e}^{-\mathrm{i}E(p)t+\mathrm{i}\vec{p}\cdot\vec{x}}, \tag{129}$$

$$V'_{\vec{p},\sigma}(x) = v_\sigma(\vec{p}_\lambda)\frac{1}{(2\pi)^{\frac{3}{2}}}\,\mathrm{e}^{\mathrm{i}E(p)t-\mathrm{i}\vec{p}\cdot\vec{x}}, \tag{130}$$

depend on the transformed momentum of components,

$$p^\mu_\lambda = \left\langle\Lambda(\lambda)^{-1}\right\rangle^{\mu\cdot}_{\cdot\nu}\, p^\nu\,, \tag{131}$$

through the spinors (26) and (27). Hereby, we deduce that $\tilde{T}_{\lambda,a} \simeq \tilde{T}^c_{\lambda,a}$ acts alike on the spaces $\tilde{\mathcal{F}}^+$ and $\tilde{\mathcal{F}}^-$, as [1, 23, 30]

$$(\tilde{T}_{\lambda,a}\,\alpha)_\sigma(\vec{p}) = \sqrt{\frac{E(p_\lambda)}{E(p)}}\,\mathrm{e}^{\mathrm{i}a\cdot p}\sum_{\sigma'} D_{\sigma\sigma'}(\lambda,\vec{p})\alpha_{\sigma'}(\vec{p}_\lambda), \tag{132}$$

and similarly, for β, because of their related matrices,

$$\begin{aligned}D_{\sigma\sigma'}(\lambda,\vec{p}) &= \mathring{u}^+_\sigma(\vec{p})w(\lambda,\vec{p})\mathring{u}_{\sigma'}(\vec{p}_\lambda)\\ &= \left[\mathring{v}^+_\sigma(\vec{p})w(\lambda,\vec{p})\mathring{v}_{\sigma'}(\vec{p}_\lambda)\right]^* .\end{aligned} \tag{133}$$

These depend on the well-known Wigner transformations

$$w(\lambda,\vec{p}) = l^{-1}_{\vec{p}}\,\lambda\, l_{\vec{p}_\lambda} \in \rho_D\,, \tag{134}$$

whose corresponding Lorentz transformations leave the representative momentum invariant,

$$\Lambda[w(\lambda,\vec{p})]\mathring{p} = L^{-1}_{\vec{p}}\Lambda(\lambda)p_\lambda = L^{-1}_{\vec{p}}\,p = \mathring{p}\,,$$

which means that $\Lambda[w(\lambda,\vec{p})]\in SO(3)$ is a rotation, and consequently $w(\lambda,\vec{p})\in\rho_D[SU(2)]$. Furthermore, bearing in mind that the $SU(2)$ rotations of ρ_D have the form (A7), we obtain the definitive expression of the matrix elements (133) as

$$D_{\sigma\sigma'}(\lambda,\vec{p}) = \xi^+_\sigma(\vec{p})\hat{l}^{-1}_{\vec{p}}\,\hat{\lambda}\,\hat{l}_{\vec{p}_\lambda}\xi_{\sigma'}(\vec{p}_\lambda)\,, \tag{135}$$

observing that these depend explicitly on the polarization spinors. As these matrices form the representation of spin $s=\frac{1}{2}$ of the little group $SU(2)$, one can say that the equivalent Wigner representation $\tilde{T}\simeq\tilde{T}^c$ is *induced* by the subgroup $T(4)Ⓢ SU(2)$ [1, 23, 30]. Note that, for rotations, $\lambda = r\in\rho_D[SU(2)]$, we obtain the usual $SU(2)$ linear representation as $E(p_\lambda)=E(p)$ and $\hat{r}\hat{l}_{\vec{p}_\lambda}\hat{r}^{-1} = \hat{l}_{\vec{p}} \Rightarrow \hat{l}^{-1}_{\vec{p}}\hat{r}\hat{l}_{\vec{p}_\lambda} = \hat{r} \Rightarrow D(r,\vec{p}) = D(\hat{r})$, where

$$D_{\sigma\sigma'}(\hat{r}) = \xi^+_{\sigma'}\hat{r}\xi_\sigma = \left(\eta^+_{\sigma'}\hat{r}\eta_\sigma\right)^*. \tag{136}$$

Thus, we understand that the specific mechanism of the induced representations acts only for the Lorentz boosts, $\lambda\in\rho_D[SL(2,\mathbb{C})/SU(2)]$.

The Wigner-induced representations are unitary with respect to the scalar product (39) [23, 25],

$$\langle\tilde{T}_{\lambda,a}\alpha,\tilde{T}_{\lambda,a}\alpha'\rangle = \langle\alpha,\alpha'\rangle\,, \tag{137}$$

and similarly for β. Bearing in mind that the covariant representations are unitary with respect to the scalar product (4), which can be decomposed as in Eq. (40), we conclude that the expansion (23) establishes the unitary equivalence $T=\tilde{T}\oplus\tilde{T}$ of the covariant representation with the *orthogonal* sum of Wigner's unitary irreducible ones [25]. Under such circumstances, the self-adjoint generators $\tilde{X}\in\mathrm{Lie}(\tilde{T})$ defined as

$$\tilde{P}_\mu = -\,\mathrm{i}\frac{\partial\tilde{T}_{1,a}}{\partial a^\mu}\bigg|_{a=0}\,,\quad \tilde{J}_{\mu\nu} = \mathrm{i}\frac{\partial\tilde{T}_{\lambda(\omega),0}}{\partial\omega^{\mu\nu}}\bigg|_{\omega=0} \tag{138}$$

are just the associated operators of the generators $X\in\mathrm{Lie}(T)$ such that

$$\begin{aligned}(X\psi)(x) = \int \mathrm{d}^3p\sum_\sigma &\left[U_{\vec{p},\sigma}(x)(\tilde{X}\alpha)_\sigma(\vec{p})\right.\\ &\left.-V_{\vec{p},\sigma}(x)(\tilde{X}\beta)^*_\sigma(\vec{p})\right] ,\end{aligned} \tag{139}$$

as we deduce by deriving Eq. (127) with respect to the corresponding group parameter $\zeta\in(\omega,a)$ in $\zeta=0$. Thus, we find that the isometry generators, whose associated operators obey $\tilde{X}^c = -\tilde{X}$, are reducible as a consequence of the fact that $\tilde{T}^c\simeq\tilde{T}$ [18].

The associated Abelian generators are trivial, being diagonal in the momentum basis,

$$\tilde{H} = -\tilde{H}^c = E(p), \qquad \tilde{P}^i = -\tilde{P}^{ci} = p^i. \tag{140}$$

For rotations, we use the Cayley-Klein parameters as in Eq. (A.7), recovering the natural splitting (14),

$$J_i = L_i + S_i \;\;\Rightarrow\;\; \tilde{J}_i = -\tilde{J}^c_i = \tilde{L}_i + \tilde{S}_i, \tag{141}$$

laying out the components of the Pryce (e) spin operator (115) and intuitive components of the orbital angular momentum operator,

$$L_i \;\;\Rightarrow\;\; \tilde{L}_i = -\tilde{L}^c_i = -\mathrm{i}\epsilon_{ijk}p^j\tilde{\partial}_k. \tag{142}$$

The sets of conserved operators $\{\tilde{L}_1, \tilde{L}_2, \tilde{L}_3\}$ and $\{\tilde{S}_1, \tilde{S}_2, \tilde{S}_3\}$ satisfying Eq. (B.5) generate the representations $\tilde{T}^o$ and $\tilde{T}^s$ of the associated factorization,

$$T^r = T^o \otimes T^s \;\;\Rightarrow\;\; \tilde{T}^r = \tilde{T}^o \otimes \tilde{T}^s, \tag{143}$$

of the $SU(2)$ restriction $\tilde{T}^r \equiv \tilde{T}\big|_{SU(2)}$ of the representation $\tilde{T}$.

For the Lorentz boosts, we perform a similar calculation with $\lambda = l(\tau)$ as in Eq. (A9), obtaining a similar splitting,

$$K_i \;\;\Rightarrow\;\; \tilde{K}_i = -\tilde{K}^c_i = \tilde{K}^o_i + \tilde{K}^s_i, \tag{144}$$

where the orbital and spin components,

$$\tilde{K}^o_i = -\tilde{K}^{o\,c}_i = \mathrm{i}E(p)\tilde{\partial}_i + \mathrm{i}\frac{p^i}{2E(p)} = \frac{1}{2}\left\{\tilde{X}^i, E(p)\right\}, \tag{145}$$

$$\tilde{K}^s_i = -\tilde{K}^{s\,c}_i = \frac{1}{E(p)+m}\epsilon_{ijk}p^j\tilde{S}_k, \tag{146}$$

no longer commute among themselves, as we can see from Eq. (D6). This means that the factorization (143) cannot be extended to the entire $SL(2,\mathbb{C})$ group. Note that the form (145) guarantees that the operators K^o_i are Hermitian with respect to the scalar product (39). [1)] The algebraic properties of these operators are presented in Appendix B, where we show how an algebra of orbital operators in MR can be selected. This is formed by the orbital subalgebra Lie$(\tilde{T}^o)$ generated by the set $\{E(p), p^i, \tilde{L}_i, \tilde{K}^o_i\}$ and the kinetic operators $\tilde{X}^i$ and $\tilde{V}^i$, which do not have spin parts.

Finally, let us turn back to the Pauli-Lubanski operator whose components are formed by products of isometry generators as in Eq. (76). After a few manipulation, we find the associated operators

$$W^0 \Rightarrow \tilde{W}^0 = \tilde{W}^{c\,0} = p^i\tilde{S}_i, \tag{147}$$

$$W^i \Rightarrow \tilde{W}^i = \tilde{W}^{c\,i} = E(p)\tilde{J}_i + \epsilon_{ijk}p^j\tilde{K}^s_k = m\tilde{S}^{(+)}_i, \tag{148}$$

expressed in terms of operators (115) and (117). Hereby, we recover the identity $P^\mu W_\mu = E(p)\tilde{W}_0 - p^i\tilde{W}^i = 0$ and the well-known invariant $\tilde{W}^\mu\tilde{W}_\mu = -\frac{3}{4}m^2 1_{2\times 2}$. In Appendix B, we give the commutation relations of the components $\tilde{W}^\mu$ with our new operators $\tilde{S}_i$ and $\tilde{X}^i$ that complete the algebraic properties we already know [1, 30].

D. Spectral representations

The correspondence $A \Leftrightarrow (\tilde{A}, \tilde{A}^c)$ defined by Eq. (100) is bijective. We have seen how A generates the operators $\tilde{A}$ and $\tilde{A}^c$, so we now have to face the inverse problem, which we try to solve by resorting to spectral representations, such as those defined in Ref. [18], in the particular case when $\tilde{A}$ and $\tilde{A}^c$ are matrix operators. In the following, we generalize these spectral representations to any equal-time associated operators whose action on the wave spinors is given by arbitrary kernels.

Let us start with the equal-time integral operator (46), whose action in CR is given by the time-dependent bilocal kernel $\mathfrak{A}$. In addition, we assume that A is reducible with its associated operators acting as

$$(\tilde{A}\alpha)_\sigma(t,\vec{p}) = \int \mathrm{d}^3p' \sum_{\sigma'} \tilde{\mathfrak{A}}_{\sigma\sigma'}(t,\vec{p},\vec{p}\,')\alpha_{\sigma'}(\vec{p}\,'), \tag{149}$$

$$(\tilde{A}^c\beta)_\sigma(t,\vec{p}) = \int \mathrm{d}^3p' \sum_{\sigma'} \tilde{\mathfrak{A}}^c_{\sigma\sigma'}(t,\vec{p},\vec{p}\,')\beta_{\sigma'}(\vec{p}\,'). \tag{150}$$

In this case, we may exploit the orthonormalization and completeness properties of the mode spinors, given by Eqs. (34), (35), and (36), to relate the kernels of the associated operators through the spectral representation

$$\begin{aligned}
&\mathfrak{A}(t,\vec{x},\vec{x}\,')\\
&= \int \mathrm{d}^3p\,\mathrm{d}^3p' \sum_{\sigma\sigma'} \Big[U_{\vec{p},\sigma}(t,\vec{x})\tilde{\mathfrak{A}}_{\sigma\sigma'}(t,\vec{p},\vec{p}\,')U^+_{\vec{p}\,',\sigma'}(t,\vec{x}\,')\\
&\quad + V_{\vec{p},\sigma}(t,\vec{x})\tilde{\mathfrak{A}}^{c\,*}_{\sigma\sigma'}(t,\vec{p},\vec{p}\,')V^+_{\vec{p}\,',\sigma'}(t,\vec{x}\,')\Big],
\end{aligned} \tag{151}$$

1) The second term of Eq. (145) was omitted in In Eq. (124) of Ref. [18] but without affecting other results.

giving the action of the operator A in CR when we know the actions of the associated operators $\tilde{A}$ and $\tilde{A}^c$.

This mechanism is useful for taking over to CR the principal properties of our operators we defined in MR, where we studied the induced Wigner representations and their generators. In spite of their manifest covariance, the operators $T_{\lambda,a}$ can be seen as equal-time operators after the transformation (128). Their kernels in CR, $\mathfrak{T}_{\lambda,a}(t,\vec{x},\vec{x}')$, may be derived according to the spectral representation (151), where we have to substitute the kernels in MR that are time-independent with the form

$$\tilde{\mathfrak{T}}_{\lambda,a}(\vec{p},\vec{p}\,') = \tilde{\mathfrak{T}}^c_{\lambda,a}(\vec{p},\vec{p}\,') = \delta^3\left(\vec{p}_\lambda - \vec{p}\,'\right) \mathrm{e}^{\mathrm{i}a\cdot p}\sqrt{\frac{E(p')}{E(p)}}D(\lambda,\vec{p}), \tag{152}$$

depending on the momentum (131) and matrix (135). In a similar manner, we may write the spectral representations of the kernels of the basis generators for which we separated the orbital parts, $\tilde{L}_i$, $\tilde{K}^o_i$, and $\tilde{X}^i$, depending on momentum derivatives. According to the results of Sec. IV.B, we may write the kernels of these operators in MR:

$$\tilde{\mathfrak{L}}_i(\vec{p},\vec{p}\,') = -\tilde{\mathfrak{L}}^c_i(\vec{p},\vec{p}\,') = -\mathrm{i}\epsilon_{ijk}p^j\tilde{\partial}_k\delta^3(\vec{p}-\vec{p}\,')1_{2\times2}, \tag{153}$$

$$\tilde{\mathfrak{K}}^o_i(\vec{p},\vec{p}\,') = -\tilde{\mathfrak{K}}^{o\,c}_i(\vec{p},\vec{p}\,') = \left[\delta^3(\vec{p}-\vec{p}\,')\frac{\mathrm{i}p^i}{2E(p)} + \mathrm{i}E(p)\tilde{\partial}_i\delta^3(\vec{p}-\vec{p}\,')\right]1_{2\times2}, \tag{154}$$

$$\tilde{\mathfrak{X}}^i(\vec{p},\vec{p}\,') = \tilde{\mathfrak{X}}^{i\,c}(\vec{p},\vec{p}\,') = i\tilde{\partial}_i\delta^3(\vec{p}-\vec{p}\,')1_{2\times2}. \tag{155}$$

Substituting this into Eq. (151) will give the kernels of the operators L_i, K^o_i, and X^i acting in CR as integral operators that may depend on time.

In the particular case when $A\in F[t]$ is a Fourier operator, the associated operators have the kernels

$$\tilde{\mathfrak{A}}(t,\vec{p},\vec{p}\,') = \delta^3(\vec{p}-\vec{p}\,')\tilde{A}(t,\vec{p}), \tag{156}$$

$$\tilde{\mathfrak{A}}^c(t,\vec{p},\vec{p}\,') = \delta^3(\vec{p}-\vec{p}\,')\tilde{A}^c(t,\vec{p}), \tag{157}$$

which solve one of the integrals of the spectral representation (151), leaving the simpler form

$$\mathfrak{A}(t,\vec{x}-\vec{x}\,') = \int \mathrm{d}^3p \sum_{\sigma\sigma'}\left[U_{\vec{p},\sigma}(t,\vec{x})\tilde{A}_{\sigma\sigma'}(t,\vec{p})U^+_{\vec{p},\sigma'}(t,\vec{x}\,') + V_{\vec{p},\sigma}(t,\vec{x})\tilde{A}^c_{\sigma\sigma'}(t,\vec{p})^*V^+_{\vec{p},\sigma'}(t,\vec{x}\,')\right], \tag{158}$$

which can be applied to all the spin parts of our operators.

In Ref. [18], we used this type of spectral representation to study the transformations (15) of the spin symmetry starting with the identities

$$\hat{r}\xi_\sigma = \sum_{\sigma'}\xi_{\sigma'}D_{\sigma'\sigma}(\hat{r}) \Rightarrow U_{\vec{p},\hat{r}\xi_\sigma}(x) = \sum_{\sigma'}U_{\vec{p},\xi_{\sigma'}}(x)D_{\sigma'\sigma}(\hat{r}), \tag{159}$$

$$\hat{r}\eta_\sigma = \sum_{\sigma'}\eta_{\sigma'}D^*_{\sigma'\sigma}(\hat{r}) \Rightarrow V_{\vec{p},\hat{r}\eta_\sigma}(x) = \sum_{\sigma'}V_{\vec{p},\eta_{\sigma'}}(x)D^*_{\sigma'\sigma}(\hat{r}), \tag{160}$$

where r are the rotations (A4) of ρ_D, while the matrices $D(\hat{r})$ are defined by Eq. (136). Under such circumstances, the operator $T^s_{\hat{r}}$ can be defined as the integral Fourier operator with the local time-independent kernel

$$\mathfrak{T}^s_{\hat{r}}(\vec{x}-\vec{x}\,') = \int \mathrm{d}^3p\,\frac{\mathrm{e}^{\mathrm{i}(\vec{p}-\vec{p}\,')\cdot\vec{x}}}{(2\pi)^3}T^s_{\hat{r}}(\vec{p}) \tag{161}$$

given by Eq. (158), where we substitute

$$\tilde{A}_{\sigma\sigma'}(t,\vec{p}) = \tilde{A}^c_{\sigma\sigma'}(t,\vec{p}) = D_{\sigma\sigma'}(\hat{r}). \tag{162}$$

The Fourier transform of $T^s_{\hat{r}}(\vec{p})$ can be derived by now considering the form of the mode spinors (26) and (27) and using the identities (159), (160), and (A15). After some calculation, we obtain

$$T^s_{\hat{r}}(\vec{p}) = \frac{m}{E(p)}\left[l_{\vec{p}}\,r\,\frac{1+\gamma^0}{2}l_{\vec{p}} + l^{-1}_{\vec{p}}\,r\,\frac{1-\gamma^0}{2}l^{-1}_{\vec{p}}\right] = l_{\vec{p}}\,r\,l^{-1}_{\vec{p}}\tilde{\Pi}_+(\vec{p}) + l^{-1}_{\vec{p}}\,r\,l_{\vec{p}}\tilde{\Pi}_-(\vec{p}). \tag{163}$$

This spectral representation was crucial for showing that the spin components defined by Eq. (16) have just the Fourier transforms (68) proposed by Pryce (e). In Ref. [18], we started with the Fourier transform (163), where we substituted $\hat{r}=\hat{r}(\theta)$ given by Eq. (A7). Then, by applying the definition (16), we found the Fourier transforms (70), which fortunately coincide with those proposed by Pryce, as we deduced after using suitable computer code.

We now have all the elements required to write the kernels of the operators $T^o_{\hat{r}}$ of the orbital representation of the $SO(3)$ group, which are no longer Fourier operators, for the first time. These operators are defined by Eq. (17), which combines the actions of $T_{r,0}$ and $T^s_{\hat{r}}$ such that, according to Eqs. (152) and (161), we may write the associated kernels in MR,

$$\tilde{\mathfrak{T}}^o_{\hat{r}}(\vec{p},\vec{p}\,') = \tilde{\mathfrak{T}}^{oc}_{\hat{r}}(\vec{p},\vec{p}\,') = \delta^3\left(\vec{p}_{\hat{r}} - \vec{p}\,'\right) D^{-1}(\hat{r})D(\hat{r},\vec{p}),$$
$$= \delta^3\left(\vec{p}_{\hat{r}} - \vec{p}\,'\right) 1_{2\times 2}, \quad p_{\hat{r}} = R(\hat{r})^{-1}p, \qquad (164)$$

as a result of Eq. (136). Substituting these kernels into Eq. (151), we obtain the kernels of the operators $T^o_{\hat{r}}$ of the orbital representation acting on the free fields in CR. Finally, substituting again $\hat{r}\to\hat{r}(\theta)$ into $T^o_{\hat{r}}$ and applying the definition (18), we obtain the kernels (153) giving the action of the operators (142) in MR directly, without resorting to Wigner's theory as in Sec. III.C.

We conclude that the action of the operators of the spin and orbital symmetries can be properly defined thanks to our spectral representations outlined in Ref. [18] for Fourier operators and generalized here to any equal-time integral operators.

V. QUANTUM THEORY

The quantization reveals the physical meaning of the quantum observables of RQM, transforming them into operators of QFT. The principal benefit of our approach is the association between the operator actions in CR and MR, allowing us to derive the expectation values of the operators defined in MR according to the general rule (101) at any time. Thus, we are able to apply the Bogolyubov method for quantizing the operators of RQM.

A. Quantization

In special relativistic QFT, each observer has its own measurement apparatus formed by the set of observables defined in its proper frame at a fixed initial time. As we already adopted the point of view of an observer staying at rest at the origin preparing the free fields in initial time $t=0$, we assume that this observer keeps the same initial condition for quantization.

Applying the Bogolyubov method of quantization [27], we first replace the wave spinors of MR with field operators, $(\alpha,\alpha^*)\to(\mathfrak{a},\mathfrak{a}^\dagger)$ and $(\beta,\beta^*)\to(\mathfrak{b},\mathfrak{b}^\dagger)$, satisfying canonical anti-commutation relations; among them, the non-vanishing ones are

$$\left\{\mathfrak{a}_\sigma(\vec{p}),\mathfrak{a}^\dagger_{\sigma'}(\vec{p}\,')\right\} = \left\{\mathfrak{b}_\sigma(\vec{p}),\mathfrak{b}^\dagger_{\sigma'}(\vec{p}\,')\right\} = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}\,'). \quad (165)$$

The Dirac free field thus becomes the field operator

$$\psi(x) = \int d^3p \sum_\sigma \left[U_{\vec{p},\sigma}(x)\mathfrak{a}_\sigma(\vec{p}) + V_{\vec{p},\sigma}(x)\mathfrak{b}^\dagger_\sigma(\vec{p})\right], \quad (166)$$

denoted with the same symbol but acting on the Fock state space equipped with the scalar product $\langle\ |\ \rangle$ and a normalized vacuum state $|0\rangle$ accomplishing

$$\mathfrak{a}_\sigma(\vec{p})|0\rangle = \mathfrak{b}_\sigma(\vec{p})|0\rangle = 0, \quad \langle 0|\mathfrak{a}^\dagger_\sigma(\vec{p}) = \langle 0|\mathfrak{b}^\dagger_\sigma(\vec{p}) = 0. \quad (167)$$

The sectors with different numbers of particles must be constructed by applying the standard method for constructing generalized momentum bases of various polarizations.

Through quantization, the expectation value of any time-dependent operator $A(t)$ of RQM becomes an operator,

$$A(t) \Rightarrow \mathsf{A} = :\langle\psi, A(t)\psi\rangle_D :|_{t=0}, \quad (168)$$

calculated respecting the normal ordering of the operator products [20] at the initial time $t=0$. This procedure allows us to write any operator A directly in terms of the operators associated to the operator $A = A(t)|_{t=0}$. We first consider the reducible operators complying with the condition (106), for which we obtain the general formula

$$\mathsf{A} = \int d^3p \left[\mathfrak{a}^\dagger(\vec{p})(\tilde{A}\mathfrak{a})(\vec{p}) - \mathfrak{b}^\dagger(\vec{p})(\tilde{A}^{c+}\mathfrak{b})(\vec{p})\right], \quad (169)$$

written with the compact notation

$$\mathfrak{a}^\dagger(\vec{p})(\tilde{A}\mathfrak{a})(\vec{p}) \equiv \sum_\sigma \mathfrak{a}^\dagger_\sigma(\vec{p})(\tilde{A}\mathfrak{a})_\sigma(\vec{p}), \quad (170)$$

and similarly for the second term. To shorten the terminology, we say here that the associated operators $A \Leftrightarrow (\tilde{A},\tilde{A}^c)$ are the *parent* operators of A. We specify that the bracket in (168) is calculated according to Eq. (101), where the last term changes its sign after introducing the normal ordering of the operator products. When $\tilde{A}^c = -\tilde{A}$, we say that the one-particle operator (169) is even (of positive charge parity), describing an additive property that is similar for particles and antiparticles as, for example, the energy, momentum, spin, *etc*. The odd operators (with negative charge parity), for which $\tilde{A}^c = \tilde{A}$, describe electrical properties depending on the opposite charges of particles and antiparticles. Thus, we introduce the operator signature, which behaves in commutation relations as the usual algebraic signs in multiplication, *e.g.*, $[A_{\rm odd}, B_{\rm odd}] = C_{\rm even}$, $[A_{\rm odd}, B_{\rm even}] = C_{\rm odd}, \ldots,$ *etc.*

Given an arbitrary operator $A\in{\rm Aut}(\mathcal{F})$ and its Hermitian conjugated A^+, we define the adjoint operator of A,

$$A^+(t) \Rightarrow \mathsf{A}^\dagger = :\langle\psi, A(t)^+\psi\rangle_D :\big|_{t=0} = :\langle A(t)\psi,\psi\rangle_D :|_{t=0}, \quad (171)$$

complying with the standard definition $\langle\alpha|\mathsf{A}^{\dagger}\beta\rangle=\langle\mathsf{A}\alpha|\beta\rangle$ on the Fock space. In to following, we shall meet only self-adjoint one-particle operators as all their parent operators of RQM are reducible and Hermitian with respect to the scalar products of the spaces in which they act. Thus, we obtain an operator algebra formed by fields and self-adjoint one-particle operators, which have the obvious properties

$$\left[\mathsf{A},\psi(x)\right]=-(A\psi)(x), \tag{172}$$

$$[\mathsf{A},\mathsf{B}]=:\langle\psi,[A,B]\psi\rangle_D:, \tag{173}$$

preserving the structures of Lie algebras but without carrying over other algebraic properties of their parent operators from RQM, as the product of two one-particle operators is no longer an operator of the same type. Therefore, we must restrict ourselves to the Lie algebras of symmetry generators and unitary transformations whose actions reduce to sums of successive commutations according to the well-known rule

$$e^{\mathsf{X}}\mathsf{Y}e^{-\mathsf{X}}=\mathsf{Y}+[\mathsf{X},\mathsf{Y}]+\frac{1}{2}[\mathsf{X},[\mathsf{X},\mathsf{Y}]]+\frac{1}{3!}[\mathsf{X},[\mathsf{X},[\mathsf{X},\mathsf{Y}]]]..., \tag{174}$$

which use in the following.

The Poincaré generators (6) give rise to the self-adjoint one-particle operators calculated at the initial time t=0,

$$\mathsf{P}_{\mu}=:\langle\psi,P_{\mu}\psi\rangle_D:,\quad \mathsf{J}_{\mu\nu}=:\langle\psi,J_{\mu\nu}\psi\rangle_D:\big|_{t=0}. \tag{175}$$

The brackets corresponding to the operators P^{μ} and S_{ij} are independent of time, but for the operators S_{0i}, we must impose the initial condition to show later how these operators evolve in time. With these generators, we may construct unitary transformations with various parametrizations, among which we choose those of the first kind, defining the unitary operators of translations and $SL(2,\mathbb{C})$ transformations as

$$\mathsf{U}(a)=\exp\left(-ia^{\mu}\mathsf{P}_{\mu}\right),\qquad a\in T(4), \tag{176}$$

$$\mathsf{U}(\omega)=\exp\left(\frac{i}{2}\omega^{\mu\nu}\mathsf{J}_{\mu\nu}\right),\ \lambda(\omega)\in\rho_D[SL(2,\mathbb{C})], \tag{177}$$

in accordance with our definition (6) of the isometry generators and the rule (172). This construction guarantees the expected isometry transformations of the field operators,

$$\mathsf{U}(a)\mathfrak{a}_{\sigma}(\vec{p})\mathsf{U}^{\dagger}(a)=\left(\tilde{T}_{1,a}\,\mathfrak{a}\right)_{\sigma}(\vec{p})=\mathrm{e}^{ia\cdot p}\mathfrak{a}_{\sigma}(\vec{p}), \tag{178}$$

$$\begin{aligned}\mathsf{U}(\omega)\mathfrak{a}_{\sigma}(\vec{p})\mathsf{U}^{\dagger}(\omega)&=\left(\tilde{T}_{\lambda(\omega),0}\,\mathfrak{a}\right)_{\sigma}(\vec{p})\\&=\sqrt{\frac{E(p_{\lambda})}{E(p)}}\sum_{\sigma'}D_{\sigma\sigma'}\left(\lambda(\omega),\vec{p}\right)\mathfrak{a}_{\sigma'}(\vec{p}_{\lambda}),\end{aligned} \tag{179}$$

where the matrix D is given by Eq. (135) and $\vec{p}_{\lambda}$ by Eq. (131). As the operators $\mathfrak{a}_{\sigma}$ and $\mathfrak{b}_{\sigma}$ transform alike under isometries, from Eq. (128), we obtain the transformations of the quantum field

$$\mathsf{U}(a)\psi(x)\mathsf{U}^{\dagger}(a)=\left(T_{1,a}\,\psi\right)(x)=\psi(x-a), \tag{180}$$

$$\begin{aligned}\mathsf{U}(\omega)\psi(x)\mathsf{U}^{\dagger}(\omega)&=\left(T_{\lambda(\omega),0}\,\psi\right)(x)\\&=\lambda(\omega)\psi\left(\Lambda^{-1}(\lambda(\omega))x\right).\end{aligned} \tag{181}$$

Moreover, the isometry generators usually transform according to the adjoint representation of the Poincaré group [30], thus assuring the relativistic covariance. In the case of Lorentz transformations $\lambda(\omega)\in\rho_D[SL(2,\mathbb{C})]$, we have

$$\mathsf{U}(\omega)P_{\mu}\mathsf{U}^{\dagger}(\omega)=\Lambda_{\mu\cdot}^{\cdot\alpha}(\omega)P_{\alpha}, \tag{182}$$

$$\mathsf{U}(\omega)J_{\mu\nu}\mathsf{U}^{\dagger}(\omega)=\Lambda_{\mu\cdot}^{\cdot\alpha}(\omega)\Lambda_{\nu\cdot}^{\cdot\beta}(\omega)J_{\alpha\beta}, \tag{183}$$

where $\Lambda(\omega)$ is defined in Appendix A. Thus, we may say that the unitary operators $\mathsf{U}(a)$ and $\mathsf{U}(\omega)$ encapsulate the entire theory of the relativistic covariance under Poincaré isometries. More specifically, the transformations

$$\mathsf{U}(\omega,a)=\mathsf{U}(\omega)\mathsf{U}(a):\mathsf{A}\to\mathsf{A}'=\mathsf{U}(\omega,a)\mathsf{A}\mathsf{U}^{\dagger}(\omega,a) \tag{184}$$

of an operator expressed in terms of particle and antiparticle operators can be derived by Eqs. (178) and (179). In general, these transformations are not manifest covariant because of their momentum-dependent transformation matrices remaining under the integral over momenta.

We have seen that the quantization is performed at the initial time $t=0$ when one obtains a set of one-particle operators, among which we may find conserved operators that commute with the energy one $\mathsf{H}=\mathsf{P}_0$ or dynamical operators whose time evolution is governed by the translation operator generated by H,

$$\mathsf{U}(t)=\exp(-it\mathsf{H}):\mathsf{A}\to\mathsf{A}(t)=\mathsf{U}^\dagger(t)\mathsf{A}\mathsf{U}(t). \tag{185}$$

Thus, the observer staying at rest at the origin recovers the time evolution of the observables obtained through quantization in initial time $t=0$.

B. Reducible operators

The reducible operators of RQM give rise to the one-particle operators of QFT. There are two such operators commuting with the entire algebra of observables, namely, the charge operator $\mathsf{Q}=\mathsf{N}_+-\mathsf{N}_-$ and that of the total number of particles $\mathsf{N}=\mathsf{N}_++\mathsf{N}_-$, formed by the particle and antiparticle number operators

$$\mathsf{N}_+=:\langle\psi,\Pi_+\psi\rangle_D:=\int d^3p\,\mathfrak{a}^\dagger(\vec p)\mathfrak{a}(\vec p), \tag{186}$$

$$\mathsf{N}_-=:-\langle\psi,\Pi_-\psi\rangle_D:=\int d^3p\,\mathfrak{b}^\dagger(\vec p)\mathfrak{b}(\vec p), \tag{187}$$

coming from the parent operators $\pm\Pi_\pm$ of RQM. Other diagonal operators in the momentum basis are the translations generators, energy and momentum,

$$\begin{aligned}\mathsf{H}&=:\langle\psi,H\psi\rangle_D:\\&=\int d^3p\,E(p)\left[\mathfrak{a}^\dagger(\vec p)\mathfrak{a}(\vec p)+\mathfrak{b}^\dagger(\vec p)\mathfrak{b}(\vec p)\right],\end{aligned} \tag{188}$$

$$\begin{aligned}\mathsf{P}^i&=:\langle\psi,P^i\psi\rangle_D:\\&=\int d^3p\,p^i\left[\mathfrak{a}^\dagger(\vec p)\mathfrak{a}(\vec p)+\mathfrak{b}^\dagger(\vec p)\mathfrak{b}(\vec p)\right],\end{aligned} \tag{189}$$

as well as our new operator of fermion polarization [1)],

$$\mathsf{W}_s=:\langle\psi,W_s\psi\rangle_D:=\frac{1}{2}\int d^3p\left[\mathfrak{a}^\dagger(\vec p)\sigma_3\mathfrak{a}(\vec p)+\mathfrak{b}^\dagger(\vec p)\sigma_3\mathfrak{b}(\vec p)\right], \tag{190}$$

which completes the set $\{\mathsf{H},\mathsf{P}^1,\mathsf{P}^2,\mathsf{P}^3,\mathsf{W}_s,\mathsf{Q}\}$ of commuting operators determining the momentum bases of the Fock state space.

Applying the general rule (169) to the associated rotation generators (141), we find the splitting of the total angular momentum

$$\mathsf{J}_i=:\langle\psi,J_i\psi\rangle_D:=:\langle\psi,L_i\psi\rangle_D:+:\langle\psi,S_i\psi\rangle_D:=\mathsf{L}_i+\mathsf{S}_i, \tag{191}$$

where the components of the orbital angular momentum, L_i, and spin operator, S_i, can be written as

$$\mathsf{L}_i=-\frac{i}{2}\int d^3p\,\epsilon_{ijk}p^j\left[\mathfrak{a}^\dagger(\vec p)\overset{\leftrightarrow}{\tilde\partial}_i\mathfrak{a}(\vec p)+\mathfrak{b}^\dagger(\vec p)\overset{\leftrightarrow}{\tilde\partial}_i\mathfrak{b}(\vec p)\right], \tag{192}$$

$$\mathsf{S}_i=\frac{1}{2}\int d^3p\left[\mathfrak{a}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{a}(\vec p)+\mathfrak{b}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{b}(\vec p)\right], \tag{193}$$

according to Eqs. (142) and (115). Here we use the special notation

$$\alpha^+\overset{\leftrightarrow}{\tilde\partial}_i\beta=\alpha^+(\partial_{p^i}\beta)-(\partial_{p^i}\alpha^+)\beta+2\alpha^+\Omega_i(\vec p)\beta, \tag{194}$$

inspired by Green's theorem, which points out explicitly that L_i are self-adjoint operators. The components L_i and S_i form the bases of two *independent* unitary representations of the $su(2)\sim so(3)$ algebra, $\left[\mathsf{L}_i,\mathsf{S}_j\right]=0$, generating the orbital and spin symmetries, respectively. These operators are *conserved* as they commute with H, while the commutation relations

$$\left[\mathsf{L}_i,\mathsf{P}^j\right]=i\epsilon_{ijk}\mathsf{P}^k,\qquad\left[\mathsf{S}_i,\mathsf{P}^j\right]=0, \tag{195}$$

show that only the spin operator is invariant under space translations. Moreover, using Eqs. (179) and (A10) and then changing the integration variable, $\vec p_\lambda\to\vec p$, we obtain the transformation of the spin operator under arbitrary transformations $\lambda(\omega)\in SL(2,\mathbb{C})$ as

$$\begin{aligned}\Lambda(\omega):\mathsf{S}_i&\to\mathsf{S}'_i=\mathsf{U}(\omega)\mathsf{S}_i\mathsf{U}^\dagger(\omega)\\&=\frac{1}{2}\int d^3p\left[\mathfrak{a}^\dagger(\vec p)\Sigma'_i(\vec p)\mathfrak{a}(\vec p)+\mathfrak{b}^\dagger(\vec p)\Sigma'_i(\vec p)\mathfrak{b}(\vec p)\right],\end{aligned} \tag{196}$$

where $\Sigma'_i(\vec p)=R_{ij}(\omega,\vec p)\Sigma_j(\vec p)$ are the transformed Σ-matrices under the Wigner rotations

$$R(\omega,\vec p)=\Lambda\left(w[\lambda(\omega),\Lambda(\omega)\vec p]\right)=L^{-1}_{\Lambda(\omega)\vec p}\Lambda(\omega)L_{\vec p}. \tag{197}$$

For genuine rotations, $\lambda(\omega)=r\in\rho_D[SU(2)]$, the matrix $R(r)$ is independent of momentum such that the spin operator transforms as a $SO(3)$ vector-operator, $\mathsf{S}_i\to R_{ij}(r)\mathsf{S}_j$. We may conclude that the quantum version of the Pryce (e) spin operator $\vec{\mathsf{S}}$ transforms covariantly only under rotations.

The generators of the Lorentz boosts have the general form (169) depending on the operators (144), which have orbital and spin terms suggesting the splitting

1) In Eqs. (115) and (149) of Ref. [18] the factor $\frac{1}{2}$ must be ignored.

$$\mathsf{K}_i =: \langle\psi, K_i\psi\rangle_D := \mathsf{K}_i^o + \mathsf{K}_i^s\,, \tag{198}$$

in orbital and spin parts that read

$$\mathsf{K}_i^o = \frac{\mathrm{i}}{2}\int \mathrm{d}^3p\, E(p)\left[\mathfrak{a}^\dagger(\vec p)\overset{\leftrightarrow}{\tilde\partial}_i\mathfrak{a}(\vec p) + \mathfrak{b}^\dagger(\vec p)\overset{\leftrightarrow}{\tilde\partial}_i\mathfrak{b}(\vec p)\right], \tag{199}$$

$$\mathsf{K}_i^s = \int \mathrm{d}^3p\left[\mathfrak{a}^\dagger(\vec p)\tilde K_i^s\mathfrak{a}(\vec p) + \mathfrak{b}^\dagger(\vec p)\tilde K_i^s\mathfrak{b}(\vec p)\right], \tag{200}$$

as results of Eqs. (145) and (146). The commutation relations

$$\left[\mathsf{H},\mathsf{K}_i^o\right] = -\mathrm{i}\mathsf{P}^i\,,\quad \left[\mathsf{P}^i,\mathsf{K}_j^o\right] = -\mathrm{i}\delta^i_j\mathsf{H}\,, \tag{201}$$

$$\left[\mathsf{H},\mathsf{K}_i^s\right] = 0\,,\qquad \left[\mathsf{P}^i,\mathsf{K}_j^s\right] = 0 \tag{202}$$

show that only the operators K_i^s are conserved and invariant under translations while K_i^o satisfy the usual orbital commutation relations evolving as

$$\mathsf{K}_i^o(t) = \mathsf{U}^\dagger(t)\mathsf{K}_i^o\mathsf{U}(t) = \mathsf{K}_i^o + \mathsf{P}^i\,t\,, \tag{203}$$

which means that the generators (198) are time-dependent,

$$\mathsf{K}_i(t) = \mathsf{U}^\dagger(t)\mathsf{K}_i\mathsf{U}(t) = \mathsf{K}_i^o(t) + \mathsf{K}_i^s = \mathsf{K}_i + \mathsf{P}^i\,t\,, \tag{204}$$

evolving linearly in time.

The operators discussed above satisfy commutation relations similar to those given in Appendix B for their associated parent operators of RQM. The set $\{\mathsf{H},\mathsf{P}^i,\mathsf{J}_i,\mathsf{K}_i\}$ generates the representation of the Lie$(\tilde P_+^\uparrow)$ algebra with values in one-particle operators, which includes the orbital subalgebra generated by $\{\mathsf{H},\mathsf{P}^i,\mathsf{L}_i,\mathsf{K}_i^o\}$. In contrast, the operators S_i and K_i^s do not close an algebra, with each commutator giving rise to a new operator thus generating an infinite Lie algebra.

The operators (147) and (148) associated to the components of the Pauli-Lubanski operator give rise to the odd one-particle operators

$$\mathsf{W}^0 = \frac12\int \mathrm{d}^3p\, p^i\left[\mathfrak{a}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{a}(\vec p) - \mathfrak{b}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{b}(\vec p)\right], \tag{205}$$

$$\mathsf{W}^i = m\frac12\int \mathrm{d}^3p\,\Theta_{ij}(\vec p)\left[\mathfrak{a}^\dagger(\vec p)\Sigma_j(\vec p)\mathfrak{a}(\vec p) - \mathfrak{b}^\dagger(\vec p)\Sigma_j(\vec p)\mathfrak{b}(\vec p)\right], \tag{206}$$

where the tensor Θ is defined in Eq. (A.13). The operator W^0 is known as the *helicity* operator; as in the momentum-helicity basis (presented in Appendix D), this takes the form

$$\mathsf{W}^0 = \frac12\int \mathrm{d}^3p\, p\left[\mathfrak{a}^\dagger(\vec p)\sigma_3\mathfrak{a}(\vec p) - \mathfrak{b}^\dagger(\vec p)\sigma_3\mathfrak{b}(\vec p)\right], \tag{207}$$

resulting from the identity (D8). A dimensionless version of this operator called the helical operator was defined recently for any peculiar polarization as [32, 33]

$$\mathsf{W}_h = \frac12\int \mathrm{d}^3p\,\frac{p^i}{p}\left[\mathfrak{a}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{a}(\vec p) - \mathfrak{b}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{b}(\vec p)\right], \tag{208}$$

becoming the odd replica of our polarization operator (190) in the momentum-helicity basis, which is even by definition.

A special set of operators, whose quantization deserves to be briefly examined, is formed by the operators (75) related to the historical Frankel and Pryce (c)-Czochor proposals. The associated operators (117) and (118) give the corresponding even one-particle operators

$$\mathsf{S}_i^{(+)} = \frac12\int \mathrm{d}^3p\,\Theta_{ij}(\vec p)\left[\mathfrak{a}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{a}(\vec p) + \mathfrak{b}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{b}(\vec p)\right], \tag{209}$$

$$\mathsf{S}_i^{(-)} = \frac12\int \mathrm{d}^3p\,\Theta^{-1}_{ij}(\vec p)\left[\mathfrak{a}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{a}(\vec p) + \mathfrak{b}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{b}(\vec p)\right]. \tag{210}$$

Similarly, the parent operators (83), (87), (89), and (93) give rise to the one-particle operators

$$\mathsf{S}_{\mathrm{Fr}\,i} = \frac12\int \mathrm{d}^3p\,\frac{E(p)}{m}\,\Theta^{-1}_{ij}(\vec p)\left[\mathfrak{a}^\dagger(\vec p)\Sigma_j(\vec p)\mathfrak{a}(\vec p + \mathfrak{b}^\dagger(\vec p)\Sigma_j(\vec p)\mathfrak{b}(\vec p)\right], \tag{211}$$

$$\mathsf{C}_{\mathrm{Fr}\,i} = \frac12\int \mathrm{d}^3p\,\frac{E(p)}{m}\,\Theta_{ij}(\vec p)\left[\mathfrak{a}^\dagger(\vec p)\Sigma_j(\vec p)\mathfrak{a}(\vec p) + \mathfrak{b}^\dagger(\vec p)\Sigma_j(\vec p)\mathfrak{b}(\vec p)\right], \tag{212}$$

$$\mathsf{S}_{\mathrm{PC}\,i} = \frac12\int \mathrm{d}^3p\,\frac{m}{E(p)}\,\Theta_{ij}(\vec p)\left[\mathfrak{a}^\dagger(\vec p)\Sigma_j(\vec p)\mathfrak{a}(\vec p) + \mathfrak{b}^\dagger(\vec p)\Sigma_j(\vec p)\mathfrak{b}(\vec p)\right], \tag{213}$$

$$\mathsf{C}_{\mathrm{PC}i} = \frac{1}{2}\int d^3p \frac{m}{E(p)} \Theta^{-1}_{ij}(\vec{p}) \left[\mathfrak{a}^\dagger(\vec{p})\Sigma_j(\vec{p})\mathfrak{a}(\vec{p}) + \mathfrak{b}^\dagger(\vec{p})\Sigma_j(\vec{p})\mathfrak{b}(\vec{p})\right], \tag{214}$$

which are conserved and translation invariant, behaving as $SO(3)$ vectors. They satisfy similar commutation relations as in Eqs. (85), (86), (91), and (92) but cannot close an algebra as each new commutator defines a new operator. Note that after quantization, the Fradkin-Good operator (95) becomes the odd version of the Pryce (e) one such that this brings nothing new.

An important set of kinetic observables is formed by the components of position and velocity operators. In Ref. [18], we showed that the original Pryce (e) operator proposed as a mass-center one becomes the dipole operator after quantization, which can be transformed into the mass-center one by changing the sign of the antiparticle term by hand. To improve this apparently arbitrary procedure, we define the mass-center operator (126) in RQM before quantization. Bearing in mind all these results, we now define the particle and antiparticle center operators at the initial time $t_0 = 0$ and the corresponding velocities as

$$\mathsf{X}^i_+ =: \langle\psi, \Pi_+ X^i\psi\rangle_D := \frac{i}{2}\int d^3p\, \mathfrak{a}^\dagger(\vec{p}) \overset{\leftrightarrow}{\tilde{\partial}}_i \mathfrak{a}(\vec{p}), \tag{215}$$

$$\mathsf{V}^i_+ =: \langle\psi, \Pi_+ V^i\psi\rangle_D := \int d^3p \frac{p^i}{E(p)} \mathfrak{a}^\dagger(\vec{p})\mathfrak{a}(\vec{p}), \tag{216}$$

$$\mathsf{X}^i_- = - : \langle\psi, \Pi_- X^i\psi\rangle_D := \frac{i}{2}\int d^3p\, \mathfrak{b}^\dagger(\vec{p}) \overset{\leftrightarrow}{\tilde{\partial}}_i \mathfrak{b}(\vec{p}), \tag{217}$$

$$\mathsf{V}^i_- = - : \langle\psi, \Pi_- V^i\psi\rangle_D := \int d^3p \frac{p^i}{E(p)} \mathfrak{b}^\dagger(\vec{p})\mathfrak{b}(\vec{p}), \tag{218}$$

using the derivative (194). These operators satisfy

$$[\mathsf{H}, \mathsf{X}^i_\pm] = -i\mathsf{V}^i_\pm, \quad [\mathsf{H}, \mathsf{V}^i_\pm] = 0, \tag{219}$$

showing that the velocity components $\mathsf{V}^i_\pm$ are conserved operators, while the position ones evolve as

$$\mathsf{X}^i_\pm(t) = \mathsf{U}^\dagger(t)\mathsf{X}^i_\pm\mathsf{U}(t) = \mathsf{X}^i_\pm + t\mathsf{V}^i_\pm. \tag{220}$$

Moreover, we can verify that $\mathsf{X}^i_\pm(t)$ satisfy canonical coordinate-momentum relations,

$$\left[\mathsf{X}^i_\pm(t), \mathsf{X}^j_\pm(t)\right] = 0, \quad \left[\mathsf{X}^i_\pm(t), \mathsf{P}^j\right] = i\delta_{ij}\mathsf{N}_\pm, \tag{221}$$

as was expected according to the Pryce (e) hypothesis, but with $\mathsf{N}_\pm$ instead of the identity operator. These position operators transform under rotations as $SO(3)$ vector operators satisfying

$$\left[\mathsf{L}_i, \mathsf{X}^j_\pm(t)\right] = i\epsilon_{ijk}\mathsf{X}^k_\pm(t), \quad \left[\mathsf{S}_i, \mathsf{X}^j_\pm(t)\right] = 0. \tag{222}$$

The transformations under Lorentz boosts are relatively complicated because of the transformation matrices, which depend on the momentum remaining under integration, as in Eq. (196). For this reason, the relativistic covariance of the position and other orbital operators will be studied elsewhere.

The above results allow us to bring the components of the *dipole* and *mass-center* operators into intuitive forms:

$$\mathsf{X}^i(t) = \mathsf{X}^i_+(t) - \mathsf{X}^i_-(t), \quad \mathsf{X}^i_{MC}(t) = \mathsf{X}^i_+(t) + \mathsf{X}^i_-(t), \tag{223}$$

whose velocities

$$\mathsf{V}^i = \mathsf{V}^i_+ - \mathsf{V}^i_-, \quad \mathsf{V}^i_{MC} = \mathsf{V}^i_+ + \mathsf{V}^i_-, \tag{224}$$

have conserved components. The dipole velocity of components V^i, known as the classical current [21], is referred to here as the conserved current. Note that the position operators at different instants $t' \neq t$ do not commute,

$$\left[\mathsf{X}^i(t), \mathsf{X}^j(t')\right] = \left[\mathsf{X}^i_{MC}(t), \mathsf{X}^j_{MC}(t')\right] = i(t'-t)\mathsf{G}_{ij}, \tag{225}$$

giving rise to the new even one-particle operator

$$\mathsf{G}_{ij} = \int \frac{d^3p}{E(p)}\left(\delta_{ij} - \frac{p^i p^j}{E(p)^2}\right) \times \left[\mathfrak{a}^\dagger(\vec{p})\mathfrak{a}(\vec{p}) + \mathfrak{b}^\dagger(\vec{p})\mathfrak{b}(\vec{p})\right], \tag{226}$$

derived according to Eq. (B15).

The principal observables of QFT we studied above are Hermitian one-particle operators, whose parent operators are reducible. These observables are either conserved, commuting with H, or evolve linearly in time, as the boost generators and position operators. The conserved spin operator of components (193) associated to position operators (223) whose velocities (224) are conserved may describe a smooth inertial motion without Zitterbewegung. However, it is not forbidden to measure the traditional observables $\vec{x}$ and $\vec{s}$ whose components are no longer reducible operators, generated after quantization oscillating terms.

C. Irreducible operators

To analyze the behaviour of the irreducible operators, it is convenient to split each Hermitian operator $\mathsf{A} =$

$\mathsf{A}_{\rm diag}+\mathsf{A}_{\rm osc}$ in its diagonal and oscillating parts, as described in Sec. III.B. After quantization, we obtain the operator $\mathsf{A}=\mathsf{A}_{\rm diag}+\mathsf{A}_{\rm osc}$, whose diagonal part is a one-particle operator expressed in terms of associated operators as

$$\mathsf{A}_{\rm diag}=\int d^3p\left[\mathfrak{a}^\dagger(\vec{p})\left(\tilde{A}^{(+)}\mathfrak{a}\right)(\vec{p})-\mathfrak{b}^\dagger(\vec{p})\left(\tilde{A}^{(-)}\mathfrak{b}\right)(\vec{p})\right], \quad (227)$$

while the oscillating term,

$$\mathsf{A}_{\rm osc}=\int d^3p\left[\mathfrak{a}^\dagger(\vec{p})\left[\left(\tilde{A}^z\mathfrak{b}\right)^\dagger(-\vec{p})\right]^T +[\mathfrak{b}(-\vec{p})]^T\left(\tilde{A}^{z+}\mathfrak{a}\right)(\vec{p})\right], \quad (228)$$

depends only on the operator $\tilde{A}^z=\tilde{A}^{(\pm)}=[\tilde{A}^{(\mp)}]^+$. This may be written either in compact notation,

$$\mathfrak{a}^\dagger(\vec{p})\left[\left(\tilde{A}^z\mathfrak{b}\right)^\dagger(-\vec{p})\right]^T=\sum_{\sigma\sigma'}\mathfrak{a}^\dagger_\sigma(\vec{p})\tilde{A}^z_{\sigma\sigma'}(\vec{p})\mathfrak{b}^\dagger_{\sigma'}(-\vec{p}), \quad (229)$$

or by explicitly using the matrix elements (113).

We focus here on the operators of QFT whose parents are either Fourier operators or simple momentum-independent matrix operators of ρ_D that can be seen as particular Fourier operators for which the Fourier transform is the operator itself. Therefore, we may derive the matrix elements of the associated operators according to Eqs. (111)−(114), where we have to substitute the operators under consideration. Thus, we obtain the diagonal terms that are one-particle operators and oscillating parts with the specific form (228). All these operators form an open algebra with obvious commutation rules, $[\mathsf{A}_{\rm diag},\mathsf{B}_{\rm diag}]=\mathsf{C}_{\rm diag}$, $[\mathsf{A}_{\rm osc},\mathsf{B}_{\rm osc}]=\mathsf{C}_{\rm diag}$, and $[\mathsf{A}_{\rm osc},\mathsf{B}_{\rm diag}]=\mathsf{C}_{\rm osc}$, showing that only the diagonal terms may form a sub-algebra.

Let us first consider the quantization of the coordinate operator $\underline{\vec{x}}=\vec{X}-\delta\vec{X}$, which can be done as we have already derived the Pryce (e) dipole operator with components (223), and we know that $\delta\vec{X}$ is a Fourier operator. Applying the canonical quantization procedure at the initial time t=0 and translating the result at an arbitrary instant t, we obtain the operators

$$\delta\mathsf{X}^i(t)=\delta\mathsf{X}^i_{\rm diag}+\delta\mathsf{X}^i_{\rm osc}(t), \quad (230)$$

with conserved odd diagonal parts

$$\delta\mathsf{X}^i_{\rm diag}=-\frac{1}{2}\int d^3p\,\frac{\epsilon_{ijk}p^j}{E(p)(E(p)+m)}\left[\mathfrak{a}^\dagger(\vec{p})\Sigma_k(\vec{p})\mathfrak{a}(\vec{p}) -\mathfrak{b}^\dagger(\vec{p})\Sigma_k(\vec{p})\mathfrak{b}(\vec{p})\right] \quad (231)$$

and oscillating terms of the form

$$\delta\mathsf{X}^i_{\rm osc}(t)=\int d^3p\sum_{\sigma,\sigma'}\left[\delta\tilde{X}^{zi}_{\sigma\sigma'}(t,\vec{p})\mathfrak{a}^\dagger_\sigma(\vec{p})\mathfrak{b}^\dagger_{\sigma'}(-\vec{p})+\text{H.c.}\right], \quad (232)$$

where, according to Eq. (A.13), we have

$$\delta\tilde{X}^{zi}_{\sigma\sigma'}(t,\vec{p})=-\frac{ie^{2iE(p)t}}{2E(p)}\Theta^{-1}_{ij}(\vec{p})\xi^+_\sigma(\vec{p})\sigma_j\eta_{\sigma'}(-\vec{p}). \quad (233)$$

Hereby, we obtain the components of the coordinate operator of QFT,

$$\underline{\mathsf{x}}^i(t)=\mathsf{X}^i(t)-\delta\mathsf{X}^i(t)=\underline{\mathsf{x}}^i_0+t\mathsf{V}^i-\delta\mathsf{X}^i_{\rm osc}(t), \quad (234)$$

with the static terms

$$\underline{\mathsf{x}}^i_0=\mathsf{X}^i-\delta\mathsf{X}^i_{\rm diag}\equiv\mathsf{X}^i_{\rm Pr(c)}, \quad (235)$$

which we interpret as the components of the *initial coordinate* operator as this is just the diagonal part of the coordinate operator (234) at the instant $t=0$. This one-particle operator, corresponding to the Pryce (c) hypothesis [5], has components that satisfy canonical coordinate-momentum commutation relations but do not commute among themselves, as we verify in Appendix C.

The oscillating term of Eq. (234) produces the Zitterbewegung discovered studying the vector current [2, 3] produced by the Dirac current density, $j^\mu(x)=:\bar{\psi}(x)\gamma^\mu\psi(x):$. Its time-like component gives rise to the conserved charge operator

$$\mathsf{Q}=\int d^3x:\bar{\psi}(t,\vec{x})\gamma^0\psi(t,\vec{x}):=:\langle\psi,\psi\rangle_D:=\mathsf{N}_+-\mathsf{N}_-, \quad (236)$$

expressed in terms of the operators (186) and (187), while its space part produces the vector current with components

$$\mathsf{I}^i_V(t)=\int d^3x:\bar{\psi}(t,\vec{x})\gamma^i\psi(t,\vec{x}):=:\langle\psi,\gamma^0\gamma^i\psi\rangle_D:\big|_t =2i:\langle\psi,s_{0i}\psi\rangle_D:\big|_t=2i\mathsf{s}_{0i}(t), \quad (237)$$

proportional to the generators (A8) we split as

$$\mathsf{I}^i_V(t)=\mathsf{I}^i_{V\,\rm diag}+\mathsf{I}^i_{V\,\rm osc}(t)\ \Rightarrow\ \mathsf{s}_{0i}(t)=\mathsf{s}_{{\rm diag}\,0i}+\mathsf{s}_{{\rm osc}\,0i}(t). \quad (238)$$

Calculating these components, we recover the well-known result

$$\mathsf{I}^i_V(t)=\frac{d}{dt}\underline{\mathsf{x}}^i(t)\Rightarrow\mathsf{I}^i_{V\,\rm diag}=\mathsf{V}^i,\quad \mathsf{I}^i_{V\,\rm osc}(t)=-\frac{d}{dt}\delta\mathsf{X}^i_{\rm osc}(t), \quad (239)$$

which was discussed in Refs. [21, 22] but using particular polarization spinors.

Besides, the conserved Dirac current density one component the axial current density $j^\mu_A(x) = -:\bar\psi(x)\gamma^5\gamma^\mu\psi(x):$, which is conserved only in the massless case. This gives rise to the axial charge

$$\mathsf{Q}_A(t) = \int d^3x\, j^0_A = :\langle\psi,\gamma^5\psi\rangle_D:\big|_t = \mathsf{Q}_{A\,\mathrm{diag}} + \mathsf{Q}_{A\,\mathrm{osc}}(t), \quad (240)$$

with a conserved diagonal part

$$\mathsf{Q}_{A\,\mathrm{diag}} = \int d^3p\, \frac{p^i}{E(p)}\left[\mathfrak{a}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{a}(\vec p) + \mathfrak{b}^\dagger(\vec p)\Sigma_i(\vec p)\mathfrak{b}(\vec p)\right], \quad (241)$$

which is an even one-particle operator, in contrast with the charge operator, which is odd. In addition, this has the oscillating term

$$\mathsf{Q}_{A\,\mathrm{osc}}(t) = -\int d^3p\, \frac{m}{E(p)}\left[e^{2iE(p)t}\mathfrak{a}^\dagger(\vec p)\left(\mathfrak{b}^\dagger(-\vec p)\right)^T + \mathrm{H.c.}\right]. \quad (242)$$

The corresponding components of axial current,

$$\begin{aligned}\mathsf{I}^i_A(t) &= -\int d^3x : \bar\psi(t,\vec x)\gamma^5\gamma^i\psi(t,\vec x):\\ &= -:\langle\psi,\gamma^0\gamma^5\gamma^i\psi\rangle_D:\big|_t\\ &= 2:\langle\psi,s_i\psi\rangle_D:\big|_t = 2\mathsf{s}_i(t),\end{aligned} \quad (243)$$

are proportional with the generators (A.6), which we split again as

$$\mathsf{I}^i_A(t) = \mathsf{I}^i_{A\,\mathrm{diag}} + \mathsf{I}^i_{A\,\mathrm{osc}}(t) \;\Rightarrow\; \mathsf{s}_i(t) = \mathsf{s}_{\mathrm{diag}\,i} + \mathsf{s}_{\mathrm{osc}\,i}(t). \quad (244)$$

Note that the conserved diagonal terms $\mathsf{I}^i_{A\,\mathrm{diag}} = 2\,\mathsf{S}_{\mathrm{PC}\,i}$ depend on the components (213) of the Pryce (c)-Czochor operator, which, by definition, is the diagonal part of Pauli's one. The oscillating parts read

$$\mathsf{I}^i_{A\,\mathrm{osc}}(t) = \int d^3p \sum_{\sigma,\sigma'}\left[\tilde I^{zi}_{A\sigma\sigma'}(t,\vec p)\mathfrak{a}^\dagger_\sigma(\vec p)\mathfrak{b}^\dagger_{\sigma'}(-\vec p) + \mathrm{H.c.}\right], \quad (245)$$

where

$$\tilde I^{zi}_{A\sigma\sigma'}(t,\vec p) = i e^{2iE(p)t}\epsilon_{ijk}\frac{p^j}{E(p)}\xi^+_\sigma(\vec p)\sigma_k\eta_{\sigma'}(-\vec p). \quad (246)$$

Thus, we have a complete image of the time evolution of the principal currents of Dirac's theory related to the operators $\mathsf{s}_{\mu\nu}(t)$ defined by Eqs. (243) and (237) that represent the generators of the operator-valued representation of QFT equivalent to $\rho_D[SL(2,\mathbb{C})]$.

Other matrix operators of RQM, irreducible on $\tilde{\mathcal{F}}$, are the generators of various transformations that can be defined in ρ_D. For example, the Foldy-Wouthuysen transformation (A17), which relates the Pauli-Dirac and Pryce spin operators as in Eq. (A19), are generated by the Hermitian matrices $-i\gamma^i$, which are the parents of the operators

$$\mathsf{F}^i(t) = -i:\langle\psi,\gamma^i\psi\rangle_D:\big|_t = \mathsf{F}^i_{\mathrm{diag}} + \mathsf{F}^i_{\mathrm{osc}}(t), \quad (247)$$

with diagonal parts

$$\mathsf{F}^i_{\mathrm{diag}} = \int d^3p\, \epsilon_{ijk}p^j\left[\mathfrak{a}^\dagger(\vec p)\Sigma_k(\vec p)\mathfrak{a}(\vec p) - \mathfrak{b}^\dagger(\vec p)\Sigma_k(\vec p)\mathfrak{b}(\vec p)\right], \quad (248)$$

which are now odd one-particle operators. The oscillating terms read

$$\mathsf{F}^i_{\mathrm{osc}}(t) = \int d^3p \sum_{\sigma,\sigma'}\left[\tilde F^{zi}_{\sigma\sigma'}(t,\vec p)\mathfrak{a}^\dagger_\sigma(\vec p)\mathfrak{b}^\dagger_{\sigma'}(-\vec p) + \mathrm{H.c.}\right], \quad (249)$$

where, by using the tensor (A.13) again, we may write

$$\tilde F^{zi}_{\sigma\sigma'}(t,\vec p) = i e^{2iE(p)t}\frac{m}{E(p)}\Theta_{ij}(\vec p)\xi^+_\sigma(\vec p)\sigma_j\eta_{\sigma'}(-\vec p). \quad (250)$$

This behaviour explains why the particular Foldy-Wouthuysen transformation (A17) can relate the conserved Pryce (e) spin operator to the non-conserved Pauli-Dirac one, as in Eq. (A19).

The Chakrabarti spin operator $\vec S_{\mathrm{Ch}}$ can be quantized starting with its Fourier transform (71), deriving the associated operators, and applying the quantization procedure. Thus, we find that the components of this operator,

$$\mathsf{S}_{\mathrm{Ch}\,i}(t) = \mathsf{S}_i + \mathsf{S}_{\mathrm{osc}\,i}(t), \quad (251)$$

are formed by those of the Pryce (e) spin operator with supplemental oscillating terms of the form

$$\mathsf{S}_{\mathrm{osc}\,i}(t) = \int d^3p \sum_{\sigma,\sigma'}\left[\tilde S^{zi}_{\sigma\sigma'}(t,\vec p)\mathfrak{a}^\dagger_\sigma(\vec p)\mathfrak{b}^\dagger_{\sigma'}(-\vec p) + \mathrm{H.c.}\right], \quad (252)$$

where

$$\tilde S^{zi}_{\sigma\sigma'}(t,\vec p) = \frac{i e^{2iE(p)t}}{m}\epsilon_{ijk}p^j\xi^+_\sigma(\vec p)\sigma_k\eta_{\sigma'}(-\vec p). \quad (253)$$

This result was expected as we know that the parent operator (71) is not conserved.

Finally, let us focus on the scalar and pseudo-scalar charges. Starting with the scalar one, we may split it as

$$\mathsf{Q}^{sc}(t) = \int d^3x : \bar{\psi}(t,\vec{x})\psi(t,\vec{x}) :=: \langle\psi,\gamma^0\psi\rangle_D :$$
$$= \mathsf{Q}^{sc}_{diag} + \mathsf{Q}^{sc}_{osc}(t), \tag{254}$$

where the conserved diagonal term

$$\mathsf{Q}^{sc}_{diag} = m\int \frac{d^3p}{E(p)}\left[\mathfrak{a}^\dagger(\vec{p})\mathfrak{a}(\vec{p}) + \mathfrak{b}^\dagger(\vec{p})\mathfrak{b}(\vec{p})\right] \tag{255}$$

is an even one-particle operator, while the oscillating part can be written as

$$\mathsf{Q}^{sc}_{osc}(t) = \int \frac{d^3p}{E(p)}\sum_{\sigma,\sigma'}\left[\tilde{Q}^{sc\,z}_{\sigma\sigma'}(t,\vec{p})\mathfrak{a}^\dagger_\sigma(\vec{p})\mathfrak{b}^\dagger_{\sigma'}(-\vec{p}) + \text{H.c.}\right],$$

$$\tilde{Q}^{sc\,z}_{\sigma\sigma'}(t,\vec{p}) = -e^{2iE(p)t}\,p^j\xi^+_\sigma(\vec{p})\sigma_j\eta_{\sigma'}(-\vec{p}). \tag{256}$$

It is interesting that the pseudoscalar charge does not have diagonal terms, reducing to the oscillating form

$$\mathsf{Q}^{ps}(t) = \int d^3x : \bar{\psi}(t,\vec{x})\gamma^5\psi(t,\vec{x}) :=: \langle\psi,\gamma^0\gamma^5\psi\rangle_D :$$
$$= -\int d^3p\sum_{\sigma,\sigma'}\left[e^{2iE(p)t}\xi^+_\sigma(\vec{p})\eta_{\sigma'}(-\vec{p})\mathfrak{a}^\dagger_\sigma(\vec{p})\mathfrak{b}^\dagger_{\sigma'}(-\vec{p})\right.$$
$$\left.+\text{H.c.}\right], \tag{257}$$

which could be of some interest in QFT.

To conclude, we may say that our method of associated operators allows us to quantize all the operators we need in QFT, including the irreducible ones. The oscillating terms of these operators give vanishing expectation values and real-valued contributions to dispersion in pure states, but they may present significant observable effects when measured in mixed states.

VI. PROPAGATION

In applications, we may turn back to RQM but considered now as the one-particle restriction of QFT. Thus, we have the advantages of the mathematical rigor and correct physical interpretations offered by QFT. We assume that the quantum states are prepared or measured by an ideal apparatus represented by a set of one-particle operators without oscillating parts, including the Pryce (e) spin and position operators.

A. Preparing and detecting wave packets

In the following, we study the propagation of the plane wave packets generated by the one-particle physical states

$$|\alpha\rangle = \int d^3p\sum_\sigma \alpha_\sigma(\vec{p})\mathfrak{a}^\dagger_\sigma(\vec{p})|0\rangle, \tag{258}$$

defined by normalized wave spinors, $\alpha \in \tilde{\mathcal{F}}^+$, which satisfy the normalization condition

$$\langle\alpha|\alpha\rangle = \langle\alpha,\alpha\rangle = \int d^3p\,\alpha^+(\vec{p})\alpha(\vec{p}) = 1. \tag{259}$$

The corresponding wave spinors in CR,

$$\Psi_\alpha(x) = \langle 0|\psi(x)|\alpha\rangle = \int d^3p\sum_\sigma U_{\vec{p},\sigma}(x)\alpha_\sigma(\vec{p}), \tag{260}$$

are normalized, $\langle\Psi_\alpha,\Psi_\alpha\rangle_D = 1$, with respect to the scalar product (4). This is a particular case of local relativistic wave function that can be obtained from the one-particle restriction of QFT. In general, one can directly construct such functions as Fourier transforms of momentum-dependent wave functions obtained by the recently proposed generalized Bargmann-Wigner approach [24] (see Ref. [34] and references therein). In this framework, wave functions for massive and massless particles of different discrete or even continuous spins may be constructed and studied without resorting explicitly to the field operators of QFT.

The wave functions are not measurable quantities but are often studied using numerical and graphical methods for extracting intuitive information about propagation in the presence of Zitterbewegung and spin dynamics produced by the traditional observables of Dirac's RQM. Such methods were used for the first time in Ref. [35].

In our approach, we avoid these effects by assuming that our apparatus measures only the reducible observables as the energy, momentum, position, velocity, spin, and polarization, which are one-particle operators. The physical meaning is then given only by the statistical quantities generated by these operators, which can be derived easily using our previous results. More specifically, for any one-particle operator A, the expectation value and dispersion in the state $|\alpha\rangle$, denoted as

$$\langle\mathsf{A}\rangle \equiv \langle\alpha|\mathsf{A}|\alpha\rangle = \langle\alpha,\tilde{A}\alpha\rangle, \tag{261}$$

$$\text{disp}(\mathsf{A}) \equiv \langle\mathsf{A}^2\rangle - \langle\mathsf{A}\rangle^2 = \langle\tilde{A}\alpha,\tilde{A}\alpha\rangle - \langle\alpha,\tilde{A}\alpha\rangle^2, \tag{262}$$

may be written in terms of associated operators acting in MR of RQM. Once we have the dispersion, we may write the uncertainty $\Delta\mathsf{A} = \sqrt{\mathrm{disp}(\mathsf{A})}$.

To exploit these formulas, we need to specify the structure of the functions α_σ. We observe first that it is important to know where the state $|\alpha\rangle$ is prepared, translating the state to that point. If the state was prepared initially at the origin, then for a state prepared by the same apparatus at the point of position vector $\vec{x}_0$, we must perform the back translation $|\alpha\rangle \to \mathsf{U}(0,-\vec{x}_0)|\alpha\rangle = e^{-i\vec{x}_0\cdot\vec{p}}|\alpha\rangle$ defined by Eq. (178). Meanwhile, we know the position operator defined with the help of the covariant derivatives (124), which can be quite complicated in the case of peculiar polarization. Therefore, for a rapid inspection of a relevant example, it is convenient to choose the simplest polarization spinors (D6) of the standard momentum-spin basis, where $\Sigma_i = \sigma_i$ and $\Omega_i = 0$.

Starting with these arguments, we assume that the wave packet with the mentioned polarization is prepared at the initial time t=0 by an observer O at the initial point $\vec{x}_0$. Therefore, we may consider the wave spinor

$$\alpha(\vec{p}) = \begin{pmatrix} \alpha_{\frac{1}{2}}(\vec{p}) \\ \alpha_{-\frac{1}{2}}(\vec{p}) \end{pmatrix} = \phi(\vec{p})e^{-i\vec{x}_0\cdot\vec{p}} \begin{pmatrix} \cos\dfrac{\theta_s}{2} \\ \sin\dfrac{\theta_s}{2} \end{pmatrix}, \quad (263)$$

where θ_s is the polarization angle, while $\phi : \mathbb{R}^3_{\vec{p}} \to \mathbb{R}$ is a real-valued scalar function that is normalized as

$$\langle\alpha|\alpha\rangle = 1 \;\Rightarrow\; \int d^3p\, \phi(\vec{p})^2 = 1\,. \quad (264)$$

With this function, we may calculate the expectation values and dispersions of the operators without spin terms, as in the scalar theory. For example, in the case of the energy operator (188), we may write

$$\langle\mathsf{H}\rangle = \int d^3p\, E(p)\phi(\vec{p})^2\,, \quad (265)$$

$$\mathrm{disp}(\mathsf{H}) = \int d^3p\, E(p)^2\phi(\vec{p})^2 - \langle\mathsf{H}\rangle^2\,, \quad (266)$$

and similarly for the momentum components (189).

The polarization angle helps us to rapidly find the measurable quantities related to the spin components (193) and polarization $\mathsf{W}_s = \mathsf{S}_3$. Considering that now $\tilde{S}_i = 1/2\sigma_i$, we obtain from Eqs. (261) and (262) the quantities

$$\langle\mathsf{S}_1\rangle = \frac{1}{2}\sin\theta_s\,, \mathrm{disp}(\mathsf{S}_1) = \frac{1}{4}\cos^2\theta_s\,,$$

$$\langle\mathsf{S}_2\rangle = 0\,, \qquad \mathrm{disp}(\mathsf{S}_2) = \frac{1}{4}\,,$$

$$\langle\mathsf{S}_3\rangle = \frac{1}{2}\cos\theta_s\,, \quad \mathrm{disp}(\mathsf{S}_3) = \frac{1}{4}\sin^2\theta_s\,,$$

with an obvious physical meaning as the polarization angle is defined on the interval $[0,\pi]$ such that for $\theta_s = 0$, the polarization is $\sigma = \frac{1}{2}$ (↑), while for $\theta_s = \pi$ it is $\sigma = -\frac{1}{2}$ (↓). In both these cases of *total* polarization, the measurements are exact with $\mathrm{disp}(\mathsf{W}_s) = \mathrm{disp}(\mathsf{S}_3) = 0$.

The propagation of the wave packet is described by the position operator of components $X^i_+(t) = X^i_+ + tV^i_+$ defined by Eqs. (215) and (216). In the momentum-spin basis we use here, we have the advantage of $\Omega = 0$, which means that the covariant derivatives (124) become the usual ones, $\tilde{\partial}_i \to \partial_{p^i}$. Thus, we find the quantities

$$\langle\mathsf{X}^i_+\rangle = \frac{i}{2}\int d^3p\, \alpha^+(\vec{p})\, \overleftrightarrow{\partial}_{p_i}\, \alpha(\vec{p}) = x^i_0 \int d^3p\, \phi(\vec{p})^2 = x^i_0\,, \quad (267)$$

$$\begin{aligned}\mathrm{disp}(\mathsf{X}^i_+) &= \int d^3p\, \partial_{p^i}\alpha^+(\vec{p})\partial_{p^i}\alpha(\vec{p})\,(\text{no sum}) - (x^i_0)^2 \\ &= \int d^3p\, \left(\partial_{p^i}\phi(\vec{p})\right)^2\,,\end{aligned} \quad (268)$$

$$\langle\mathsf{V}^i_+\rangle = \int d^3p\, \frac{p^i}{E(p)}\phi(\vec{p})^2\,, \quad (269)$$

$$\mathrm{disp}(\mathsf{V}^i_+) = \int d^3p \left(\frac{p^i}{E(p)}\right)^2 \phi(\vec{p})^2 - \langle\mathsf{V}^i_+\rangle^2\,, \quad (270)$$

which depend only on the scalar function ϕ. Finally, we obtain the remarkable but expected result

$$\mathrm{disp}(\mathsf{X}^i_+(t)) = \mathrm{disp}(\mathsf{X}^i_+) + t^2\mathrm{disp}(\mathsf{V}^i_+)\,, \quad (271)$$

which lays out the dispersive character of this type of wave packets that spread as other scalar or non-relativistic wave packets [28]. A similar calculation can be performed for the angular momentum, which is conserved in our approach but less relevant in analyzing the inertial motion.

Let us imagine now that another observer, O', detects the above prepared wave packet, performing measurement with a similar apparatus at the point $\vec{x}'_0$. We denote by $\vec{x}_0 - \vec{x}'_0 = \vec{n}d$ the relative position vector assuming that the observers O and O' use the same Cartesian coordinates and therefore same observables. The wave packet evolves causally until the detector measures some of its parameters, selecting (or filtering) only the fermions coming from the source O whose momenta are in a narrow domain $\Delta\subset\mathbb{R}^3_{\vec p}$ along the direction $\vec n$. Therefore, the measured state $|\alpha'\rangle$ is given now by the corresponding projection operator Λ_Δ as

$$|\alpha'\rangle=\Lambda_\Delta|\alpha\rangle=\int_\Delta \mathrm{d}^3p\,\alpha(\vec p)\,\mathfrak{a}^\dagger(\vec p)|0\rangle. \tag{272}$$

This state is strongly dependent on the domain Δ of measured momenta. Here, we assume that this is a cone of axis $\vec n$ and a very small solid angle $\Delta\Omega$ such that we may apply the mean value theorem,

$$\int_\Delta \mathrm{d}^3pF(\vec p)\simeq\Delta\Omega\int_0^\infty \mathrm{d}pp^2F(\vec np), \tag{273}$$

in spherical coordinates $\vec p=(p,\vartheta,\varphi)$ to all the integrals over Δ. We first evaluate the quantity

$$\langle\alpha|\Lambda_\Delta|\alpha\rangle=\int_\Delta \mathrm{d}^3p\,\alpha^+(\vec p)\alpha(\vec p)=\int_\Delta \mathrm{d}^3p\,\phi(\vec p)^2$$
$$\simeq\Delta\Omega\int_0^\infty \mathrm{d}p\,p^2\phi(\vec np)^2=\Delta\Omega\kappa, \tag{274}$$

giving the probability $P_\Delta=|\langle\alpha|\Lambda_\Delta|\alpha\rangle|^2$ of measuring any momentum $\vec p\in\Delta$. Obviously, when one measures the whole continuous spectrum, $\Delta=\mathbb{R}^3_k$, then Λ_Δ becomes the identity operator and $P_\Delta=1$.

Under such circumstances, the observer O' measures new expectation values

$$\langle\mathsf{A}\rangle'=\langle\alpha'|\mathsf{A}|\alpha'\rangle=\frac{\langle\alpha|\Lambda_\Delta\mathsf{A}|\alpha\rangle}{\langle\alpha|\Lambda_\Delta|\alpha\rangle} \tag{275}$$

for all the common observables of O and O' that depend on momentum. Applying the above calculation rules, we obtain the expectation values

$$\langle\mathsf{H}\rangle'=\frac{1}{\kappa}\int_0^\infty p^2\mathrm{d}p\,E(p)\phi(\vec np)^2, \tag{276}$$

$$\langle\mathsf{P}^i\rangle'=n^i\frac{1}{\kappa}\int_0^\infty p^2\mathrm{d}p\,p\phi(\vec np)^2=n^i\langle\mathsf{P}\rangle', \tag{277}$$

$$\langle\mathsf{V}^i_+\rangle'=n^i\frac{1}{\kappa}\int_0^\infty p^2\mathrm{d}p\,\frac{p}{E(p)}\phi(\vec np)^2=n^i\langle\mathsf{V}_+\rangle', \tag{278}$$

which show that O' in fact observes a one-dimensional motion along the direction $\vec n$ measuring the new observables

$$\mathsf{P}=\int \mathrm{d}^3p\,p\left[\mathfrak{a}^\dagger(\vec p)\mathfrak{a}(\vec p)+\mathfrak{b}^\dagger(\vec p)\mathfrak{b}(\vec p)\right], \tag{279}$$

$$\mathsf{V}_+=\int \mathrm{d}^3p\,\frac{p}{E(p)}\mathfrak{a}^\dagger(\vec p)\mathfrak{a}(\vec p), \tag{280}$$

whose expectation values result from Eqs. (277) and (278). We say that these operators and V_-, defined similarly for antiparticles, are the *radial* observables of the common list of observables of O and O'.

Therefore, O' measures a one-dimensional wave packet $|\alpha'\rangle$ whose wave spinors depend now on the new normalized scalar function

$$\phi'(p)=\frac{1}{\sqrt{\kappa}}p\phi(\vec np), \tag{281}$$

allowing us to write the statistical quantities of the radial operators measured by O' as

$$\langle\mathsf{H}\rangle'=\int_0^\infty \mathrm{d}p\,E(p)\phi'(p)^2, \tag{282}$$

$$\mathrm{disp}(\mathsf{H})'=\int_0^\infty \mathrm{d}p\,E(p)^2\phi'(p)^2-\langle\mathsf{P}\rangle'^2, \tag{283}$$

$$\langle\mathsf{P}\rangle'=\int_0^\infty \mathrm{d}p\,p\phi'(p)^2, \tag{284}$$

$$\mathrm{disp}(\mathsf{P})'=\int_0^\infty \mathrm{d}p\,p^2\phi'(p)^2-\langle\mathsf{P}\rangle'^2, \tag{285}$$

$$\langle\mathsf{V}_+\rangle'=\int_0^\infty \mathrm{d}p\,\frac{p}{E(p)}\phi'(p)^2, \tag{286}$$

$$\mathrm{disp}(\mathsf{V}_+)'=\int_0^\infty \mathrm{d}p\left(\frac{p}{E(p)}\right)^2\phi'(p)^2-\langle\mathsf{V}_+\rangle'^2. \tag{287}$$

The expectation values of the operators X^i_+ are not affected by the projection on the domain Δ, $\langle\mathsf{X}^i_+\rangle'=\langle\mathsf{X}^i_+\rangle=x^i_0$, but the dispersions may be different as O' measures

$$\mathrm{disp}(\mathsf{X}^i_+)'=\frac{1}{\kappa}\int_0^\infty \mathrm{d}p\,p^2\left(\partial_{p^i}\phi(\vec p)\right)^2\Big|_{\vec p=\vec np}. \tag{288}$$

The only operators whose measurement is independent of the momentum filtering are the spin components, for which we have $\langle \mathsf{S}_i\rangle' = \langle \mathsf{S}_i\rangle$ and $\mathrm{disp}(S_i)' = \mathrm{disp}(S_i)$.

In this manner, we have derived all the statistical quantities of prepared or detected wave packets using only analytical methods without resorting to a visual study of the packet profile in CR, which might be intuitive but is sterile from the perspective of QFT.

B. Isotropic wave packet

As a simple example, we consider now an isotropic wave-packet for which it is convenient to use spherical coordinates in momentum space with $\vec{p} = p\vec{n}_p$ and

$$\vec{n}_p = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)\,. \tag{289}$$

We assume that at the initial time $t_0 = 0$, the observer O prepares the wave packet (258) whose wave spinor (263) is equipped with the isotropic function

$$\phi(\vec{p}) \to \phi(p) = N p^{\gamma\bar{p}-\frac{3}{2}} \mathrm{e}^{-\gamma p}\,, \quad \gamma, \bar{p} > 0\,, \tag{290}$$

depending on the real parameters γ and $\bar{p}$ and the normalization factor

$$N = \frac{(2\gamma)^{\gamma\bar{p}}}{2\sqrt{\pi\Gamma(2\gamma\bar{p})}}, \tag{291}$$

which guarantees that

$$\int \mathrm{d}^3p\phi(p)^2 = 4\pi\int_0^\infty \mathrm{d}p\, p^2\, \phi(p)^2 = 1\,. \tag{292}$$

The parameter $\bar{p}$ is just the expectation value of the radial momentum (279) such that

$$\langle \mathsf{P}\rangle = 4\pi\int_0^\infty \mathrm{d}p\, p^3\, \phi(p)^2 = \bar{p}\,, \tag{293}$$

$$\mathrm{disp}(\mathsf{P}) = 4\pi\int_0^\infty \mathrm{d}k\, p^4\, \phi(k)^2 - \bar{p}^2 = \frac{\bar{p}}{2\gamma}\,. \tag{294}$$

In this isotropic case, the Cartesian momentum and velocity components measured by O have vanishing expectation values, $\langle \mathsf{P}^i\rangle = 0$ and $\langle \mathsf{V}_+^i\rangle = 0$, but relevant dispersions that read

$$\mathrm{disp}(\mathsf{P}^i) = \frac{4\pi}{3}\langle \mathsf{P}^2\rangle = \frac{4\pi}{3}\left(\bar{p}^2 + \frac{\bar{p}}{2\gamma}\right), \tag{295}$$

$$\mathrm{disp}(\mathsf{V}_+^i) = \frac{4\pi}{3}\langle \mathsf{V}_+^2\rangle, \tag{296}$$

as the angular integrals give $\int (n_p^i)^2 \mathrm{d}\Omega = \frac{4\pi}{3}$. Moreover, the observer O measures the components of the initial position operator with expectation values (267) and dispersions (268) that now read

$$\mathrm{disp}(\mathsf{X}_+^i) = \frac{1}{6}\frac{\gamma^2}{\gamma\bar{p}-1} \quad \Rightarrow \quad \gamma\bar{p} > 1\,, \tag{297}$$

imposing a mandatory condition for our parameters.

The observer O' detects the one-dimensional wave packet with

$$\kappa = \frac{1}{4\pi} \quad \Rightarrow \quad \phi'(p) = \sqrt{4\pi}\, p\,\phi(p)\,, \tag{298}$$

which means that the statistical quantities of the operators (283)−(288) coincide with those given by Eqs. (265)−(271) measured by the observer O. To write the expressions of these quantities, we consider integrals of general form

$$G(\nu,\rho;\mu) = \int_0^\infty \mathrm{d}p\, p^{2\nu-1}\left(p^2+m^2\right)^{\rho-1}\mathrm{e}^{-\mu p} = \frac{m^{2\nu+2\rho-2}}{2\sqrt{\pi}\Gamma(1-\rho)} G^{31}_{13}\left(\frac{m^2\mu^2}{4}\,\middle|\, \begin{matrix} 1-\nu \\ 1-\rho-\nu, 0, \frac{1}{2}\end{matrix}\right), \tag{299}$$

which can be solved in terms of Meijer's G-functions according to Eq. (3.389) of Ref. [36]. With their help, we may write

$$\langle \mathsf{H}\rangle' = \langle \mathsf{H}\rangle = 4\pi N^2 G\left(\gamma\bar{p}, \frac{3}{2}; 2\gamma\right), \tag{300}$$

$$\langle \mathsf{V}_+\rangle' = \langle \mathsf{V}_+\rangle = 4\pi N^2 G\left(\gamma\bar{p}+\frac{1}{2}, \frac{1}{2}; 2\gamma\right), \tag{301}$$

$$\langle \mathsf{V}_+^2\rangle' = \langle \mathsf{V}_+^2\rangle = 4\pi N^2 G(\gamma\bar{p}+1, 0; 2\gamma)\,, \tag{302}$$

while for H^2, we find the closed expression

$$\langle \mathsf{H}^2\rangle' = \langle \mathsf{H}^2\rangle = \bar{p}^2 + m^2 + \frac{\bar{p}}{2\gamma} = E(\bar{p})^2 + \frac{\bar{p}}{2\gamma}\,. \tag{303}$$

We now have all we need to write the dispersions (296) and those of the radial operators H and V_+.

The analytical results derived above are less intuitive because of the functions G, which are relatively complicated. Therefore, to demonstrate that these results are plausible, we must resort to a brief graphical analysis comparing the above expectation values with the corresponding classical quantities $E(\bar{p})$ and $V(\bar{p}) = \frac{\bar{p}}{E(\bar{p})}$. In

Fig. 1, we plot the ratios $\frac{\langle \mathsf{H}\rangle}{E(\bar p)}$ and $\frac{2\gamma\,\mathrm{disp}(\mathsf{H})}{\bar p}$ as functions of $q=\gamma\bar p>1$, observing that $\langle \mathsf{H}\rangle$ is very close to $E(\bar p)$, while the dispersion $\mathrm{disp}(\mathsf{H})<\frac{\bar p}{2\gamma}$ tends asymptotically to its maximal value. In Fig. 2, we plot the ratio $\frac{\langle \mathsf{V}_+\rangle}{V(\bar p)}$ and $\mathrm{disp}(\mathsf{V}_+)$, observing again that $\langle \mathsf{V}_+\rangle$ is very close to the classical velocity with a small dispersion. Thus, we see that in the case of Dirac's massive fermions, the quantum corrections to the classical motion are relatively small but not negligible. Note that these corrections diminish as $\bar p$ increases, vanishing in the ultra-relativistic limit when the velocity approaches the speed of light. This behaviour convinces us that the above model properly describes a plausible physical reality.

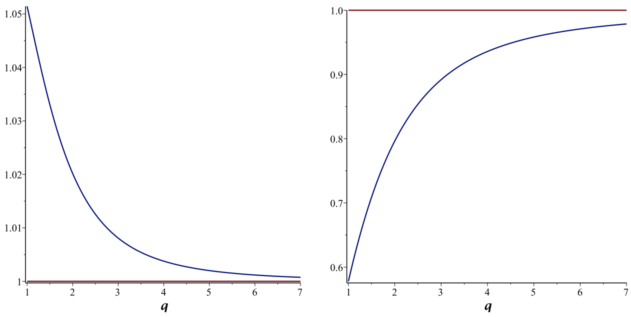

Fig. 1. (color online) Ratios $\frac{\langle \mathsf{H}\rangle}{E(\bar p)}\to 1_+$ (left panel) and $\frac{2\gamma\,\mathrm{disp}(\mathsf{H})}{\bar p}\to 1_-$ (right panel) as functions of $q=\gamma\bar p$ in the domain $1<q\leq 7$ for $\gamma m=1$.

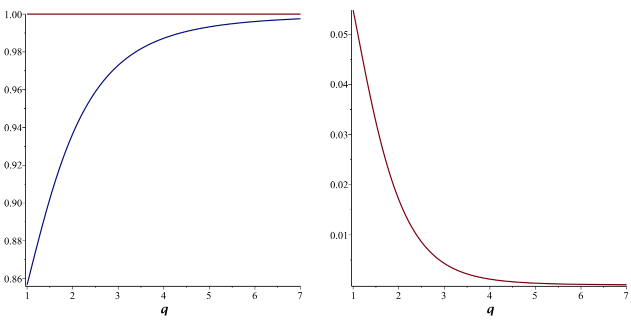

Fig. 2. (color online) Ratio $\frac{\langle \mathsf{V}_+\rangle}{V(\bar p)}\to 1_-$ (left panel) and velocity dispersion $\mathrm{disp}(\mathsf{V}_+)$ (right panel) as functions of $q=\gamma\bar p$ in the domain $1<q\leq 7$ for $\gamma m=1$.

VII. CONCLUDING REMARKS

Here, we improved the quantum theory of Dirac's free field focusing on the spin and position operators of the Pryce (e) version as fundamental observables of QFT. We succeeded in this by using the method of associated operators, allowing us to derive the principal operators of QFT. The original results at the level of RQM, presented in Secs. III.C, III.D, and IV.A−IV.D, prepare the quantization procedure, leading to the new results reported in Secs. V.B and V.C. A study of the wave packet was presented here in Sec. VI for the first time.

In QFT, we have the benefit of a correct physical interpretation that is not similar to the interpretations at the level of RQM or even classical theory. An example is the position operator of the Pryce (e) version, which was proposed as a mass-center position operator satisfying the canonical coordinate-momentum commutation relations [5]. The quantization preserves this property but transforms the would-be mass-center operator into the dipole one, interpreting the antiparticle term correctly. For this reason, we separately defined the position operators of particle and antiparticle centers, (215) and (217), respectively, whose linear combinations give both the dipole and mass-center operators. Besides these operators, we showed that the one-particle operator (235), interpreted as the initial coordinate operator, complies with Pryce's hypothesis (c), being related to the Pryce (c)-Czochor one-particle operator (213). Similarly, the Frankel spin operator (211) corresponding to the Pryce (d) hypothesis is related to a specific position operator that does not yet have an obvious physical meaning. In addition, we note that the orbital boost generators (200) may be interpreted as components of a position operator with spin-induced non-commutativity [37] but with orbital angular momentum instead of spin.

Released on QFT, we do not abandon the RQM, but we reconsider each particular system we investigate as a restriction of QFT, thus keeping the correct physical interpretation. An example is the Dirac wave packet we studied in Sec. V, where all the statistical quantities were derived using associated operators in MR and wave spinors. It is worth noting that in the one-particle RQM derived from QFT, the associated spin, orbital angular momentum, and position operators in MR have familiar forms such that in momentum-spin basis, they become just the corresponding operators, $\tilde S_i=\frac{1}{2}\sigma_i$, $\tilde L_i=-\mathrm{i}\epsilon_{ijk}p^j\partial_{p^k}$, and $\tilde X^i=\mathrm{i}\partial_{p^i}$, of the original non-relativistic Pauli's theory but now describing relativistic systems, such as, for example, the spin-orbit interactions of photons and electrons [38].

In momentum bases with peculiar polarization, these operators become more complicated, depending explicitly on polarization through the matrices (116) and (125), which can have non-trivial forms, as in the case of the momentum-helicity basis where these quantities are given by Eqs. (D7) and (D9). The matrices (116) are the Pauli operators written in a new basis, but the role of the matrices (125) defining the covariant derivatives remains obscure for now until we study concrete examples of orbital operators in bases with peculiar polarization. Unfortunately, we do not have other momentum bases with peculiar polarization, as the helicity is the only one used so far. We hope that our approach will offer an opportunity for defining new types of peculiar polarization that could be observed in further experiments.

However, one may ask why the theory of quantum free field deserves this effort based on its relatively complicated mathematics. This is because we cannot analytically solve the equations of interacting fields to obtain closed forms of interacting quantum fields or other operators of QFT. Instead, we may resort to perturbations in terms of *in* and *out* fields, which are just free fields for which we constructed the approach presented here. For example, using perturbations, we can calculate the expectation values of the new position operators (223) or the traditional one (234) in *out* states if we know the incident wave packets in *in* states. In this manner, we may better understand the role of the radiative corrections in the fermion propagation affected by Zitterbewegung.

We conclude that our approach may present new directions for investigating traditional processes, such as the QED processes involving polarized fermions, studied for the first time in terms of momentum-spin basis a long time ago [39]. Moreover, we hope that by solving the inherent new technical problems and presenting various examples of systems of free or interacting polarized fermions, we may improve the theory, filling the gap between the actual notorious successes of Dirac's theory, the Hydrogen atom, and QED.

APPENDX A: DIRAC REPRESENTATION

The Dirac γ-matrices, which satisfy $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$, give rise to the generators $s^{\mu\nu}=\frac{\mathrm{i}}{4}\left[\gamma^\mu,\gamma^\nu\right]=\overline{s^{\mu\nu}}$ of the Dirac reducible representation $\rho_D=(1/2,0)\oplus(0,1/2)$ of the $SL(2,\mathbb{C})$ group in the four-dimensional space $\mathcal{V}_D=\mathcal{V}_P\oplus\mathcal{V}_P$ of Dirac spinors. Remarkably, this space hosts the fundamental representation of the group $SU(2,2)$ [40] in which $SL(2,\mathbb{C})$ is a subgroup. A basis of the Lie algebra $su(2,2)$ may be formed by those of the $sl(2,\mathbb{C})$ subalgebra, $\sigma_{\mu\nu}$, and the matrices γ^μ, $\gamma^5\gamma^\mu$, and $\mathrm{i}\gamma^5$.

All these matrices, including the $SL(2,\mathbb{C})$ generators, are Dirac self-adjoint such that the transformations

$$\lambda(\omega)=\exp\left(-\frac{\mathrm{i}}{2}\omega^{\alpha\beta}s_{\alpha\beta}\right)\in\rho_D[SL(2,\mathbb{C})],\tag{A1}$$

with real-valued parameters, $\omega^{\alpha\beta}=-\omega^{\beta\alpha}$, leave the Hermitian form $\overline{\psi}\psi$ invariant as $\overline{\lambda(\omega)}=\lambda^{-1}(\omega)=\lambda(-\omega)$. The corresponding Lorentz transformations, $\Lambda^{\mu\cdot}_{\cdot\nu}(\omega)\equiv\Lambda^{\mu\cdot}_{\cdot\nu}[\lambda(\omega)]=\delta^\mu_\nu+\omega^{\mu\cdot}_{\cdot\nu}+\frac{1}{2}\omega^{\mu\cdot}_{\cdot\alpha}\omega^{\alpha\cdot}_{\cdot\nu}+\cdots$, satisfy the identities

$$\lambda^{-1}(\omega)\gamma^\alpha\lambda(\omega)=\Lambda(\omega)^{\alpha\cdot}_{\cdot\beta}\gamma^\beta,\tag{A2}$$

which encapsulate the canonical homomorphism [30].

In the chiral representation we consider here, the Dirac matrices are expressed in terms of Pauli matrices, σ_i, and $\mathbf{1}=1_{2\times2}$ as

$$\gamma^0=\begin{pmatrix}0&\mathbf{1}\\\mathbf{1}&0\end{pmatrix},\quad\gamma^i=\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix},\quad\gamma^5=\begin{pmatrix}-\mathbf{1}&0\\0&\mathbf{1}\end{pmatrix},\tag{A3}$$

such that the transformations $\lambda(\omega)$ generated by the matrices $s^{\mu\nu}$ are reducible to the subspaces of Pauli spinors $\mathcal{V}_P$ carrying the irreducible representations $(1/2,0)$ and $(0,1/2)$ of ρ_D [1, 30]. We denote by

$$r=\mathrm{diag}(\hat{r},\hat{r})\in\rho_D[SU(2)]\tag{A4}$$

the transformations we simply call rotations, and by

$$l=\mathrm{diag}(\hat{l},\hat{l}^{-1})\in\rho_D[SL(2,\mathbb{C})/SU(2)]\tag{A5}$$

the Lorentz boosts. For rotations, we use the generators

$$s_i=\frac{1}{2}\epsilon_{ijk}s^{jk}=\mathrm{diag}(\hat{s}_i,\hat{s}_i)=-\frac{1}{2}\gamma^0\gamma^5\gamma^i,\quad\hat{s}_i=\frac{1}{2}\sigma_i,\tag{A6}$$

and Cayley-Klein parameters $\theta^i=\frac{1}{2}\epsilon_{ijk}\omega^{jk}$ such that

$$r(\theta)=\mathrm{diag}(\hat{r}(\theta),\hat{r}(\theta)),\qquad\hat{r}(\theta)=\mathrm{e}^{-\mathrm{i}\theta^i\hat{s}_i}=\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^i\sigma_i}.\tag{A7}$$

Similarly, we choose the parameters $\tau^i=\omega^{0i}$ and generators

$$s_{i0}=s^{0i}=\mathrm{diag}(-\mathrm{i}\hat{s}_i,i\hat{s}_i)=\frac{\mathrm{i}}{2}\gamma^0\gamma^i\tag{A8}$$

for the Lorentz boosts that read

$$l(\tau)=\mathrm{diag}(\hat{l}(\tau),\hat{l}^{-1}(\tau)),\quad\hat{l}(\tau)=\mathrm{e}^{\tau^i\hat{s}_i}=\mathrm{e}^{\frac{1}{2}\tau^i\sigma_i}.\tag{A9}$$

The corresponding transformations of the group $L^\uparrow_+$ will be denoted as $R(r)\equiv R(\hat{r})=\Lambda(r)$ and $L(l)\equiv L(\hat{l})=\Lambda(l)$. We say that $\vec{s}$ is the Pauli-Dirac spin operator reducible to a pair of Pauli spin operators, $\vec{\hat{s}}$. Note that these operators satisfy the identities

$$\hat{r}^{-1}\sigma_i\hat{r}=R_{ij}(\hat{r})\sigma_j\;\Rightarrow\;r^{-1}\sigma_i r=R_{ij}(\hat{r})\sigma_j,\tag{A10}$$

resulting from the canonical homomorphism.

The boosts (A9) with parameters $\tau^i = -\frac{p^i}{p}\tanh^{-1}\frac{p}{E(p)}$ can be written as [1]

$$l_{\vec{p}} = \frac{E(p)+m+\gamma^0\vec{\gamma}\cdot\vec{p}}{\sqrt{2m(E(p)+m)}} = l^+_{\vec{p}}, \quad l^{-1}_{\vec{p}} = l_{-\vec{p}} = \bar{l}_{\vec{p}}. \qquad (A11)$$

They give rise to the Lorentz boosts $L_{\vec{p}} = \Lambda(l_{\vec{p}})$ with the matrix elements

$$\langle L_{\vec{p}}\rangle^{0\cdot}_{\cdot 0} = \frac{E(p)}{m}, \quad \langle L_{\vec{p}}\rangle^{0\cdot}_{\cdot i} = \langle L_{\vec{p}}\rangle^{i\cdot}_{\cdot 0} = \frac{p^i}{m},$$

$$\langle L_{\vec{p}}\rangle^{i\cdot}_{\cdot j} = \delta_{ij} + \frac{p^i p^j}{m(E(p)+m)}, \qquad (A12)$$

which transform the representative momentum $\mathring{p} = (m,0,0,0)$ into the desired momentum $\vec{p} = L_{\vec{p}}\mathring{p}$. Hereby, it is convenient to separate the three-dimensional tensor

$$\Theta_{ij}(\vec{p}) \equiv < l_{\vec{p}} >^{i\cdot}_{\cdot j} \Rightarrow \Theta^{-1}_{ij}(\vec{p}) = \delta_{ij} - \frac{p^i p^j}{E(p)(E(p)+m)} \qquad (A13)$$

we need when we study space components. Θ^{-1} denotes the inverse of Θ on $\mathbb{R}^3$, which is different from the space part of $L^{-1}_{\vec{p}} = L_{-\vec{p}}$.

In Dirac's theory, there are applications where we may use some properties such as

$$l^2_{\vec{p}} = \frac{E(p)+\gamma^0\vec{\gamma}\cdot\vec{p}}{m}, \qquad l^2_{-\vec{p}} = \frac{E(p)-\gamma^0\vec{\gamma}\cdot\vec{p}}{m}, \qquad (A14)$$

giving rise to the following identities:

$$\frac{1\pm\gamma^0}{2} l^2_{\vec{p}} \frac{1\pm\gamma^0}{2} = \frac{1\pm\gamma^0}{2} l^2_{-\vec{p}} \frac{1\pm\gamma^0}{2} = \frac{E(p)}{m}\frac{1\pm\gamma^0}{2}, \qquad (A15)$$

which help us to recover the operators (57) and (58) and to evaluate the quantities

$$\mathring{u}^+_\sigma(\vec{p}) l^2_{\vec{p}} \mathring{u}_{\sigma'}(\vec{p}) = \mathring{v}^+_\sigma(\vec{p}) l^2_{\vec{p}} \mathring{v}_{\sigma'}(\vec{p}) = \frac{E(p)}{m}\delta_{\sigma\sigma'}, \qquad (A16)$$

which we need to normalize the mode spinors.

Among the transformations of the set $SU(2,2)/SL(2,\mathbb{C})$, notorious ones include the Foldy-Wouthuysen unitary transformations [6]. In particular,

$$U_{\rm FW}(\vec{p}) = U^+_{\rm FW}(-\vec{p}) = \frac{E(p)+m+\vec{\gamma}\cdot\vec{p}}{\sqrt{2E(p)(E(p)+m)}} \qquad (A17)$$

brings the Fourier transform of Dirac's Hamiltonian in diagonal form,

$$U_{\rm FW}(\vec{p})\hat{H}_D(\vec{p})U_{\rm FW}(-\vec{p}) = \gamma^0 E(p), \qquad (A18)$$

and transforms the Fourier transform of the Pryce (e) spin operator into the Pauli-Dirac one [6],

$$U_{\rm FW}(\vec{p})\vec{\hat{S}}(\vec{p})U_{\rm FW}(-\vec{p}) = \vec{s}. \qquad (A19)$$

Note that Pryce previously proposed a similar transformation that differs from (A17) only through a parity, $U_{\rm Pryce}(\vec{p}) = \gamma^0 U_{\rm FW}(\vec{p})$ [5].

APPENDIX B: ALGEBRAIC PROPERTIES OF ASSOCIATED OPERATORS

The generators $\{H, P^i, J_i, K_i\}$ form a basis of the Lie(T) algebra. Among them, the $sl(2,\mathbb{C})$ ones satisfy

$$su(2)\sim so(3): \left[J_i, J_j\right] = {\rm i}\epsilon_{ijk}J_k, \quad \left[J_i, K_j\right] = {\rm i}\epsilon_{ijk}K_k, \qquad (B1)$$

$$\left[K_i, K_j\right] = -{\rm i}\epsilon_{ijk}J_k, \qquad (B2)$$

commuting with the Abelian generators as

$$[H, J_i] = 0, \qquad \left[P^i, J_j\right] = {\rm i}\epsilon_{ijk}J_k, \qquad (B3)$$

$$[H, K_i] = -{\rm i}P^i, \qquad \left[P^i, K_j\right] = -{\rm i}\delta^i_j H. \qquad (B4)$$

In CR, we cannot separate an orbital subalgebra as the operators $\underline{\vec{x}}\wedge\vec{P}$ and $\vec{s}$ are not conserved. For this reason, it is convenient to analyze the algebraic properties in MR, where the Abelian generators are diagonal, as in Eq. (140).

In MR, the generators $\{E(p), p^i, \tilde{J}_i, \tilde{K}_i\}$ of the algebra Lie$(\tilde{T})$ associated to Lie(T) satisfy similar commutation rules, allowing the splittings (141) and (144), which separate the orbital parts from the spin ones. In the case of rotations, both the angular momentum and spin operator are conserved separately, with their components forming two independent $su(2)\sim so(3)$ algebras,

$$\left[\tilde{L}_i, \tilde{L}_j\right] = {\rm i}\epsilon_{ijk}\tilde{L}_k, \quad \left[\tilde{S}_i, \tilde{S}_j\right] = {\rm i}\epsilon_{ijk}\tilde{S}_k, \quad \left[\tilde{L}_i, \tilde{S}_j\right] = 0. \quad (B5)$$

In contrast, the operators $\tilde{K}^o$ and $\tilde{K}^s$ do not commute among themselves,

$$\left[\tilde{K}^o_i,\tilde{K}^s_j\right]=-\frac{\mathrm{i}}{E(p)+m}\left[E(p)\epsilon_{ijk}\tilde{S}_k+p^i\tilde{K}^s_j\right],\qquad\text{(B6)}$$

which means that the factorization (143) cannot be extended to the entire $sl(2,\mathbb{C})$ algebra. Nevertheless, the commutation relations

$$\left[\tilde{L}_i,\tilde{K}^o_j\right]=\mathrm{i}\epsilon_{ijk}\tilde{K}^o_k\,,\quad\left[\tilde{K}^o_i,\tilde{K}^o_j\right]=-\mathrm{i}\epsilon_{ijk}\tilde{L}_k\,,\qquad\text{(B7)}$$

$$\left[\tilde{L}_i,E(p)\right]=0\,,\qquad\left[\tilde{L}_i,p^j\right]=\mathrm{i}\epsilon_{ijk}p^k\,,\qquad\text{(B8)}$$

$$\left[\tilde{K}^o_i,E(p)\right]=\mathrm{i}p^i\,,\qquad\left[\tilde{K}^o_i,p^j\right]=\mathrm{i}\delta^i_jE(p)\qquad\text{(B9)}$$

convince us that the operators $\{E(p),p^i,\tilde{L}_i,\tilde{K}^o_i\}$ generate an orbital representation of the Poincaré algebra, known as the natural or scalar representation, but now in MR instead of the CR. Note that $\tilde{S}_i$ commute with this entire algebra. Other useful relations in the spin sector,

$$\left[\tilde{S}_i,\tilde{K}^s_j\right]=\frac{\mathrm{i}}{E(p)+m}\left[p^i\tilde{S}_j-\delta_{ij}\vec{p}\cdot\vec{\tilde{S}}\right],\qquad\text{(B10)}$$

$$\left[\tilde{K}^s_i,\tilde{K}^s_j\right]=\frac{\mathrm{i}}{(E(p)+m)^2}\epsilon_{ijk}p^k\vec{p}\cdot\vec{\tilde{S}}\,,\qquad\text{(B11)}$$

do not have an obvious physical meaning.

The position operator in MR at time t, $\vec{\tilde{X}}(t)=\vec{\tilde{X}}+t\vec{\tilde{V}}$, whose components are given by Eqs. (122) and (123), do not have spin terms that are genuine orbital operators satisfying

$$\left[\tilde{X}^i(t),\tilde{X}^j(t)\right]=0\,,\qquad\left[\tilde{X}^i(t),p^j\right]=\mathrm{i}\delta_{ij}\,,\qquad\text{(B12)}$$

$$\left[\tilde{X}^i(t),E(p)\right]=\mathrm{i}\tilde{V}^i\,,\qquad\left[\tilde{V}^i,E(p)\right]=0\,,\qquad\text{(B13)}$$

$$\left[\tilde{K}^o_i,\tilde{X}^j\right]=\delta_{ij}\frac{1}{2E(p)}-\mathrm{i}\frac{p^j}{E(p)}\tilde{X}^i-\frac{p^ip^j}{2E(p)^3}\,,\qquad\text{(B14)}$$

$$\left[\tilde{K}^o_i,\tilde{V}^j\right]=E(p)\left[\tilde{X}^i,\tilde{V}^j\right]=\mathrm{i}\left[\delta_{ij}-\frac{p^ip^j}{E(p)^2}\right].\qquad\text{(B15)}$$

As expected, $\vec{\tilde{X}}(t)$ behaves as an $SO(3)$ vector commuting as

$$\left[\tilde{L}_i,\tilde{X}^j(t)\right]=\mathrm{i}\epsilon_{ijk}\tilde{X}^k(t)\,,\qquad\left[\tilde{S}_i,\tilde{X}^j(t)\right]=0\,,\qquad\text{(B16)}$$

with the components of the angular momentum and spin operators. In contrast, the commutators

$$\left[\tilde{K}^s_i,\tilde{X}^j\right]=\frac{\mathrm{i}}{E(p)+m}\left[-\epsilon_{ijk}\tilde{S}_k+\frac{p^j}{E(p)}\tilde{K}^s_i\right],\qquad\text{(B17)}$$

do not have an intuitive interpretation.

The components (147) and (148) of the Pauli-Lubanski operator have well-known algebraic properties that we complete here with the commutation relations with our new operators $\tilde{S}_i$ and $\tilde{X}^i$, which read as

$$\left[\tilde{S}_i,\tilde{W}^0\right]=\mathrm{i}(E(p)+m)\tilde{K}^s_i\,,\,,\qquad\text{(B18)}$$

$$\left[\tilde{S}_i,\tilde{W}^j\right]=\mathrm{i}\,m\,\epsilon_{ijk}\tilde{S}_k+\mathrm{i}p^j\tilde{K}^s_i\,,\qquad\text{(B19)}$$

$$\left[\tilde{X}^i,\tilde{W}^0\right]=\mathrm{i}\tilde{S}_i\,,\qquad\text{(B20)}$$

$$\left[\tilde{X}^i,\tilde{W}^j\right]=\frac{\mathrm{i}}{E(p)+m}\left[\delta_{ij}\tilde{W}^0+p^i\tilde{S}^{(-)}_j\right],\qquad\text{(B21)}$$

where $\tilde{S}^{(-)}$ are defined by Eq. (118). The operators $\tilde{V}^i$ are multiplicative commuting with all the components $\tilde{W}^\mu$.

APPENDX C: ASSOCIATED PRYCE'S (C) AND (D) POSITION OPERATORS

The operators associated to the position operators (97) can be derived by considering that the Pryce (e) position operator is associated to the operators (122) and using the Fourier transforms (98) and (99). Thus, we obtain the associated operators

$$\begin{aligned}X^i_{\mathrm{Pr(c)}}\Rightarrow\tilde{X}^i_{\mathrm{Pr(c)}}=\tilde{X}^{ci}_{\mathrm{Pr(c)}}&=\mathrm{i}\tilde{\partial}_i+\frac{\epsilon_{ijk}p^j\tilde{S}_k}{E(p)(E(p)+m)}\\&=\frac{1}{2}\left\{\tilde{K}_i,\frac{1}{E(p)}\right\},\end{aligned}\qquad\text{(C1)}$$

$$X^i_{\mathrm{Pr(d)}}\Rightarrow\tilde{X}^i_{\mathrm{Pr(d)}}=\tilde{X}^{ci}_{\mathrm{Pr(d)}}=\mathrm{i}\tilde{\partial}_i-\frac{\epsilon_{ijk}p^j\tilde{S}_k}{m(E(p)+m)}\,.\qquad\text{(C2)}$$

The components of these operators do not commute among themselves such that the commutators

$$\left[\tilde{X}^i_{\mathrm{Pr(c)}},\tilde{X}^j_{\mathrm{Pr(c)}}\right]=-\mathrm{i}\epsilon_{ijk}\tilde{Y}^k_{\mathrm{Pr(c)}}\,,\qquad\text{(C3)}$$

$$\left[\tilde{X}^i_{\mathrm{Pr(d)}},\tilde{X}^j_{\mathrm{Pr(d)}}\right]=\mathrm{i}\epsilon_{ijk}\tilde{Y}^k_{\mathrm{Pr(d)}}\qquad\text{(C4)}$$

generate new associated components

$$Y^i_{\rm Pr(c)} \Rightarrow \tilde{Y}^i_{\rm Pr(c)} = -\tilde{Y}^{c\,i}_{\rm Pr(c)} = \frac{m}{E(p)^3}\tilde{S}^{(+)} = \frac{1}{E(p)^3}\tilde{W}^i\,, \tag{C5}$$

$$Y^i_{\rm Pr(d)} \Rightarrow \tilde{Y}^i_{\rm Pr(d)} = -\tilde{Y}^{c\,i}_{\rm Pr(d)} = \frac{1}{mE(p)}\tilde{S}^{(+)}_i = \frac{1}{m^2E(p)}\tilde{W}^i\,, \tag{C6}$$

proportional with those defined by Eqs. (117) and (148), giving rise to new even one-particle operators.

These operators have interesting algebraic properties, but here, we restrict ourselves to derive the commutation relations with the associated isometry generators, *i.e.*, the translation generators, $E(p)$ and p^i, and the $SL(2,\mathbb{C})$ ones, $\tilde{J}_i$ and $\tilde{K}_i$, defined by Eqs. (141) and (144), whose terms are given in Eqs. (142), (115), (145), and (146). We obtain the commutation rules with the components of the Pryce (c) position operator,

$$\begin{aligned}
\left[E(p),\tilde{X}^j_{\rm Pr(c)}\right] &= -{\rm i}\tilde{V}^i\,,\\
\left[p_i,\tilde{X}^j_{\rm Pr(c)}\right] &= -{\rm i}\delta_{ij}1_{2\times2}\,,\\
\left[\tilde{J}_i,\tilde{X}^j_{\rm Pr(c)}\right] &= {\rm i}\epsilon_{ijk}\tilde{X}^k_{\rm Pr(c)}\,,\\
\left[\tilde{K}_i,\tilde{X}^j_{\rm Pr(c)}\right] &= \frac{1}{2E(p)}\left(\delta_{ij}-\frac{p^ip^j}{E(p)^2}\right)1_{2\times2}\\
&\quad -\frac{\rm i}{E(p)^2}p^i\tilde{X}^j_{\rm Pr(c)} - \frac{\rm i}{E(p)}\epsilon_{ijk}\tilde{J}_k\,,
\end{aligned}\tag{C7}$$

and of those of the Pryce (d) ones,

$$\begin{aligned}
\left[E(p),\tilde{X}^j_{\rm Pr(d)}\right] &= -{\rm i}\tilde{V}^i\,,\\
\left[p_i,\tilde{X}^j_{\rm Pr(d)}\right] &= -{\rm i}\delta_{ij}1_{2\times2}\,,\\
\left[\tilde{J}_i,\tilde{X}^j_{\rm Pr(d)}\right] &= {\rm i}\epsilon_{ijk}\tilde{X}^k_{\rm Pr(d)}\,,\\
\left[\tilde{K}_i,\tilde{X}^j_{\rm Pr(d)}\right] &= \frac{1}{2E(p)}\left(\delta_{ij}-\frac{p^ip^j}{E(p)^2}\right)1_{2\times52}\\
&\quad -\frac{\rm i}{E(p)}p^j\tilde{X}^i_{\rm Pr(d)}\,,
\end{aligned}\tag{C8}$$

drawing the conclusion that the components of these operators satisfy canonical momentum-coordinate commutation relations and behave as $SO(3)$ vectors, except with different commutation rules from the boost generators.

The corresponding components of the one-particle operators, $\mathsf{X}^i_{\rm Pr(c)}$, $\mathsf{X}^i_{\rm Pr(d)}$, $\mathsf{Y}^i_{\rm Pr(c)}$, and $\mathsf{Y}^i_{\rm Pr(d)}$ mudt be derived by substituting the associated operators (C1)−(C6) into Eq. (169).

APPENDX D: SPIN AND HELICITY MOMENTUM BASES

In general, the Pauli polarization spinors, $\xi_\sigma(\vec{p})$, and $\eta_\sigma(\vec{p})={\rm i}\sigma_2\xi^*_\sigma(\vec{p})$, which may depend on momentum, form related orthonormal systems,

$$\xi^+_\sigma(\vec{p})\xi_{\sigma'}(\vec{p}) = \eta^+_\sigma(\vec{p})\eta_{\sigma'}(\vec{p}) = \delta_{\sigma\sigma'}\,, \tag{D1}$$

which are complete,

$$\sum_\sigma \xi_\sigma(\vec{p})\xi^+_\sigma(\vec{p}) = \sum_\sigma \eta_\sigma(\vec{p})\eta^+_\sigma(\vec{p}) = 1_{2\times2}\,, \tag{D2}$$

representing bases in the subspaces of Pauli spinors, $\mathcal{V}_P$, of the space of Dirac spinors, $\mathcal{V}_D=\mathcal{V}_P\oplus\mathcal{V}_P$.

In the case of arbitrary common polarization, the spin projection is measured along a unit vector $\vec{n}$. In this case, the Pauli polarization spinors $\xi_\sigma(\vec{n})$ and $\eta_\sigma(\vec{n})={\rm i}\sigma_2\xi_\sigma(\vec{n})^*$ satisfy the eigenvalues problems

$$(\vec{n}\cdot\hat{\vec{s}})\xi_\sigma(\vec{n}) = \sigma\xi_\sigma(\vec{n}) \;\Rightarrow\; (\vec{n}\cdot\hat{\vec{s}})\eta_\sigma(\vec{n}) = -\sigma\eta_\sigma(\vec{n})\,, \tag{D3}$$

where the matrices $\hat{s}_i$ are defined in Eq. (A6). These spinors have the form

$$\xi_{\frac12}(\vec{n}) = \sqrt{\frac{1+n^3}{2}}\begin{pmatrix}1\\ \dfrac{n^1+in^2}{1+n^3}\end{pmatrix},$$
$$\xi_{-\frac12}(\vec{n}) = \sqrt{\frac{1+n^3}{2}}\begin{pmatrix}\dfrac{-n^1+in^2}{1+n^3}\\ 1\end{pmatrix}, \tag{D4}$$

satisfy the normalization and completeness conditions, and have the property

$$\sum_\sigma \sigma\xi_\sigma(\vec{n})\xi^+_\sigma(\vec{n}) = \sum_\sigma \sigma\eta_\sigma(\vec{n})\eta^+_\sigma(\vec{n}) = n^i\sigma_i\,, \tag{D5}$$

which we may use in concrete calculations.

A well-known example is the momentum-spin basis [20] with $\vec{n}=\vec{e}_3$ and

$$\xi_{\frac12} = \begin{pmatrix}1\\0\end{pmatrix},\quad \xi_{-\frac12} = \begin{pmatrix}0\\1\end{pmatrix}, \tag{D6}$$

which is widely used in applications.

The only peculiar polarization used so far is the helicity giving rise to the momentum-helicity basis in which the spinors $\xi_\sigma(\vec{n}_p)$ have the forms (D4) with $\vec{n}=\vec{n}_p=\frac{\vec{p}}{p}$. To write the spin components (193) in this basis, we derive the matrices (116) that read [18]

$$\Sigma_1(\vec{p}) = \frac{p^1}{p}\sigma_3 - p^1\frac{p^1\sigma_1 + p^2\sigma_2}{p(p+p^3)} + \sigma_1\,,$$
$$\Sigma_2(\vec{p}) = \frac{p^2}{p}\sigma_3 - p^2\frac{p^1\sigma_1 + p^2\sigma_2}{p(p+p^3)} + \sigma_2\,, \tag{D7}$$
$$\Sigma_3(\vec{p}) = \frac{p^3}{p}\sigma_3 - \frac{p^1\sigma_1 + p^2\sigma_2}{p}\,,$$

verifying that these satisfy

$$p^i\Sigma_i(\vec{p}) = p\sigma_3\,. \tag{D8}$$

The form of the covariant derivatives $\tilde{\partial}_i = \partial_{p^i}1_{2\times2} + \Omega_i(\vec{p})$ is determined by the matrices (125) [18],

$$\Omega_1(\vec{p}) = \frac{-\mathrm{i}}{2p^2(p+p^3)}\Big[p^1p^2\sigma_1 + pp^2\sigma_3 + (pp^3 + {p^2}^2 + {p^3}^2)\sigma_2\Big]\,,$$
$$\Omega_2(\vec{p}) = \frac{\mathrm{i}}{2p^2(p+p^3)}\Big[p^1p^2\sigma_2 + pp^1\sigma_3 + (pp^3 + {p^1}^2 + {p^3}^2)\sigma_1\Big]\,, \tag{D9}$$
$$\Omega_3(\vec{p}) = \frac{\mathrm{i}}{2p^2}\left(p^1\sigma_2 - p^2\sigma_1\right)\,,$$

satisfying $p^i\Omega_i(\vec{p}) = 0$. Thus, we obtain apparently complicated matrices Σ_i and Ω_i but whose algebra is the same as in the momentum-spin basis where $\Omega_i = 0$ and $\Sigma_i = \sigma_i$.

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