Molecular components in *J/ψ* and *ρ-π* puzzle[*]

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Abstract: Motivated by the large branching fractions of $J/\psi \to f_0(1710)\omega / f_0(1710)\phi$ and the light exotic candidates, we found that there may exist molecular states composed of $f_0(1710)\omega$ and $f_0(1710)\phi$, which correspond to $X(2440)$ and $X(2680)$ observed a few decades ago. The branching fractions of $X(2440)$ and $X(2680)$ to various PV and $KK\omega(\phi)$ channels were estimated in the molecular scenario. In addition, the large branching fractions of $J/\psi \to f_0(1710)\omega / f_0(1710)\phi$ indicate sizable molecular components in the J/ψ state. Thus, we consider J/ψ as the superposition of the $c\bar{c}(1S)$, $f_0(1710)\omega$, and $f_0(1710)\phi$ molecular states. These molecular components have a significant impact on the light hadron decays of J/ψ, which may elucidate the long standing $\rho - \pi$ puzzle.

Keywords: ρ-π puzzle, J/ψ, molecular state

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I. INTRODUCTION

The OZI suppressed light hadron pair decays of vector charmonia occur via annihilation of three gluons, whereas the dilepton decays occur via a virtual photon. An estimation based on perturbative QCD yields

$$R = \frac{\mathcal{B}(\psi' \to h)}{\mathcal{B}(J/\psi \to h)} = \frac{\mathcal{B}(\psi' \to \ell^+\ell^-)}{\mathcal{B}(J/\psi \to \ell^+\ell^-)} \sim 12\%, \qquad (1)$$

which is referred to as the "12% rule". Severe violation of the "12% rule" was first observed in the $\rho\pi$ channel, which was measured to be $(0.19 \pm 0.08)\%$ by Mark II Collaboration in 1983 [1]; subsequently, it was observed in more channels. This anomalous phenomenon is named "$\rho - \pi$ puzzle." In Table 1, we list the measured branching fractions of $J/\psi \to VP$ and $\psi' \to VP$ along with their ratios.

To solve the "$\rho - \pi$ puzzle", various schemes have been proposed. In essence, there are two different major methods to address the discrepancy between experimental measurements and the "12% rule" expectation: introducing some additional mechanisms in the decays of either ψ' or J/ψ. For example, in Ref. [2], the author suggested that ψ' is a $2S - 1D$ mixing state rather than a pure $2S$ state. The destructive interference greatly suppresses the branching fractions of $\psi' \to \rho\pi$. This type of suppression could also result from the possible final state interaction [3−14]. The estimations in previous studies suggested that the branching fractions of light hadron decays of J/ψ are enhanced by some mechanisms. For example, Freund and Nambu [15] considered that J/ψ might mix with a 1^{--} glueball with a mass of $1.4 \sim 1.8$ GeV that could also transit into $\rho\pi$. Thus, the rate of $J/\psi \to \rho\pi$ could be enhanced by constructive interference [16−24].

As an important light hadron production platform, the J/ψ decays exhibit anomalous phenomena besides the "$\rho - \pi$ puzzle". For example, the branching ratio of $J/\psi \to f_0(1710)\phi \to K\bar{K}\phi$ was measured to be $(3.6\pm 0.6)\times10^{-4}$ [25], which is higher than that of $J/\psi \to f_0(980)\phi$, which is $(3.2\pm0.9)\times10^{-4}$ [26]. Similarly, the branching ratio for $J/\psi \to f_0(1710)\omega \to K\bar{K}\omega$ is also greater than that of $J/\psi \to f_0(980)\omega$, which are $(4.8\pm$

Table 1. Measured branching fractions of $J/\psi \to PV$ and $\psi' \to PV$, where P and V refer to pseudoscalar and vector light mesons, respectively [26].

Channel	Branch ratio	Channel	Branch ratio	Ratio
$J/\psi \to \rho\pi$	$(1.69 \pm 0.15) \times 10^{-2}$	$\psi' \to \rho\pi$	$(3.2 \pm 1.2) \times 10^{-5}$	$(0.19 \pm 0.08)\%$
$J/\psi \to \rho^0\pi^0$	$(5.6 \pm 0.7) \times 10^{-3}$	$\psi' \to \rho^0\pi^0$		
$J/\psi \to K^{*+}\bar{K}^- + c.c.$	$(6.0^{+0.8}_{-1.0}) \times 10^{-3}$	$\psi' \to K^{*+}\bar{K}^- + c.c.$	$(2.9 \pm 0.4) \times 10^{-5}$	$(0.48 \pm 0.10)\%$
$J/\psi \to K^{*0}\bar{K}^0 + c.c.$	$(4.2 \pm 0.4) \times 10^{-3}$	$\psi' \to K^{*0}\bar{K}^0 + c.c.$	$(1.09 \pm 0.20) \times 10^{-4}$	$(2.60 \pm 0.54)\%$
$J/\psi \to \omega\eta$	$(1.74 \pm 0.20) \times 10^{-3}$	$\psi' \to \omega\eta$	$< 1.1 \times 10^{-5}$	$< (0.63 \pm 0.07)\%$
$J/\psi \to \phi\eta$	$(7.4 \pm 0.8) \times 10^{-4}$	$\psi' \to \phi\eta$	$(3.10 \pm 0.31) \times 10^{-5}$	$(4.19 \pm 0.62)\%$
$J/\psi \to \phi\eta'$	$(4.6 \pm 0.5) \times 10^{-4}$	$\psi' \to \phi\eta'$	$(1.54 \pm 0.20) \times 10^{-5}$	$(3.35 \pm 0.57)\%$
$J/\psi \to \omega\pi$	$(4.5 \pm 0.5) \times 10^{-4}$	$\psi' \to \omega\pi^0$	$(2.1 \pm 0.6) \times 10^{-5}$	$(4.67 \pm 1.43)\%$
$J/\psi \to \rho\eta$	$(1.93 \pm 0.23) \times 10^{-4}$	$\psi' \to \rho\eta$	$(2.2 \pm 0.6) \times 10^{-5}$	$(11.40 \pm 3.40)\%$
$J/\psi \to \phi\pi$	3×10^{-6}	$\psi' \to \phi\pi^0$	$< 4 \times 10^{-7}$	$< 13.33\%$
$J/\psi \to \omega\eta'$	$(1.89 \pm 0.18) \times 10^{-4}$	$\psi' \to \omega\eta'$	$3.2^{+2.5}_{-2.1} \times 10^{-5}$	$(16.93 \pm 13.33)\%$
$J/\psi \to \rho\eta'$	$(8.1 \pm 0.8) \times 10^{-5}$	$\psi' \to \rho\eta'$	$1.9^{+1.7}_{-1.2} \times 10^{-5}$	$(23.46 \pm 21.12)\%$

$1.1) \times 10^{-4}$ [25] and $(1.4 \pm 0.5) \times 10^{-4}$ [27], respectively. Generally, the branching ratios for processes involving higher excited states are smaller than those only involving ground states in the J/ψ decays owing to the effect of nodes and a smaller phase space for the former processes. Thus, larger $f_0(1710)\omega$ and $f_0(1710)\phi$ branching ratios indicate anomalous strong coupling between J/ψ and $f_0(1710)\omega/f_0(1710)\phi$. Additionally, the thresholds of $f_0(1710)\omega$ and $f_0(1710)\phi$ are 2487 MeV and 2723 MeV, respectively. In the vicinity of the $f_0(1710)\omega$ threshold, a resonance $X(2440)$ with $M = 2440 \pm 10$ MeV and $\Gamma = 310 \pm 20$ MeV [28] was reported. Recently, a resonance state around 2.4 GeV was observed in $\pi^+\pi^-\phi$ and $f_0(980)\phi$ invariant mass spectra with quantum number 1^{--} [29−34]. We hypothesize that this state may be $X(2440)$ observed decades ago and consider it a molecular state in this study. In the vicinity of the $f_0(1710)\phi$ threshold, a resonance $X(2680)$ with $M = 2676 \pm 27$ MeV, $\Gamma = 150$ MeV [35] was reported. Therefore, these two states could be molecular candidates of $f_0(1710)\omega$ and $f_0(1710)\phi$, respectively.

Additionally, Table 1 shows that the branching ratios of $\psi(2S)$ to various PV final states are in the ranges of $(1.5 \sim 5.5) \times 10^{-5}$ (expect $\psi(2S) \to \phi\pi$). The branching ratios of J/ψ to various PV final states vary from 5.6×10^{-3} to 8.1×10^{-5} (expect $J/\psi \to \phi\pi$). Recently, $\mathcal{B}(\psi(3686) \to \phi K_S^0 K_S^0)/\mathcal{B}(J/\psi \to \phi K_S^0 K_S^0) = 6.0 \pm 1.6\%$ was experimentally measured [36]. This ratio was also suppressed in relation to the 12 % rule. There may be some type of mechanistic effect in the decays of J/ψ to PV. Thus, we suppose that the experimentally observed state J/ψ may contain extremely small molecular state components, which only affect some hadronic decay channels of J/ψ and not others such as leptonic decays. In this scenario, we expect that a mixing scheme for J/ψ could elucidate the long standing "$\rho - \pi$ puzzle".

In this paper, we propose that the experimentally observed J/ψ is a mixture of $c\bar{c}$ and hadronic molecules $X_{f_0(1710)\omega}$ and $X_{f_0(1710)\phi}$ (hereinafter denoted as $X_{f_0\omega}$ and $X_{f_0\phi}$, respectively). In the J/ψ light hadron decays, the molecular components break down into on-shell $f_0(1710)$ and $\phi(\omega)$, which then transition into light hadron pairs by exchanging an appropriate light hadron. It is important to note that the exchanged light hadron is also on-shell, potentially increasing the rescattering contributions. Consequently, although the proportion of hadronic molecules $X_{f_0\phi}$ and $X_{f_0\omega}$ in J/ψ state may be small, they still have a significant impact on the light hadron decays.

The rest of this paper is organized as follows. After the introduction, we analyze the mixing between $c\bar{c}$ and hadronic molecules in Sec. II. Then, we estimate the fractions of $X_{f_0\phi}$ and $X_{f_0\omega}$ components in J/ψ in Sec. III. In Sec. IV, we calculate branching fractions of $J/\psi \to VP$ considering the components $X_{f_0\phi}$ and $X_{f_0\omega}$. The last section presents a short summary.

II. SPECTRA OF $c\bar{c}$ BOUND STATE, MOLECULES, AND THEIR MIXING

Based on the ansatz that the physical states $X(2440)$, $X(2680)$, and J/ψ are the mixtures of the molecular states $X_{f_0\omega}$, $X_{f_0\phi}$ and the charmonium $c\bar{c}(1S)$ through a unitary matrix U transformation, we have the following relation:

$$\begin{pmatrix} |X(2440)\rangle \\ |X(2680)\rangle \\ |J/\psi\rangle \end{pmatrix} = U \begin{pmatrix} |X_{f_0\omega}\rangle \\ |X_{f_0\phi}\rangle \\ |c\bar{c}(1S)\rangle \end{pmatrix} \quad (2)$$

where U is a unitary matrix with compact form as

$$U = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}. \tag{3}$$

Thus, we have

$$|J/\psi\rangle = c_{31}|X_{f_0\omega}\rangle + c_{32}|X_{f_0\phi}\rangle + c_{33}|c\bar{c}(1S)\rangle. \tag{4}$$

The mass of ψ' notably differs from that of the molecular states $X_{f_0\omega}$ and $X_{f_0\phi}$. Thus, the mixing between $c\bar{c}(2S)$ and the molecular states should be dismissible. One can suppose that the magnitudes of $c\bar{c}(1S) \to PV$ and $c\bar{c}(2S) \to PV$ satisfy the "12% rule", while the violations at least partly result from the mixing of the charmonium and light meson-meson molecular states. To estimate the mixing between the charmonium and the molecular states, we first investigated the spectra of charmonium and molecular states individually.

A. Spectrum of $c\bar{c}$ bound state

According to Ref. [37], the mass spectra of charmonia can be obtained by solving Schrödinger equation with one-gluon-exchange plus a linear confinement potential which manifests the non-perturbative QCD effects. The Hamiltonian for 1^{--} states is

$$H = H_0 + H' \tag{5}$$

with

$$H_0 = \sum_{i=1}^{2}\left(m_i + \frac{p^2}{2m_i}\right) + \frac{-4}{3}\frac{\alpha_s(r)}{r} + \kappa r + c \tag{6}$$

and

$$H' = \frac{1}{4}\frac{32\pi}{9m_1m_2}\alpha_s(r)\left(\frac{\sigma}{\sqrt{\pi}}\right)^3 e^{-\sigma^2 r^2}, \tag{7}$$

where $\kappa = 0.18$ GeV^2, $\sigma = 3.0996$ GeV, and $m_c = 1.628$ GeV [37]. The constant c denotes zero-point energy set by fitting experimental data. From the aforementioned Hamiltonian, the mass of the $1S$ and $2S$ states can be obtained, while their mass difference is independent of the constant c. By setting $\psi(2S)$ to be a pure $c\bar{c}(2S)$ state, i.e., $m_{c\bar{c}(2S)} = (m_{\psi(2S)})_{\text{exp}}$, one has $c = (448.93 \pm 0.06)$ MeV. For this value of c, $m_{c\bar{c}(1S)} = (3088.19 \pm 0.06)$ MeV, which is approximately 10 MeV below the PDG average [26]. Additionally, the masses of the ground and first excited states of $c\bar{c}(0^-)$ are $m_{\eta_c} = 3022.02$ MeV and $m_{\eta_c(2S)} = 3641.90$ MeV, respectively.

B. Masses of light meson-meson bound states $X_{f_0\omega}$ and $X_{f_0\phi}$

The mass spectra of the light meson-meson bound states can also be estimated by solving the non-relativistic Schrödinger equation, where the potentials between the meson components could be induced by one-boson exchange, as shown in Fig. 1 [38, 39]. In Refs. [40, 41], the authors considered σ and $f_0(980)$ as mixture of $(u\bar{u}+d\bar{d})/\sqrt{2}$ and $s\bar{s}$. Thus, we consider that σ can couple with $f_0(1710)$ and ϕ. Taking the $f_0(1710)\phi$ system as an example, one can transform the scattering amplitude of the elastic scattering process in the momentum space into a non-relativistic effective potential in the coordinate space. To obtain the amplitude corresponding to the diagram in Fig. 1, the following effective Lagrangians are employed:

$$\mathcal{L}_{f_0f_0\sigma} = g_\sigma m_{f_0}\langle f_0 f_0 \sigma\rangle, \tag{8}$$

$$\mathcal{L}_{\phi\phi\sigma} = g_\sigma m_\phi\langle \phi\phi\sigma\rangle. \tag{9}$$

Unlike hadronic scattering, where incoming and outgoing mesons are on their mass shells, the ingredient mesons in the bound state are off-shell; thus, a form factor which partly compensates the off-shell effect is introduced as follows:

$$\mathcal{F}(q^2, m_\sigma^2) = \frac{\Lambda^2 - m_\sigma^2}{\Lambda^2 - q^2}, \tag{10}$$

where m_σ and q are the mass and four momenta of the exchanged σ meson, respectively. The resultant effective potential reads as [38]

$$V(\Lambda, m_\sigma, r) = -g_\sigma^2\left[\frac{1}{4\pi r}\left(e^{-m_\sigma r} - e^{-\Lambda r}\right) - \frac{\Lambda^2 - m_\sigma^2}{8\pi\Lambda}e^{-\Lambda r}\right]. \tag{11}$$

According to Ref. [38], the relations $g_\sigma = 2/3 g_{\sigma NN}$ and $g_{\sigma NN}^2/4\pi = 5.69$ are assumed. However, note that in Ref. [38] the mesons in the corresponding effective Lagrangians are all ground states, namely $f_0(980)$. However, $f_0(1710)$ is a higher excited state in the f_0 family. For σ exchanging, the interaction between heavy and light
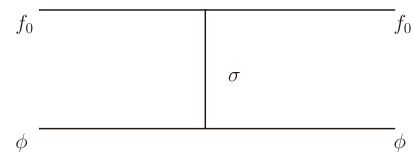

Fig. 1. Feynman diagram for $f_0(1710)\phi$ scattering by exchanging a σ meson.

quarks is ignored, in general; in other words, only the contributions of light quarks are considered [42]. Thus, for meson scattering, the effective coupling constant is 2/3 times smaller than that from baryons. This is suitable for $f_0(1710)$ as well. Moreover, according to Refs. [43, 44], we have $g^2/4\pi = 3.20$, 4.45, and 0.085 for three $N^*(1440)N\sigma$, $N^*(1680)N\sigma$, and $N^*(1710)N\sigma$ vertices, respectively. Thus, to obtain an estimate, we adopt $g_\sigma(f_0(1710)) \sim g_\sigma(f_0(980))$ here. According to Ref. [39], we set $\Lambda = 1.5 \sim 2.0$ GeV.

After setting $m_{f_0(1710)} = 1704$ MeV and $m_\phi = 1019$ MeV, we obtain $m_{X_{f_0\phi}} = 2647 \sim 2701$ MeV. Adopting a similar method, we obtain $m_{X_{f_0\omega}} = 2440 \sim 2477$ MeV.

C.   Mixing of charmonium and molecular states

In the present estimation, we impose the following conditions for the unitray matrix as mandatory conditions: the determinant of the matrix must be unity, and all the matrix elements must be real. According to Eq. (2), the unitary matrix U transforms the unphysical states $|X_{f_0\omega}\rangle$, $|X_{f_0\phi}\rangle$ and $|c\bar{c}(1S)\rangle$ into the physical eigenstates $|X(2440)\rangle$, $|X(2680)\rangle$ and $|J/\psi\rangle$ and simultaneously diag-

onalizes the mass matrix $\tilde{M}_q$ as

$$M_{\text{mass}} = U\tilde{M}_q U^\dagger \tag{12}$$

with

$$M_{\text{mass}} = \begin{pmatrix} m_{X(2440)} & 0 & 0 \\ 0 & m_{X(2680)} & 0 \\ 0 & 0 & m_{J/\psi} \end{pmatrix}, \tag{13}$$

and

$$\tilde{M}_q = \begin{pmatrix} m_{X_{f_0\omega}} & \lambda_1 & \lambda_2 \\ \lambda_1 & m_{X_{f_0\phi}} & \lambda_3 \\ \lambda_2 & \lambda_3 & m_{c\bar{c}(1S)} \end{pmatrix}. \tag{14}$$

Namely, $m_{X(2440)}$, $m_{X(2680)}$, and $m_{J/\psi}$ are the three roots of equation,

$$m^3 - m^2 \left(m_{c\bar{c}(1S)} + m_{X_{f_0\phi}} + m_{X_{f_0\omega}} \right) + m \left(m_{c\bar{c}(1S)}m_{X_{f_0\omega}} + m_{c\bar{c}(1S)}m_{X_{f_0\phi}} + m_{X_{f_0\omega}}m_{X_{f_0\phi}} - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 \right)$$
$$+ \left(\lambda_1^2 m_{c\bar{c}(1S)} + \lambda_2^2 m_{X_{f_0\phi}} + \lambda_3^2 m_{X_{f_0\omega}} - 2\lambda_1\lambda_2\lambda_3 - m_{X_{f_0\omega}}m_{X_{f_0\phi}}m_{c\bar{c}(1S)} \right) = 0. \tag{15}$$

Generally, we have three unknown variables in the Hermitian matrix $\tilde{M}_q$, namely λ_1, λ_2, and λ_3. While there are three independent equations by which we set these three unknown variables. In principle, we could simultaneously set the values of three non-diagonal matrix elements by setting the physical masses of $m_{X(2440)}$, $m_{X(2680)}$, and $m_{J/\psi}$ as the eigen-values of the mass matrix. However, note that the secular equation cannot be solved in a normal manner. Thus, we adopt an alternative meth-

od to obtain the ranges of three non-diagonal matrix elements. We pre-determine the ranges of the elements of the unitary transformation matrix U that diagonalizes the mass matrix $\tilde{M}_q$ and then substitute them into the secular equation to check whether the equation can be satisfied, that is, whether all the requirements (unitarity, *etc.*) are fulfilled. Then, we can obtain the unitary transformation matrix U and mass matrix $\tilde{M}_q$ as

$$U = \begin{pmatrix} -0.990 \sim -0.985 & -0.147 \sim -0.049 & -0.099 \sim 0.141 \\ 0.056 \sim 0.159 & -0.976 \sim -0.997 & 0.050 \sim 0.150 \\ 0.061 \sim 0.133 & 0.057 \sim 0.161 & 0.985 \sim 0.990 \end{pmatrix} \tag{16}$$

and

$$M_q = \begin{pmatrix} 2438 \sim 2463 & -33 \sim -7 & 45 \sim 86 \\ -33 \sim -7 & 2645 \sim 2708 & 24 \sim 74 \\ 45 \sim 86 & 24 \sim 51 & 3079 \sim 3087 \end{pmatrix} \text{ MeV.} \tag{17}$$

From these two matrices, the pre-determined ranges of the mass of $X_{f_0\omega}$ and $X_{f_0\phi}$ are $(2438 \sim 2463)$ MeV and $(2645 \sim 2708)$ MeV. respectively, which are very close to those obtained by potential model estimations. Moreover, the value of $m_{c\bar{c}(1S)}$ is fitted to be $(3079 \sim 3087)$ MeV, which is also very close to the one obtained from the quark model, which is (3088.19 ± 0.06) MeV.

III. CONTRIBUTIONS OF MOLECULAR COMPONENTS TO PV CHANNELS

Besides the contributions from $c\bar{c}$ annihilation, the molecular components in J/ψ should also have an important effect on the decay $J/\psi \to PV$. For the molecular state, its components transit into light pseoduscalar and vector mesons exchanging a proper light meson. This meson loop connects the molecular and final states. Taking $|f_0\phi\rangle$ as an example, the Feynman diagrams contributing to the decays $X_{f_0\phi} \to PV$ are presented in Fig. 2. In the present study, these diagram were estimated at the hadron level, and the relevant vertices read [38]

$$\langle X_{f_0V}|f_0V\rangle = g_X\,\Lambda_0\,X_{f_0V}Vf_0,$$
$$\langle f_0|PP\rangle = g_{f_0}m_P\,\mathrm{Tr}[PP]f_0,$$
$$\langle f_0|VV\rangle = g_{f_0}m_V\,\mathrm{Tr}[VV]f_0,$$
$$\langle \phi|PV\rangle = \frac{G}{\sqrt{2}}\,\varepsilon^{\lambda\nu\alpha\beta}\mathrm{Tr}[\partial_\lambda V_\nu \partial_\alpha V_\beta P], \qquad (18)$$

$\Lambda_0 = 1$ GeV is a dimensional parameter to ensure that the effective coupling constant g_X is dimensionless; m_V and m_P denote the masses of vector and pseudoscalar mesons, respectively. The matrix form of the psedoscalar and vector mesons are

$$P = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} + \frac{\eta_1}{\sqrt{3}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} + \frac{\eta_1}{\sqrt{3}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} + \frac{\eta_1}{\sqrt{3}} \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} + \frac{\omega_1}{\sqrt{3}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} + \frac{\omega_1}{\sqrt{3}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & -\frac{2\omega_8}{\sqrt{6}} + \frac{\omega_1}{\sqrt{3}} \end{pmatrix}.$$

with mixing parameters

$$\eta_8 = \eta\cos\theta + \eta'\sin\theta, \quad \eta_1 = -\eta\sin\theta + \eta'\cos\theta,$$
$$\omega_8 = \omega\cos\varphi + \phi\sin\varphi, \quad \omega_1 = -\omega\sin\varphi + \phi\cos\varphi, \qquad (19)$$

and mixing angles θ and φ fulfilling $\sin\theta = -0.31 \pm 0.11$ [26] and $\sin\varphi = -0.76$ [45].

According to the above vertices, we can obtain the amplitudes corresponding to diagrams (a) and (b) in Fig. 2,
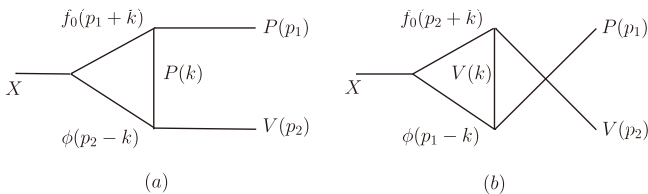
$$\mathcal{M}_a = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{g_{f_0}m_P}{(p_1+k)^2 - m_{f_0}^2} \frac{g_X g^{\mu\nu}\varepsilon_X^\mu}{(p_2-k)^2 - m_\phi^2}$$
$$\times \frac{g_{\phi PV}\varepsilon^{\lambda\nu\alpha\beta}(p_2-k)^\lambda p_2^\alpha \varepsilon_V^{*\beta}}{k^2 - m_P^2}$$

$$\mathcal{M}_b = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{g_{\phi PV}\varepsilon^{\lambda\nu\alpha\sigma}(p_1-k)^\lambda k^\alpha}{(p_1-k)^2 - m_\phi^2} \frac{g_X g^{\mu\nu}\varepsilon_X^\mu}{(p_2+k)^2 - m_{f_0}^2}$$
$$\times \frac{g_{f_0}m_V g^{\sigma\beta}\varepsilon_V^{*\beta}}{k^2 - m_V^2}, \qquad (20)$$

respectively.

IV. NUMERICAL RESULTS AND DISCUSSIONS

A. Coupling Constants

Considering the $SU(3)$ symmetry, the coupling constants in the hadron vertex in Eq. (18) satisfy $G = (3g^2)/(4\pi^2 f_\pi)$, $f_\pi = 93$ MeV, and $g = (12)/(2\sqrt{2})$ [38]. In the mixing scenario, $X(2240)$ and $X(2680)$ are molecular

Fig. 2. Feynman diagrams contributing to $X_{f_0\phi} \to PV$.

states with a very small $c\bar{c}(1S)$ component, and the decays of $c\bar{c}$ components into light PV are further suppressed by the OZI suppression rule. Thus, we estimate $X(2240)/X(2680) \to PV$; the $c\bar{c}(1S)$ components can be ignored and $X(2240)/X(2680)$ can be considered as a pure $f_0(1710)\omega/f_0(1710)\phi$ molecular state. In the molecular scenario, the coupling constant of the molecular state and its components can be estimated as follows [46, 47]:

$$g_X^2 = \frac{16\pi(m_{f_0(1710)} + m_V)^2}{\mu} \sqrt{2\mu E_B} \frac{1}{\Lambda_0^2}. \tag{21}$$

where $\mu = (m_{f_0(1710)} m_V)/(m_{f_0(1710)} + m_V)$ and $E_B = m_{f_0(1710)} + m_V - m_X$. Setting $m_{f_0(1710)} = 1704$ MeV, $m_\omega = 783$ MeV, $m_\phi = 1019$ MeV, $m_{X_{f_0\omega}} = (2440 \sim 2477)$ MeV and $m_{X_{f_0\phi}} = (2647 \sim 2701)$ MeV, we obtain $g_{X_{f_0\omega}} = 9.6 \pm 1.8$ and $g_{X_{f_0\phi}} = 11.7 \pm 1.8$.

Concerning the value of the coupling constant g_{f_0}, reproducing the experimental data $\Gamma(f_0(1710)) = 123$ MeV [26] and $\mathcal{B}(f_0(1710) \to KK) = 0.38 \pm 0.19$ [48], $\mathcal{B}(f_0(1710) \to \eta\eta) = 0.22 \pm 0.12$ [49] and $\mathcal{B}(f_0(1710) \to \pi\pi) = 0.039 \pm 0.024$ [48], we have $g_{f_0} = 2.24 \pm 0.56$, $g_{f_0} = 3.17 \pm 0.86$, and $g_{f_0} = 4.68 \pm 1.44$, respectively. As a matter of fact [50], the three experimental values were not reliably measured, and the values of g_{f_0} obtained from different experimental data are not consistent with each other. Thus, in our calculations, we considered g_{f_0} as a free parameter for the first step.

It should be noted that some channels in Table 1 are isospin violated. Thus, in the present study, some isospin violated vertices were also considered. From the branching fraction $\mathcal{B}(\phi \to \omega\pi)_{\exp} = (4.7 \pm 0.5) \times 10^{-5}$, we obtain $g_{\phi\omega\pi} = 0.04$. Concerning the other involved isospin violation vertices, for example $g_{\phi\rho\eta}$, $g_{\phi\rho\eta'}$, $g_{\omega\rho\eta}$, $g_{\omega\rho\eta'}$ and $g_{\omega\omega\pi}$, given the lack of corresponding experimental data, we consider that they are of the same order as $g_{\phi\omega\pi}$ and set $g_{\phi\rho\eta} = g_{\phi\rho\eta'} = g_{\omega\rho\eta} = g_{\omega\rho\eta'} = g_{\omega\omega\pi} = 0.04$ in the present calculations.

B.   Branching fractions of $X(2240)/X(2680) \to PV$

As we discussed in the previous subsection, the coupling constant g_{f_0} is considered as a free parameter for the first step. The partial width of $X(2440) \to VP$ satisfies

$$\Gamma(X(2440) \to VP)_{\text{Theory}} = \frac{1}{2 m_{X(2440)}} \int d\Pi_2$$

$$|c_{11} \mathcal{M}(X_{f_0\omega} \to VP)$$

$$+ c_{12} \mathcal{M}(X_{f_0\phi} \to VP)$$

$$+ c_{13} \mathcal{M}(c\bar{c}(1S) \to VP)|^2, \tag{22}$$

with the amplitude for $c\bar{c}(1S) \to PV$ being

$$\mathcal{M}(c\bar{c}(1S) \to VP) = g_{c\bar{c}PV} \varepsilon^{\lambda\nu\alpha\beta} p_\psi^\lambda \varepsilon_\psi^\nu p_V^\alpha \varepsilon_V^{*\beta}. \tag{23}$$

The constants $g_{c\bar{c}PV}$ are obtained by fitting $\mathcal{B}(c\bar{c}(1S) \to PV)_{\text{Exp}}$ data with the relation

$$\mathcal{B}(c\bar{c}(1S) \to PV)_{\text{Exp}} = \mathcal{B}(\psi' \to PV)_{\text{Exp}}/(13.3\% \times c_{33}^2), \tag{24}$$

where 13.3 % comes from $\mathcal{B}(\psi' \to l^+l^-)_{\text{Exp}}/\mathcal{B}(J/\psi \to l^+l^-)_{\text{Exp}}$ with $l = e, \mu$. Table 2 lists the corresponding data and values. The partial width of $X(2680) \to VP$ is similar, and we omit it for simplification.

Setting $\Gamma_{X(2440)} = 310$ MeV and $\Gamma_{X(2680)} = 150$ MeV [26], one can obtain the branching fractions of $X(2240)/X(2680) \to PV$. Next, we briefly discuss the effective coupling constant g_{f_0}. Although its value is not well measured, its approximate range is $2 \sim 4$. Here, we set $g_{f_0} = 2$ and 3 as examples and list the estimated results in Table 3.

For $X(2440)$, our estimations indicate that the branching ratio of $X(2440) \to \rho\pi^0$ is up to the order of 10^{-2}, and the one for $X(2440) \to K^{*0}\bar{K}^0$ is approximately one order of magnitude smaller than that of $X(2440) \to \rho\pi^0$. The branching fractions of the other seven PV channel are even smaller. Regarding $X(2680)$, the branching fractions of $K^*\bar{K}^0$ channel are of the order of 10^{-2}, which is approximately one order of magnitude larger than that of $X(2680) \to \rho\pi^0$.

Besides the two body PV decay process, the branching ratios of the three body decay processes, that is, $X(2440) \to \omega K\bar{K}$ and $X(2680) \to \phi K\bar{K}$, are also estimated, where $K\bar{K}$ are the daughter particles of $f_0(1710)$. Our estimations indicate that the branching fractions of these three body decay processes are of the order of 10^{-1}. To date, the decay properties of $X(2440)$ and $X(2680)$ have been poorly measured; we hope our results shown in Table 3 can be confirmed by future measurements from BESIII, BELLE, LHCb, and probably the future charm-tau factory (FCTF).

C.   Branching fractions of $J/\psi \to PV$

In the mixing scheme, the physical J/ψ state is the

Table 2.   Experimental data of J/ψ and ψ' to lepton pair [26].

Decay channel	Branch ratio	Decay channel	Branch ratio	Ratio
$J/\psi \to e^+e^-$	$(5.971 \pm 0.032) \times 10^{-2}$	$\psi' \to e^+e^-$	$(7.93 \pm 0.17) \times 10^{-3}$	$(13.28 \pm 0.29)\%$
$J/\psi \to \mu^+\mu^-$	$(5.961 \pm 0.033) \times 10^{-2}$	$\psi' \to \mu^+\mu^-$	$(8.0 \pm 0.6) \times 10^{-3}$	$(13.42 \pm 1.01)\%$

Table 3. Branching fractions of $X(2440)$ and $X(2680)$ decaying into a light pseudoscalar and vector meson, where we set $g_{f_0} = 2$ and 3 as examples. The branching fractions of the decay channels $X(2440) \to \omega K\bar{K}$ and $X(2680) \to \phi K\bar{K}$ are also listed. The uncertainties of the present estimations result from c_{ij}.

Channel	$\mathcal{B}(X(2440) \to PV)_{\text{theory}}$		$\mathcal{B}(X(2680) \to PV)_{\text{theory}}$	
	$g_{f_0} = 2$	$g_{f_0} = 3$	$g_{f_0} = 2$	$g_{f_0} = 3$
$\rho^0\pi^0$	$(7.97 \pm 0.14) \times 10^{-3}$	$(1.78 \pm 0.03) \times 10^{-2}$	$(1.45 \pm 0.73) \times 10^{-3}$	$(3.21 \pm 1.59) \times 10^{-3}$
$K^{*+}\bar{K}^-$	$(1.05 \pm 0.18) \times 10^{-3}$	$(2.31 \pm 0.40) \times 10^{-3}$	$(1.29 \pm 0.09) \times 10^{-2}$	$(2.88 \pm 0.19) \times 10^{-2}$
$K^{*0}\bar{K}^0$	$(1.13 \pm 0.20) \times 10^{-3}$	$(2.42 \pm 0.43) \times 10^{-3}$	$(1.31 \pm 0.10) \times 10^{-2}$	$(2.92 \pm 0.21) \times 10^{-2}$
$\omega\eta$	$(2.89 \pm 0.13) \times 10^{-4}$	$(6.22 \pm 0.25) \times 10^{-4}$	$(1.35 \pm 0.51) \times 10^{-4}$	$(2.85 \pm 1.04) \times 10^{-4}$
$\phi\eta$	$(4.85 \pm 1.38) \times 10^{-5}$	$(9.07 \pm 2.52) \times 10^{-5}$	$(9.16 \pm 0.87) \times 10^{-5}$	$(1.98 \pm 0.16) \times 10^{-3}$
$\phi\eta'$	$(2.15 \pm 0.83) \times 10^{-6}$	$(2.39 \pm 0.96) \times 10^{-6}$	$(1.19 \pm 0.40) \times 10^{-5}$	$(2.08 \pm 0.55) \times 10^{-5}$
$\omega\pi$	$(3.19 \pm 0.97) \times 10^{-6}$	$(3.68 \pm 1.07) \times 10^{-6}$	$(3.75 \pm 2.42) \times 10^{-6}$	$(4.85 \pm 2.87) \times 10^{-6}$
$\rho\eta$	$(2.62 \pm 0.87) \times 10^{-6}$	$(2.71 \pm 0.89) \times 10^{-6}$	$(2.48 \pm 1.87) \times 10^{-6}$	$(2.69 \pm 1.98) \times 10^{-6}$
$\omega\eta'$	$(1.41 \pm 0.26) \times 10^{-5}$	$(2.21 \pm 0.34) \times 10^{-5}$	$(1.13 \pm 0.74) \times 10^{-5}$	$(1.74 \pm 1.07) \times 10^{-5}$
$\rho\eta'$	$(2.16 \pm 0.73) \times 10^{-6}$	$(2.19 \pm 0.74) \times 10^{-6}$	$(1.95 \pm 1.52) \times 10^{-6}$	$(2.04 \pm 1.56) \times 10^{-6}$
$\omega K\bar{K}$	$(9.92 \pm 0.04) \times 10^{-2}$	$(2.23 \pm 0.01) \times 10^{-1}$		
$\phi K\bar{K}$			$(2.02 \pm 0.01) \times 10^{-1}$	$(4.55 \pm 0.02) \times 10^{-1}$

mixture of $|c\bar{c}(1S)\rangle$, $|X_{f_0\omega}\rangle$ and $|X_{f_0\phi}\rangle$ molecular states, as shown in Eq. (4). Considering the fact that the molecular state $X_{f_0\omega}/X_{f_0\phi}$ can decay into a light pseudoscalar and a light vector meson, as shown in the previous subsection, it can be concluded that the $X_{f_0\omega}/X_{f_0\phi}$ molecular components in the J/ψ state should also contribute to the processes $J/\psi \to PV$, and the partial widths of $J/\psi \to VP$ satisfy

$$
\begin{aligned}
\Gamma(J/\psi \to VP)_{\text{Theory}} = &\frac{1}{2\,m_{J/\psi}} \int \mathrm{d}\Pi_2 \\
&\times |c_{31}\,\mathcal{M}(X_{f_0\omega} \to VP) \\
&+ c_{32}\,\mathcal{M}(X_{f_0\phi} \to VP) \\
&+ c_{33}\,\mathcal{M}(c\bar{c}(1S) \to VP)|^2.
\end{aligned} \quad (25)
$$

Given that the mass of ψ' is notably larger than those of the $X_{f_0\omega}$ and $X_{f_0\phi}$ molecular states, ψ' could be considered as a pure $c\bar{c}(2S)$ charmonium state. Then, the partial width of $\psi' \to PV$ should be dominated by $c\bar{c}$ annihilation. In the present calculations, we estimate $\Gamma(c\bar{c}(1S) \to VP)$ using $\Gamma(\psi' \to VP)$ and the "12% rule". The terms of $\Gamma(X_{f_0\omega} \to VP)$ and $\Gamma(X_{f_0\phi} \to VP)$ are estimated in the same manner as the molecular decay. However, in the physical J/ψ state, $X_{f_0\omega}$ and $X_{f_0\phi}$ are off-shell. Thus, the coupling constants g_X estimated in Eq. (21) are not valid, and we set g_X as an undetermined parameter. Regarding the factors c_{31}, c_{32}, and c_{33}, they are determined by the unitary transformation matrix U in Eq. (16) as follows:

$$
\begin{aligned}
c_{31}^2 &= 0.0107 \pm 0.0070, \\
c_{32}^2 &= 0.0146 \pm 0.0114, \\
c_{33}^2 &= 0.9747 \pm 0.0050.
\end{aligned}
$$

For better understanding of the contributions of the different terms in Eq. (25), we have the following relations and definitions:

$$
\mathcal{B}(c\bar{c}(1S) \to PV)_{\text{Theory}} = \mathcal{B}(\psi' \to PV)_{\text{Exp}}/(13.3\% \times c_{33}^2),
$$

$$
\mathcal{B}(J/\psi \to PV)_{\text{Theory}} = \mathcal{B}(c\bar{c}(1S) \to PV)_{\text{Theory}} + \mathcal{B}(X \to PV)_{\text{Theory}} + \mathcal{B}(INT \to PV)_{\text{Theory}}, \quad (26)
$$

where INT denotes the interference terms that form molecular states with $c\bar{c}(1S)$. The estimated branching fractions defined above are listed in Table 4. In this table, the branching fraction of the process $\psi' \to \rho^0\pi^0$ is estimated by $\mathcal{B}(\psi' \to \rho^0\pi^0) = \mathcal{B}(\psi' \to \rho\pi)/3$. The decay processes $\psi' \to \phi\pi$ and $J/\psi \to \phi\pi$ are not included because the branching fractions of these two channels are extremely small. Concerning the decay channels $J/\psi(\psi') \to \rho\eta$, $J/\psi(\psi') \to \omega\eta'$, and $\psi' \to \rho\eta'$, according to Table 1, they satisfy the 12 % rule within the experimental error range. Thus, we estimate that the components of the $X \to \rho\eta$, $X \to \omega\eta'$, and $X \to \rho\eta'$ contributions in J/ψ are not larger than 10^{-5}. Therefore, our estimations in Table 4 are consistent with the expectations.

To better understand the contributions of molecular states to J/ψ hadronic decays, we define the ratio $\mathcal{R}$ as

$$\mathcal{R} = \frac{\mathcal{B}(J/\psi \to PV)_{\text{Theory}}}{\mathcal{B}(J/\psi \to PV)_{\text{Exp}}}. \tag{27}$$

Next, we briefly discuss the effective coupling constant g_X and g_{f_0}, which are different from those of the decay processes $X(2680) \to PV$ and $X(2440) \to PV$ in which the effective coupling constant g_X is evaluated through the wave function of the physical state. In the $J/\psi \to PV$ process, we set g_X as a free undetermined coupling constants for the off-shell effect. From the effective Lagrangian in Eq. (18), one has $\mathcal{B}(J/\psi \to f_0(1710)\phi)_{\text{theory}} = g_X^2 (6.92 \pm 5.40) \times 10^{-1}$ and $\mathcal{B}(J/\psi \to f_0(1710)\omega)_{\text{theory}} = g_X^2 (10.38 \pm 6.79) \times 10^{-1}$, respectively, with the uncertainties resulting from c_{ij}. In other words, the upper limit of

g_X^2 is of the order of 10^{-1}; thus, one has $g_X^2 g_{f_0}^2 \sim 1$.

The ratios $\mathcal{R}$ defined above are listed in Table 5, where $g_X g_{f_0}$ varies in the range of $0.5 \sim 1.5$. According to Table 5, the estimated branching fractions of $J/\psi \to \rho^0 \pi^0$, $J/\psi \to K^* \bar{K}$, $J/\psi \to \phi\eta$, $J/\psi \to \rho\eta$, $J/\psi \to \omega\eta'$ and $J/\psi \to \rho\eta'$ are consistent with the experimental expectations. The channels $J/\psi \to \omega\eta$ can only meet experimental expectations within 3σ error ranges. Regarding $J/\psi \to \phi\eta'$ and $J/\psi \to \omega\pi$, the theoretical values are several times smaller than the corresponding experimental expectations.

Note that the experimental data for $J/\psi \to f_0(1710)\phi \to \phi K \bar{K}$ and $J/\psi \to f_0(1710)\omega \to \omega K \bar{K}$ may have a confinement to $c_{ij} g_X g_{f_0}$ through the $f_0(1710) \to K\bar{K}$ process. The corresponding coupling constants can be fitted through the following relations and equations derived from Eq. (18):

$$\mathcal{M}(J/\psi \to f_0(1710)\phi \to \phi K \bar{K}) = 2g_{f_0} m_K \frac{1}{q^2 - m_{f_0}^2} \epsilon_\phi^{*\mu} g_{\mu\nu} \epsilon_X^\nu g_X \Lambda_0 \sqrt{3} \cos\varphi, \tag{28}$$

$$\Gamma(J/\psi \to f_0(1710)\phi \to \phi K \bar{K}) = \frac{1}{2 m_{J/\psi}} \int d\Pi_3 |c_{32} \mathcal{M}(J/\psi \to f_0(1710)\phi \to \phi K \bar{K})|^2. \tag{29}$$

By fitting $\mathcal{B}(J/\psi \to f_0(1710)\phi \to \phi K \bar{K})_{\text{Exp}} = (3.6 \pm 0.6) \times 10^{-4}$ [25], we obtain $g_X g_{f_0} = 0.09 \pm 0.02$. Likewise, by fitting $\mathcal{B}(J/\psi \to f_0(1710)\omega \to \omega K \bar{K})_{\text{Exp}} = (4.8 \pm 1.1) \times 10^{-4}$ [25] we have $g_X g_{f_0} = 0.08 \pm 0.02$. When taking the two values into the relation $\mathcal{B}(J/\psi \to PV)_{\text{Theory}}$, we obtain numerical results which are independent of c_{ij}, given that c_{ij}

and $g_X g_{f_0}$ exist in both Eqs. (25) and (29).

Finally, note that the values of $g_X g_{f_0}$ fitted from the experiments are much smaller. Although $g_X g_{f_0}$ was fitted using experimental data of the final state of three J/ψ hadrons, g_X and g_{f_0} are off shell values, and the running of those effective coupling constants is difficult to handle.

Table 4. Experimental values of the branching ratios of $J/\psi \to PV$ and $X \to PV$, where X stands for both $X_{f_0\omega}$ and $X_{f_0\phi}$. The uncertainties of $\mathcal{B}(X \to PV)_{\text{Theory}}$ and $\mathcal{B}(INT \to PV)_{\text{Theory}}$ result from c_{ij}. The uncertainties of $\mathcal{B}(c\bar{c}(1S) \to PV)_{\text{Theory}}$ result from experimental data through the "12% rule".

Channel	$\mathcal{B}(\psi' \to PV)_{\text{Exp}}$	$\mathcal{B}(J/\psi \to PV)_{\text{Exp}}$	$\mathcal{B}(c\bar{c}(1S) \to PV)_{\text{Theory}}$	$\mathcal{B}(X \to PV)_{\text{Theory}}/(g_{f_0}g_X)^2$	$\mathcal{B}(INT \to PV)_{\text{Theory}}/(g_{f_0}g_X)$
$\rho^0\pi^0$	$(1.1 \pm 0.4) \times 10^{-5}$	$(5.6 \pm 0.7) \times 10^{-3}$	$(7.82 \pm 2.93) \times 10^{-5}$	$(3.03 \pm 2.02) \times 10^{-3}$	$(9.89 \pm 3.76) \times 10^{-4}$
$K^{*+}\bar{K}^-$	$(1.5 \pm 0.2) \times 10^{-5}$	$(3.0 \pm 0.5) \times 10^{-3}$	$(1.06 \pm 0.15) \times 10^{-4}$	$(3.29 \pm 2.42) \times 10^{-3}$	$(1.21 \pm 0.53) \times 10^{-3}$
$K^{*0}\bar{K}^0$	$(5.5 \pm 1.0) \times 10^{-5}$	$(2.1 \pm 0.2) \times 10^{-3}$	$(3.99 \pm 0.73) \times 10^{-4}$	$(3.29 \pm 2.42) \times 10^{-3}$	$(2.34 \pm 1.02) \times 10^{-3}$
$\omega\eta$	$< 1.1 \times 10^{-5}$	$(1.74 \pm 0.20) \times 10^{-3}$	$< 8.06 \times 10^{-5}$	$(1.96 \pm 1.33) \times 10^{-4}$	$(2.52 \pm 0.98) \times 10^{-4}$
$\phi\eta$	$(3.10 \pm 0.31) \times 10^{-5}$	$(7.4 \pm 0.8) \times 10^{-4}$	$(2.27 \pm 0.23) \times 10^{-4}$	$(2.10 \pm 1.57) \times 10^{-4}$	$(4.40 \pm 1.98) \times 10^{-4}$
$\phi\eta'$	$(1.54 \pm 0.20) \times 10^{-5}$	$(4.6 \pm 0.5) \times 10^{-4}$	$(1.13 \pm 0.15) \times 10^{-4}$	$(2.26 \pm 1.74) \times 10^{-6}$	$(3.29 \pm 1.55) \times 10^{-5}$
$\omega\pi$	$(2.1 \pm 0.6) \times 10^{-5}$	$(4.5 \pm 0.5) \times 10^{-4}$	$(1.54 \pm 0.44) \times 10^{-4}$	$(8.18 \pm 5.88) \times 10^{-8}$	$(7.30 \pm 3.12) \times 10^{-6}$
$\rho\eta$	$(2.2 \pm 0.6) \times 10^{-5}$	$(1.93 \pm 0.23) \times 10^{-4}$	$(1.61 \pm 0.44) \times 10^{-4}$	$(6.54 \pm 4.71) \times 10^{-9}$	$(2.00 \pm 0.85) \times 10^{-6}$
$\omega\eta'$	$(3.1 \pm 2.5) \times 10^{-5}$	$(1.89 \pm 0.18) \times 10^{-4}$	$(2.35 \pm 1.83) \times 10^{-4}$	$(9.39 \pm 6.41) \times 10^{-6}$	$(9.60 \pm 3.13) \times 10^{-5}$
$\rho\eta'$	$(1.9 \pm 1.7) \times 10^{-5}$	$(8.1 \pm 0.8) \times 10^{-5}$	$(1.39 \pm 1.25) \times 10^{-4}$	$(2.41 \pm 1.72) \times 10^{-9}$	$(1.16 \pm 0.49) \times 10^{-6}$

Table 5. Values of $\mathcal{R}$ with $g_X g_{f_0}$ set to 0.5, 1.0, and 1.5 as examples.

Channel	$g_X g_{f_0} = 0.5$	$g_X g_{f_0} = 1.0$	$g_X g_{f_0} = 1.5$
$\rho^0 \pi^0$	0.24 ± 0.13	0.73 ± 0.44	1.50 ± 0.94
$K^{*+} \bar{K}^-$	0.52 ± 0.31	1.55 ± 1.03	3.11 ± 2.15
$K^{*0} \bar{K}^0$	1.14 ± 0.58	2.89 ± 1.71	5.39 ± 3.40
$\omega \eta$	0.15 ± 0.05	0.30 ± 0.14	0.52 ± 0.26
$\phi \eta$	0.68 ± 0.23	1.19 ± 0.53	1.84 ± 0.93
$\phi \eta'$	0.28 ± 0.06	0.32 ± 0.08	0.36 ± 0.10
$\omega \pi$	0.35 ± 0.11	0.36 ± 0.11	0.37 ± 0.12
$\rho \eta$	0.84 ± 0.25	0.85 ± 0.25	0.85 ± 0.26
$\omega \eta'$	1.51 ± 1.09	1.80 ± 1.21	2.11 ± 1.31
$\rho \eta'$	1.73 ± 1.55	1.73 ± 1.55	1.74 ± 1.56

It may affect the numerical values. This point have also been mentioned above, especially for g_X. Regarding the value of g_{f_0}, there are no experimental data, and more data about $f_0(1710) \to PP(VV)$ and $f_0(1710)$ hadronic decays are needed. Moreover, we expect that many more J/ψ hadronic and radiative decays containing the $f_0(1710)$ resonance peak will be experimentally observed and measured to fit the effective coupling constants. Finally, for the $\omega - \phi$ mixing, we assume $\sin\varphi = -0.76$, although its value is not well studied.

V. SUMMARY

Motivated by recent observations of a series of light exotic candidates and the anomalous large branching fraction of $J/\psi \to f_0(1710)\phi / f_0(1710)\omega$, we suppose that the experimentally observed resonances $X(2440)$ and $X(2680)$ are of the 1^{--} state and mainly composed of $X_{f_0\omega}$ and $X_{f_0\phi}$ molecular states, respectively. Meanwhile, the two molecular states $X_{f_0\omega}$ and $X_{f_0\phi}$ can mix with $c\bar{c}(1S)$ to form experimentally observed states $X(2440)$, $X(2680)$, and J/ψ. In the present study, we first evaluated the mass spectra of $c\bar{c}$ bound states with the Godfrey-Isgur model [37]. Then, the masses of two molecular states $X_{f_0\omega}$ and $X_{f_0\phi}$ were also evaluated within the framework of the OBE model [38, 39]. Finally, we investigated the mixing of the $c\bar{c}(1S)$ bound state with the molecular states $X_{f_0\omega}$ and $X_{f_0\phi}$ and obtained the mixing parameters by reproducing experimental data.

Moreover, we estimated the branching fraction of $X(2680)/X(2440) \to PV$ and the three body final states $K\bar{K}\phi(\omega)$. Our estimations indicate that the results strongly depend on the effective coupling constants g_{f_0} and g_X as well as on the $\omega - \phi$ mixing angle $\sin\varphi$. With a certain parameter range, the rate of some $J/\psi \to PV$ channels can be naturally understood with the present ansatz.

The experimental data for $\mathcal{B}(J/\psi \to f_0(1710)\phi \to \phi K\bar{K})$ [25] and $\mathcal{B}(J/\psi \to f_0(1710)\omega \to \omega K\bar{K})$ [25] were observed in the invariant mass spectra of $J/\psi \to \phi K\bar{K}$ and $J/\psi \to \omega K\bar{K}$. Therefore, we expect that the $f_0(1710)$ resonance will be observed in other three body hadronic decays of J/ψ, such as $\mathcal{B}(/\psi \to \phi(\omega)\pi\pi)$, $\mathcal{B}(J/\psi \to \phi(\omega)\eta\eta)$, $\mathcal{B}(J/\psi \to \phi(\omega)\rho\rho)$, and $\mathcal{B}(J/\psi \to \phi(\omega)\omega\omega)$.

Unfortunately, the measurement of the hadronic decays of the resonances still presents large uncertainties, which hinder setting definite values of the coupling constants. Therefore, it is still too early for drawing conclusions about the present ansatz. More precise measurements on those hadronic decays in future will provide crucial information to test the present ansatz.

ACKNOWLEDGMENTS

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