Neutrino oscillations in the Non-Kerr black hole with quantum phenomenon*

Husan Alibekov[1,2†] Farruh Atamurotov[3,4‡] Ahmadjon Abdujabbarov[1,5,6§] Vokhid Khamidov[6,7♮]

[1]Ulugh Beg Astronomical Institute, Astronomy St. 33, Tashkent 100052, Uzbekistan
[2]University of Tashkent for Applied Sciences, Str. Gavhar 1, Tashkent 100149, Uzbekistan
[3]New Uzbekistan University, Movarounnahr street 1, Tashkent 100000, Uzbekistan
[4]Tashkent State Technical University, Tashkent 100095, Uzbekistan
[5]Institute of Theoretical Physics, National University of Uzbekistan, Tashkent 100174, Uzbekistan
[6]Institute of Fundamental and Applied Research, National Research University TIIAME, Kori Niyoziy 39, Tashkent 100000, Uzbekistan
[7]Tashkent University of Information Technologies named after Muhammad al Khwarizmi, Amir Temur 108, Tashkent 100014, Uzbekistan

Abstract: In this study, we have investigated the mathematical components of the Dirac equation in curved spacetime and how they can be applied to the analysis of neutrino oscillations. More specifically, we have developed a method for calculating the phase shift in flavor neutrino oscillations by utilizing a Taylor series expansion of the action that takes into account Δm^4 orders. In addition, we have used this method to assess how the phase difference in neutrino mass eigenstates changes according to the gravitational field described by the Johannsen spacetime.

Keywords: Dirac equation, neutrino flavour oscillation, curved spacetime

DOI: 10.1088/1674-1137/ad3c2c

I. INTRODUCTION

In the field of neutrino physics, a phenomenon called neutrino oscillation has been reported for which neutrinos change from one flavor to another while moving through space. This oscillation results from the interaction between the three recognized neutrino flavors: electron neutrino ν_e, muon neutrino ν_μ, and tau neutrino ν_τ. Pontecorvo [1] first proposed the concept of neutrino oscillation. This author also suggested that neutrinos possess a mass that was previously thought to be nonexistent. The mathematical description of neutrino oscillation involves the use of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [2], which connects the flavor states of neutrinos to their mass eigenstates. This matrix includes four parameters: three mixing angles and one phase. The mixing angles determine the likelihood of a neutrino transitioning from one flavor to another, while the phase influences the relative probabilities of oscillation between different flavors. Neutrino oscillation has significant implications in astrophysics, particle physics, and cosmology, facilitating our understanding of neutrino properties and their role in the universe [3, 4], including their contribution to dark matter. It is a captivating phenomenon that has opened up new avenues of research in particle physics and astrophysics [5]. The discovery of neutrino oscillation has challenged our knowledge on neutrinos and provided valuable insights into the nature of the universe The significant breakthrough of this discovery has received recognition through various awards, notably the 2015 Nobel Prize in Physics. This prestigious honor was bestowed upon Takaaki Kajita and Arthur B. McDonald for their remarkable contributions to the Super-Kamiokande [6] and SNO experiments.

Sudbury Neutrino Observatory (SNO) [7], The Super-Kamiokande [8], and MINOS experiments [9, 10] played a vital role in enabling the discovery of neutrino oscillation. These experiments observed the phenomenon of different types of neutrinos disappearing and reappearing as they passed through the Earth's atmosphere or matter.

Different research studies aim to understand the phenomenon of neutrino lensing caused by gravitational sources. They are revealing an intriguing connection between the probability of neutrino oscillation and the individual masses of neutrinos. This connection is explained through an analysis of the impact of weak lensing induced by a Schwarzschild mass [11]. This analysis explores the implications of gravitationally modified neutrino oscillations in realistic scenarios involving two or three flavors, such as the influence of the gravitational field of a a supernova on the travel of emitted neutrinos.

Received 7 January 2024; Accepted 8 April 2024; Published online 9 April 2024
* Supported by the Grants F-FA-2021-510 from the Uzbekistan Ministry for Innovative Development
† E-mail: alibekov@astrin.uz
‡ E-mail: atamurotov@yahoo.com
§ E-mail: ahmadjon@astrin.uz
♮ E-mail: vkhamidov@tuit.uz

This influence could have observable effects on the neutrino signal [12]. Furthermore, the propagation of neutrinos in a strong gravitational field regime has been studied by considering electromagnetic interactions using the WKB approximation [13]. The behavior of neutrino oscillations in the Schwarzschild spacetime has also been investigated taking into account spin precession in the presence of a magnetic field [14, 15]. Notably, both radial and nonradial propagations of neutrinos in the Schwarzschild spacetime have been examined [16]. Finally, the effects of universe expansion and torsion on neutrino oscillations have also been studied [17]. The mass hierarchy of neutrinos refers to how the three types of neutrinos are arranged in terms of their relative sizes. There are electron, muon, and tau neutrinos, each with their own antineutrinos. Despite being light compared to other particles, neutrinos do have small but nonzero masses according to current knowledge on neutrino physics. The mass hierarchy of neutrinos can be classified as normal or inverted. In the normal hierarchy, the masses are ordered as $m_1 < m_2 < m_3$, with m_1 corresponding to the lightest, m_2 corresponding to the second lightest, and m_3 corresponding to the heaviest neutrino [18]. Conversely, in the inverted hierarchy, the masses are arranged as $m_3 < m_1 < m_2$. Determining the neutrino mass hierarchy is an important topic in neutrino physics because it affects various astrophysical and cosmological phenomena. Current evidence for the neutrino mass hierarchy comes from the observation of neutrino oscillations, which refer to the ability of neutrinos to change their type as they travel in space owing to quantum mechanical mixing between the three types of neutrinos. The probability of oscillation is influenced by the differences in the squared masses of the three neutrino types and the mixing angles between them. Experiments such as Super-Kamiokande and Daya Bay have provided valuable information about the neutrino mass differences and mixing angles. According to this information, it is highly probable that the neutrino mass hierarchy is normal [19]. However, future experiments such as the Deep Underground Neutrino Experiment (DUNE) will provide more accurate measurements, allowing for a definitive determination of the mass hierarchy.

The quantum field theory of neutrinos coupled to gravity serves as the theoretical framework for studying neutrino oscillation in curved spacetime [14]. In this framework, the probability of oscillation depends on various factors such as neutrino energy, mass-squared differences, and curvature of spacetime. The metric tensor describes the curvature of spacetime, which is influenced by the gravitational field as well as the distribution of matter and energy. Numerous studies have explored the effects of curved spacetime on neutrino oscillation probability, considering aspects such as the gravitational redshift and curvature-induced potential. These studies have demonstrated that the gravitational field can modify the oscillation probability, leading to potentially observable consequences. Examination of neutrino oscillation in curved spacetime is an active area of research with significant implications for astrophysics and cosmology [17, 20]. The development of theoretical models that describe the quantum field theory of neutrinos coupled to gravity and the impact of curved spacetime on the oscillation probability have been addressed in several studies (see, for example, [21]).

The rotation of spacetime under weak gravity conditions has been extensively investigated in relation to neutrino oscillations, particularly when neutrinos travel along the equatorial plane. By using the asymptotic form of the Kerr metric, it has been demonstrated that the rotation of a gravitational source significantly changes the phase of neutrinos. Specifically, when neutrinos are generated near a black hole with angular momentum and detected on the same side without the influence of gravitational lensing, the probability of oscillation differs greatly to that observed in the Schwarzschild spacetime [18]. The effects of gravitational lensing on neutrino oscillations within the framework of the γ-spacetime have also been studied employing a quantum-mechanical approach for relativistic neutrinos [22]. This study examined both radial and nonradial propagations, taking into account the phase of neutrino oscillations within this specific spacetime. Additionally, the presence of massive objects in the universe can impact the probability of neutrino oscillation, which has implications in the prediction of the cosmic neutrino background.

II. DIRAC EQUATION IN CURVED SPACETIME

It is crucial to first understand the fundamental methods used for solving the Dirac equation in curved spacetime before delving into the properties of neutrinos in the presence of gravity. The Dirac equation, which governs the behavior of a massive spinor field on a torsion-free pseudo-Riemannian manifold, can be easily extended as [23, 24]

$$\mathrm{i}\hbar\gamma^{\mu}\,\mathcal{D}_{\mu}\,\psi(x) = mc\,\psi(x)\,, \tag{1}$$

and using the relation $\mathcal{D}_{\mu} = \left(\partial_{\mu} + \Gamma_{\mu}\right)$, which is the covariant derivative for a spinor field, Eq. (1) can be expressed as [25, 26]

$$\left[\mathrm{i}\hbar\gamma^{\mu}\,\left(\partial_{\mu} + \Gamma_{\mu}\right) - mc\right]\psi(x) = 0\,. \tag{2}$$

Here, Γ_{μ} represents the spin connection and γ^{μ} is associated with the covariant Dirac matrices, which are linked to spacetime through the following relations [27, 28]:

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}\ . \tag{3}$$

In this context, the anti-commutation operation is denoted by curly brackets, and spin connection Γ_μ is determined by the following condition [23]:

$$\frac{\partial\gamma_\nu}{\partial x^\mu}-\Gamma^\lambda_{\nu\mu}\gamma_\lambda-\Gamma_\mu\gamma_\nu+\gamma_\nu\Gamma_\mu=0. \tag{4}$$

Let us now define constant Dirac matrices $\gamma^{(a)}$ as

$$\gamma^{(a)}=e^{(a)}_\mu\gamma^\mu. \tag{5}$$

Here, $e^{(a)}_\mu$ is the orthogonal tetrad that fulfills the following relationship:

$$g_{\mu\nu}=e^{(a)}_\mu e^{(b)}_\nu\eta_{ab}. \tag{6}$$

In the convention of flat metric, where $\eta_{ab}=\mathrm{diag}(-c^2,1,1,1)$, the expression for the spin connection can be expressed using these constant Dirac matrices [25]:

$$\Gamma_\mu=\frac{\{\gamma^{(a)},\gamma^{(b)}\}}{8}\ g_{\nu\lambda}e^\nu_{(a)}\ \nabla_\mu e^\lambda_{(b)}. \tag{7}$$

The action corresponding to Eq. (1) is

$$\mathcal{S}=\int \mathrm{d}^4x\ \sqrt{g}\ \mathcal{L}_D. \tag{8}$$

Here, g is defined as $g=g^{\mu\nu}g_{\mu\nu}$, and we can express the Lagrangian as [29]

$$\mathcal{L}_D=\frac{\mathrm{i}}{2}\left[\bar{\psi}\gamma^\mu\ \mathcal{D}_\mu\ \psi-\left(\mathcal{D}_\mu\bar{\psi}\right)\gamma^\mu\psi\right]-m\bar{\psi}\psi. \tag{9}$$

III. WKB APPROXIMATION FOR DIRAC EQUATION

We are searching for an approximate solution to Eq. (1) by employing the Wentzel–Kramers–Brillouin (WKB) approximation method [30]. Different forms of the WKB approximation have been proposed. Investigation on complex spinor $\psi(x)$ revealed that it can be decomposed into two components: amplitude $\xi=\xi(x)$ and semi-classical phase $\mathcal{S}=\mathcal{S}(x)$ as [31]

$$\psi(x)=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{S}(x)}\xi(x)\ . \tag{10}$$

The wave function in the presence of spin connection can be expressed as [32, 33, 29]

$$\psi(x)=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{S}(x)}\mathrm{e}^{-\Gamma_\mu x^\mu}\sum_{n=0}^{\infty}\left(\frac{\hbar}{i}\right)^n\xi_n(x)\ . \tag{11}$$

The Dirac matrix product in the spin connection term is expressed as [34, 27]

$$\gamma^a\{\gamma^b,\gamma^c\}=2\eta^{ab}\gamma^c-2\eta^{ac}\gamma^b-2\mathrm{i}\epsilon^{dabc}\gamma_5\gamma_d\ . \tag{12}$$

Here, η^{ab} represents the metric in a flat space, while ϵ^{abcd} denotes the totally antisymmetric tensor in the same flat space. The spin connection can be expressed in terms of the matrix that violates parity as

$$\Gamma_\mu=\frac{\gamma_5}{2i}\sqrt{-g}\ \mathcal{A}_\mu, \tag{13}$$

where

$$\mathcal{A}_\mu=\frac{\sqrt{-g}}{4}e^\mu_a\epsilon^{abcd}\left(e_{b\nu,\sigma}-e_{b\sigma,\nu}\right)e^\nu_c e^\sigma_d. \tag{14}$$

One can infer that the additional phase factor in Eq. (11) effectively confirms the interaction between the metric and the spin orientation of the spinor.

By substituting Eq. (10) into Eq. (1) and equating terms with equal powers of $\hbar$, we obtain a series of recursive equations for amplitudes ξ_n:

$$\left[\gamma^\nu\partial_\nu\mathcal{S}(x)+mc\right]\xi_0(x)=0\ , \tag{15}$$

$$\left[\gamma^\nu\partial_\nu\mathcal{S}(x)+mc\right]\xi_n(x)=\left[\gamma^\nu\partial_\nu\mathcal{S}(x)+mc\right]\xi_{n-1}(x)\ . \tag{16}$$

As a result of multiplying by $\left[\gamma^\nu\partial_\nu\mathcal{S}(x)-m\right]$ the left-hand side of Eq. (15), the Hamilton-Jacobi equation for a massive particle in a curved spacetime can be expressed as [35, 36, 37]

$$g^{\mu\nu}\partial_\mu\mathcal{S}(x)\partial_\nu\mathcal{S}(x)-m^2c^2=0. \tag{17}$$

As long as the four-momentum of the particle is known, we can link this expression to the classical action of a particle with mass m on a torsion-free pseudo-Riemannian manifold, allowing us to associate phase $\mathcal{S}(x)$ with this action:

$$p_\mu = mg_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}. \tag{18}$$

If we recognize Eq. (15) and establish this identification, it becomes equivalent to the mass-shell condition,

$$p_\mu = \partial_\mu \mathcal{S}(x)\ , \tag{19}$$

and the solution of Eq. (15) can be expressed as

$$\mathcal{S}(x) = \int p_\mu \mathrm{d}x^\mu\ . \tag{20}$$

The Lagrangian expression, which characterizes the geodesic motion, can be formulated as $\mathcal{L} = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$, and the geodesic equation can be expressed as

$$\ddot{x}^\mu + \Gamma^\nu_{\mu\sigma}\dot{x}^\mu \dot{x}^\sigma = 0\ . \tag{21}$$

A. Dynamics of neutrino spin in external fields within a curved spacetime

The study of spin oscillations of massive Dirac neutrinos in the presence of background matter, electromagnetic fields, and gravitational fields is a complex and ongoing research area in theoretical physics. Neutrinos are fundamental particles with extremely weak interaction with matter, making their study challenging and intriguing. Next, we delve deeper into some of the key aspects of studying spin oscillations of massive Dirac neutrinos in background matter, electromagnetic fields, and gravitational fields:

● Matter Effects and Neutrino Oscillations: When neutrinos propagate through a medium, such as the dense matter found in the core of a star or the early universe, their interactions with the medium can modify their oscillation behavior. This is known as the matter effect or MSW effect, named after the physicists who first studied it (Mikheyev, Smirnov, and Wolfenstein).

This matter effect arises from the presence of charged particles in the medium. Neutrinos can experience forward scattering interactions with these charged particles, leading to an effective potential that depends on the neutrino flavor. As a result, the flavor oscillation probability of neutrinos can be significantly altered with respect to their behavior in a vacuum. The matter effect can induce resonances, which maximally modifies the oscillation probability, leading to interesting phenomena in neutrino oscillation experiments.

● Electromagnetic Fields and Spin Precession: Neutrinos, being electrically neutral particles, do not directly interact with electromagnetic fields. However, they possess a magnetic dipole moment, which allows for an indirect interaction with magnetic fields. When neutrinos propagate through regions with magnetic fields, such as in astrophysical environments or laboratory experiments, they can experience spin precession.

Spin precession refers to the rotation of the neutrino's spin around the direction of the magnetic field. This precession can modify the flavor oscillation probability of neutrinos and introduce new effects that depend on the relative orientation between the neutrino's momentum, magnetic field, and direction of propagation. The study of spin precession in neutrinos requires a careful treatment of the neutrino's magnetic properties and their interactions with magnetic fields.

● Gravitational Fields and General Relativity: Neutrinos, like all particles, are influenced by gravitational fields according to the principles of general relativity. In the presence of a gravitational field, the curvature of spacetime affects the propagation of neutrinos. This can lead to modifications in their oscillation behavior and introduce additional complexities.

The gravitational interaction can cause the trajectory of neutrinos to deviate and induce effects such as gravitational redshift and time dilation. These gravitational effects can impact the neutrino oscillation probability and potentially generate spin oscillations as well. The study of neutrino oscillations in the context of general relativity requires a combination of quantum field theory, general relativity, and development of suitable theoretical frameworks.

● Experimental Probes and Future Directions: Experimental efforts play a crucial role in studying the spin oscillations of massive Dirac neutrinos in various physical environments. Neutrino oscillation experiments, conducted at particle accelerators, underground laboratories, or using astrophysical neutrino sources, provide valuable data for testing theoretical predictions and exploring the properties of neutrinos.

Future experiments, such as the Deep Underground Neutrino Experiment (DUNE) and Jiangmen Underground Neutrino Observatory (JUNO), aim to study neutrino oscillations with higher precision and investigate matter effects, electromagnetic field interactions, and the impact of gravitational fields on neutrino's behavior.

In [37], neutrino spin oscillations within external fields in curved spacetime were investigated. The contributions of this study have been highly valuable to the field. The researchers analyzed the evolution of neutrino's spin in the presence of background matter and an external electromagnetic field within a curved spacetime. The primary motivation behind this study was to provide evidence supporting the validity of the quasiclassical equation governing neutrino spin evolution. They successfully derived a covariant equation for this purpose, starting from the Dirac equation that describes the interaction between a massive neutrino and external fields in a curved spacetime as

$$\left[\mathrm{i}\hbar\gamma^{\mu}\,\mathcal{D}_{\mu}-\frac{\mu}{2}F_{\mu\nu}\sigma^{\mu\nu}-\frac{V^{\mu}}{2}\gamma_{\mu}\left(1-\gamma^{5}\right)\right]\psi(x)=mc\,\psi(x)\,, \quad (22)$$

where $F_{\mu\nu}$ is the Faraday tensor defined as

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}\,, \quad (23)$$

and A_{μ} is the four-vector potential of the electromagnetic field. In Eq. (15), $\mathcal{D}_{\mu}=\left(\partial_{\mu}+\Gamma_{\mu}\right)$ represents the covariant derivative, where Γ_{μ} denotes the spin connection. The symbols $\gamma^{\mu}=\gamma^{\mu}(x)$, $\sigma^{\mu\nu}=\frac{\mathrm{i}}{2}\left(\gamma_{\mu}\gamma_{\nu}-\gamma_{\nu}\gamma_{\mu}\right)$, and $\gamma^{5}=-\frac{\mathrm{i}}{4!}E^{\mu\nu\alpha\beta}\gamma_{\mu}\gamma_{\nu}\gamma_{\alpha}\gamma_{\beta}$ refer to the coordinate-dependent Dirac matrices. Here, $E^{\mu\nu\alpha\beta}=\varepsilon^{\mu\nu\alpha\beta}/\sqrt{-g}$ represents the covariant antisymmetric tensor in the context of curved spacetime, where $g=\det(g_{\mu\nu})$ and $g_{\mu\nu}$ denote the metric tensor. Symbol μ represents the magnetic moment of a neutrino. The expression for $V^{\mu}=(V^{0},\mathbf{V})$, which serves as the effective potential governing the interaction of a neutrino with arbitrarily polarized and moving matter, can be found in [38].

The relationship between Eqs. (15) and (1) becomes apparent when we observe that neglecting the impact of background matter and electromagnetic fields results in considering only gravitational effects. Consequently, Eq. (15) can be expressed equivalently to Eq. (1) as

$$\left[\mathrm{i}\hbar\gamma^{\mu}\,\left(\partial_{\mu}+\Gamma_{\mu}\right)-mc\right]\psi(x)=0\,. \quad (24)$$

The expression for the Dirac equation in a locally Minkowskian frame was derived in [38]. Recognizing the work of these authors in this area, we consider unnecessary to duplicate their efforts. Instead, we focus on obtaining the Dirac equation that specifically characterizes the interaction between a massive neutrino and external fields within a curved spacetime, tailored to our unique case:

$$\left[\mathrm{i}\hbar\gamma^{\mu}\,\partial_{\mu}+\frac{\hbar}{2}\gamma^{\mu}\gamma_{5}\sqrt{-g}\,\mathcal{A}_{\mu}\right. \\ \left.-\frac{\mu}{2}F_{\mu\nu}\sigma^{\mu\nu}-\frac{V^{\mu}}{2}\gamma_{\mu}\left(1-\gamma^{5}\right)-mc\right]\psi(x)=0\,, \quad (25)$$

and, according to Ref. [39], the covariant equation governing the quasiclassical evolution of the neutrino spin, denoted as S^{μ} in the presence of general external fields, can be derived. This derivation relies on the Heisenberg equation applied to the corresponding spin operator, taking into account the influence of the external fields. Subsequently, the equation is subjected to an averaging process over the neutrino wave packet. By employing Eq. (25), we obtain the Lorentz invariant expression for the evolution equation of neutrino spin S^{μ}, which accounts for the general interactions with external fields as

$$\frac{\mathrm{d}S^{\mu}}{\mathrm{d}\tau}=2\mu\left(F^{\mu\nu}S_{\nu}-u^{\mu}F^{\lambda\rho}u_{\lambda}S_{\rho}\right)+\sqrt{2}G_{F}\mathcal{K}^{\mu\nu}S_{\nu}+\mathcal{G}^{\mu\nu}S_{\nu}, \quad (26)$$

where $G_{F}=1.17\times10^{-5}\,\mathrm{GeV}^{-2}$ is the Fermi constant. Additionally, the expressions for tensors $\mathcal{K}^{\mu\nu}$ and $\mathcal{G}^{\mu\nu}$ are

$$\mathcal{K}^{\mu\nu}=\varepsilon^{\mu\nu\alpha\beta}V_{\alpha}u_{\beta}\,, \\ \mathcal{G}^{\mu\nu}=\left(\gamma^{\mu\nu\lambda}+\gamma^{\lambda\mu\nu}+\gamma^{\nu\lambda\mu}\right)u_{\lambda}\,. \quad (27)$$

IV. NEUTRINOS

The production and detection of neutrinos occur in various flavor eigenstates represented by $|\nu_{\alpha}\rangle$. These flavor eigenstates are combinations of mass eigenstates represented by $|\nu_{i}\rangle$. Therefore, a flavor eigenstate can be expressed in terms of mass eigenstates as discussed in [16, 40],

$$|\nu_{\alpha}\rangle=\sum_{i}U^{*}_{\alpha i}|\nu_{i}\rangle\,, \quad (28)$$

for a set of three neutrino flavors $\alpha=\{e,\mu,\tau\}$ and a set of three generations $i=\{1,2,3\}$. Maki-Nakagawa-Sakata-Pontecorvo (MNSP) matrix U, also known as the neutrino flavor mixing unitary matrix, plays a role similar to the Cabibbo-Kobayashi-Maskawa matrix in governing quark mixing. In the case of three generations of neutrinos, the MNSP matrix is characterized by three mixing angles θ_{i}, phase δ that describes CP-violation, and two additional phases α_{1} and α_{2}, which can only be non-zero if neutrinos are Majorana particles. If neutrinos are Dirac particles, then $\alpha_{1}=\alpha_{2}=0$.

In a process of propagation, the neutrino moves from source S to detector D positioned at x_{S} and x_{D}, respectively. The amplitude associated with the event of detecting a neutrino of flavor α at position x_S and observing it as a neutrino of flavor β at position x_D is described by

$$\mathcal{A}_{\beta\alpha} = |\langle\nu_\beta(x_D)|\nu_\alpha(x_S)\rangle| = \sum_i U^*_{\alpha i} U_{\beta i} |\langle\nu_i(x_D)|\nu_i(x_S)\rangle|. \quad (29)$$

To estimate spinor ν_i, the WKB approximation in Eq. (11) incorporates both action $\mathcal{S}_i(x)$ for the i-th mass eigenstate and spin connection Γ_μ. Previous studies [41, 42, 43] have explored three distinct scenarios that give rise to neutrino oscillation: (a) a flat spacetime, (b) a curved spacetime within a non-rotating frame, and (c) a curved spacetime within a rotating frame. In scenarios (a) and (b), the phase difference of the neutrinos relies solely on $\mathcal{S}(x)$ and does not involve the spin connection, resulting in the following phase difference [44]:

$$\mathcal{S}(m_i, x_D - x_S) \approx \mathcal{S}_i(x_D) - \mathcal{S}_i(x_S) = \int_{x_S}^{x_D} p_\mu \mathrm{d}x^\mu. \quad (30)$$

The presence of parity-violating matrix γ_5 in the representation of the spin connection [14] in Eq. (13) reveals that in scenario (c), there is an additional contribution to the phase shift when there exist differences in spin orientation between the two mixing eigenstates ν_i. Moving forward, we will exclusively examine massive neutrinos with identical spin orientations, disregarding any contributions stemming from different spin orientations. This decision is based on the fact that our research is primarily centered around neutrino flavor oscillations that occur under the influence of action $\mathcal{S}(x)$. Consequently, we will solely consider the same spin orientation for massive neutrinos, dismissing any interaction between the neutrino spin and the metric that arises from Γ_μ.

The expression for the amplitude of the neutrino flavor transition can be formulated as follows:

$$\mathcal{A}_{\beta\alpha} = \sum_i U^*_{\alpha i} U_{\beta i} \mathrm{e}^{-\mathrm{i}\mathcal{S}(m_i, x_D - x_S)}. \quad (31)$$

The difference in phase between two mass eigenstates can be expressed in the following form [37]:

$$\Phi_{ij} = \mathcal{S}(m_i, x_D - x_S) - \mathcal{S}\left(m_j, x_D - x_S\right). \quad (32)$$

The probability of a transition in neutrino flavor from the initially produced α flavor to the detection points can be calculated as [15, 45]

$$\mathcal{P}_{\beta\alpha} = \left|\mathcal{A}_{\beta\alpha}\right|^2 = \sum_{i,j} U_{\beta i} U^*_{\beta j} U_{\alpha j} U^*_{\alpha i} \mathrm{e}^{-\mathrm{i}\Phi_{ij}}. \quad (33)$$

It is important to note that the action assumes the following form when m is small:

$$\mathcal{S}(m_i, x_D - x_S) = \sum_{n=0}^{\infty} \frac{(m_i^2)^n}{n!} \mathcal{S}^{(n)}(x_D - x_S), \quad (34)$$

where

$$\mathcal{S}^{(n)}(x_D - x_S) = \frac{\partial^{(n)} \mathcal{S}(m_i, x_D - x_S)}{\partial^{(n)} m_i^2}. \quad (35)$$

As a result, using the Taylor series, one can express the phase difference as

$$\Phi_{ij} = \Delta m_{ij}^2 \mathcal{S}^{(1)}(x_D - x_S) + \frac{\Delta m_{ij}^4}{2} \mathcal{S}^{(2)}(x_D - x_S) + ..., \quad (36)$$

where

$$\Delta m_{ij}^2 = m_i^2 - m_j^2, \qquad \Delta m_{ij}^4 = \left(m_i^2 + m_j^2\right) \Delta m_{ij}^2. \quad (37)$$

V. TWO-FLAVOUR NEUTRINO OSCILLATIONS

The result expressed by Eq. (33) is applicable to all numbers of neutrino generations and neutrino energies. However, the likelihood of conversion into a specific neutrino flavor is reduced in certain scenarios, including solar neutrino mixing. In these situations, the MNSP matrix [46, 47] simplifies to a member of the $SO(2)$ group and can be represented by a single mixing angle, Θ, as follows:

$$U = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}. \quad (38)$$

In the case of two flavors of neutrino oscillations, there is a single difference in mass denoted as $\Delta m_{12}^2 = m_1^2 - m_2^2$ and another difference $\Delta m_{12}^4 = \left(m_1^2 + m_2^2\right) \Delta m_{12}^2$. By simplifying Eq. (33), the final form of the probability of neutrino oscillation can be expressed as [48, 49]

$$\mathcal{P}_{\beta\alpha} = \begin{cases} \sin^2\Theta \sin^2\Phi_{12}, & \beta \neq \alpha \\ 1 - \sin^2\Theta \sin^2\Phi_{12}, & \beta = \alpha \end{cases}. \quad (39)$$

Here, phase shift Φ_{12} up to second order can be determined using Eq. (36) as

$$\Phi_{12} = \Delta m_{12}^2 \mathcal{S}^{(1)}(r_D - r_S) + \frac{\Delta m_{12}^4}{2} \mathcal{S}^{(2)}(r_D - r_S). \quad (40)$$

Therefore, the probability expression takes the following form:

$$\mathcal{P}_{\beta\alpha} = \begin{cases} \sin^2\Theta\sin^2\left[\mathcal{S}^{(1)}(r_D - r_S)\Delta m_{12}^2 \right. \\ \quad \left. + \dfrac{1}{2}\mathcal{S}^{(2)}(r_D - r_S)\Delta m_{12}^4\right], \quad \beta \neq \alpha \\ 1 - \sin^2\Theta\sin^2\left[\mathcal{S}^{(1)}(r_D - r_S)\Delta m_{12}^2 \right. \\ \quad \left. + \dfrac{1}{2}\mathcal{S}^{(2)}(r_D - r_S)\Delta m_{12}^4\right], \quad \beta = \alpha \end{cases} \tag{41}$$

and here, we can insert the expressions for $\mathcal{S}^{(1)}(r_D - r_S)$ and $\mathcal{S}^{(2)}(r_D - r_S)$ given by the Eqs. (53) and (54), respectively. As a result, the Eq. (41) becomes a general formula to calculate the probability of a neutrino undergoing flavor transition as it propagates from its source to a receiver.

VI. JOHANNSEN SPACETIME

We next focus on evaluating the Johannsen spacetime, which is a more general version of the Kerr spacetime, and it can be defined using the following metric [50]:

$$\begin{aligned} ds^2 = &-\frac{\tilde{\Sigma}\left(\Delta - a^2A_2^2(r)\sin^2\theta\right)}{B^2}dt^2 + \frac{\tilde{\Sigma}}{A_5(r)\Delta}dr^2 + \frac{\tilde{\Sigma}}{A_6(\theta)}d\theta^2 - \\ &-\frac{2a\left[\left(r^2+a^2\right)A_1(r)A_2(r) - \Delta\right]\tilde{\Sigma}\sin^2\theta}{B^2}dtd\phi + \\ &+\frac{\left[\left(r^2+a^2\right)^2A_1^2(r) - a^2\Delta\sin^2\theta\right]\tilde{\Sigma}\sin^2\theta}{B^2}d\phi^2, \end{aligned} \tag{42}$$

where

$$\begin{aligned} &B = A_1(r)A_3(\theta)(r^2+a^2) - A_2(r)A_4(\theta)a^2\sin^2\theta \\ &\tilde{\Sigma} = \Sigma + f(r) + g(\theta), \quad \Delta = r^2 - 2Mr + a^2, \\ &\Sigma = r^2 + a^2\cos^2\theta. \end{aligned} \tag{43}$$

Generally speaking, it is not possible to locate the stationary points of the effective potential of the function within the given background spacetime expressed by Eq. (42). Nevertheless, if we specifically select the profile functions as follows, it may become feasible:

$$\begin{aligned} &A_1(r) = 1 + \sum_{k=3}^{\infty}\alpha_{1k}\left(\frac{M}{r}\right)^k, \quad A_2(r) = 1 + \sum_{k=2}^{\infty}\alpha_{3k}\left(\frac{M}{r}\right)^k, \\ &A_5(r) = 1 + \sum_{k=2}^{\infty}\alpha_{5k}\left(\frac{M}{r}\right)^k, \quad f(r) = r^2\sum_{k=3}^{\infty}\epsilon_k\left(\frac{M}{r}\right)^k, \\ &A_3(\theta) = A_4(\theta) = A_6(\theta) = 1, \quad g(\theta) = 0, \end{aligned} \tag{44}$$

in which the equatorial plane at $\theta_0 = \pi/2$ is where the stationary points of the function effective potential can be found. It is crucial to highlight that the Johannsen spacetime is distinguished by a set of parameters, namely, α_{1k}, α_{3k}, α_{5k}, and ϵ_k, which vary as a function of the mass and spin of the black hole. It is also important to emphasize the following key features of this parametrization: (a) the metric retains its smoothness at all points both inside and outside the event horizon, and (b) it has been convincingly demonstrated that certain black hole solutions can be accurately reproduced in alternative theories of gravity by appropriately choosing the deformation parameters [51]. In addition, the spacetime is derived by enforcing the separability of the Hamilton-Jacobi equations, despite lacking a theoretical basis for doing so. It is worth noting that alternative theories of gravity exist in which non-Kerr black hole solutions do not meet this condition. However, maintaining separability may aid in certain calculations. This spacetime model finds application in conducting phenomenological calculations in the field of black hole astrophysics [52, 53].

The equation for massive neutrinos, known as the Hamilton-Jacobi equation, can be expressed as [54]

$$-2\frac{\partial\mathcal{S}}{d\tau} = g^{\mu\nu}\frac{\partial\mathcal{S}}{\partial x^\mu}\frac{\partial\mathcal{S}}{\partial x^\nu}. \tag{45}$$

Using the metric geometry expressed by Eq. (42), Eq. (45) takes the form

$$\begin{aligned} -2\frac{\partial\mathcal{S}}{d\tau} = &g^{tt}\left(\frac{\partial\mathcal{S}_t}{\partial t}\right)^2 + 2g^{t\phi}\frac{\partial\mathcal{S}_t}{\partial t}\frac{\partial\mathcal{S}_\phi}{\partial\phi} + g^{\phi\phi}\left(\frac{\partial\mathcal{S}_\phi}{\partial\phi}\right)^2 \\ &+ g^{rr}\left(\frac{\partial\mathcal{S}_r}{\partial r}\right)^2 + g^{\theta\theta}\left(\frac{\partial\mathcal{S}_\theta}{\partial\theta}\right)^2 \quad . \end{aligned} \tag{46}$$

The given form of the Hamilton-Jacobi function is [50]

$$\mathcal{S} = \frac{1}{2}m^2\tau - Et + L\phi + \mathcal{S}_r + \mathcal{S}_\theta. \tag{47}$$

After performing certain algebraic calculations, the HJ equation can be separated as follows:

$$\frac{1}{\Delta}\left[(r^2+a^2)A_1(r)E - aA_2(r)L\right]^2$$

$$-m^2\left[r^2+f(r)\right]-A_5(r)\Delta\left(\frac{\partial\mathcal{S}_r}{\partial r}\right)^2 = \left(\frac{\partial\mathcal{S}_\theta}{\partial\theta}\right)^2+\left(\frac{A_3(\theta)L}{\sin\theta}-aA_4(\theta)E\sin\theta\right)^2+m^2a^2\cos^2\theta\,, \tag{48}$$

and the same constant K, which is known as the Carter constant, can be assigned to both sides of the equation:

$$\left(\frac{\partial\mathcal{S}_\theta}{\partial\theta}\right)^2 = K-\left(\frac{A_3(\theta)L}{\sin\theta}-aA_4(\theta)E\sin\theta\right)^2-m^2a^2\cos^2\theta\,,$$
$$\left(\frac{\partial\mathcal{S}_r}{\partial r}\right)^2=\frac{\left[(r^2+a^2)A_1(r)E-aA_2(r)L\right]^2-\Delta\left[K-m^2\left(r^2+f(r)\right)\right]}{A_5(r)\Delta^2} \tag{49}$$

It is possible to achieve motion on a plane with fixed angle $\theta=\theta_0$ by selecting the indicated angle:

$$K=\left(\frac{A_3(\theta_0)L}{\sin\theta_0}-aA_4(\theta_0)E\sin\theta_0\right)^2+m^2a^2\cos^2\theta_0. \tag{50}$$

If this is the case, the action is interpreted as

$$\mathcal{S}=-Et+L\phi+\int\frac{\sqrt{R(r)}}{\Delta}\mathrm{d}r, \tag{51}$$

where

$$R(r)=\frac{\left[(r^2+a^2)A_1(r)E-aA_2(r)L\right]^2-\Delta\left[K-m^2\left(r^2+f(r)\right)\right]}{A_5(r)}. \tag{52}$$

As mentioned in Eq. (34), we can consider the action up to the second order in m^2 and expand it from r_S (source distance) to r_D (detector distance) as

$$\mathcal{S}^{(1)}(r_D-r_S)=\int_{r_S}^{r_D}\frac{\left(r^2+f(r)\right)\sqrt{A_5(r)}}{2\sqrt{((r^2+a^2)A_1(r)E-aA_2(r)L)^2-\Delta K}}\mathrm{d}r, \tag{53}$$

and

$$\mathcal{S}^{(2)}(r_D-r_S)=\int_{r_S}^{r_D}\frac{\left(r^2+f(r)\right)^2A_5^{3/2}(r)}{4\left[((r^2+a^2)A_1(r)E-aA_2(r)L)^2-\Delta K\right]^{3/2}}\mathrm{d}r. \tag{54}$$

Thus, the Johannsen spacetime reduces to the Kerr spacetime when the parameterizations in Eq. (44) are not taken into account. The expression for $\mathcal{S}^{(1)}$ becomes the same as the one obtained in [36], where the first-order result for neutrino oscillations in a Kerr metric was derived. Next, we explore the process of determining the probability for the two-flavor case of neutrino oscillations.

A. Radial propagation

The reason why purely radial motion is not possible in the Kerr spacetime is the influence of the black hole's rotation, which introduces a phenomenon known as frame-dragging. This phenomenon is a consequence of the spacetime curvature caused by the rotating black hole. In general relativity, the presence of mass or energy curves the surrounding spacetime, affecting the motion of objects within it. In the case of a rotating black hole described by the Kerr spacetime, the rotation creates a twisting or dragging effect on the nearby spacetime. When a test particle moves radially, aiming directly toward or away from the black hole, it is still subject to the curvature of spacetime caused by the rotating black hole. This curvature, combined with the rotation, causes the test particle to experience an additional angular momentum component. As a result, the trajectory of the particle deviates from a purely radial path, leading to a combination of radial and angular motion. This can be viewed as if the rotating black hole "drags" the nearby spacetime around with it, causing objects to be influenced by the rotation even if they are initially moving radially. This effect prevents purely radial motion in the Kerr spacetime and results in a spiraling or helical trajectory for the test particle. Therefore, in the Kerr spacetime, a test particle cannot move along purely radial trajectories. Owing to the curvature of spacetime caused by the rotating black hole, the particle will experience a combination of radial and angular motion, even if it initially moves radially.

The Johannsen spacetime, on the other hand, is a modification of the Kerr spacetime that introduces additional parameters to account for potential deviations from the standard Kerr geometry. The specific effects of these deviations on the motion of test particles depend on the particular form of the Johannsen metric and the values of the parameters. Therefore, it is possible that in certain cases or parameter regimes, the Johannsen spacetime may allow for different types of motion, including potentially radial motion. The specific effects of these deviations on the motion of test particles depend on the particular form of the Johannsen metric and the values of the parameters. Without knowing the specific form of the Johannsen metric and the parameter values, it is challenging to provide a definitive answer regarding the existence or nature of purely radial motion in the Johannsen spacetime. The additional parameters introduced in the Johannsen metric could potentially alter the gravitational field in such a way that purely radial motion becomes possible. However, it is also possible that the additional forces or curvature effects introduced by the deviations from the Kerr geometry would still prevent purely radial motion.

The specific form of the Johannsen metric, which incorporates these additional parameters, determines the exact nature of the deviations from the Kerr spacetime and how they affect the motion of test particles. Depending on the values of these parameters, it is conceivable that the additional forces or curvature effects introduced in the Johannsen spacetime could allow for purely radial motion in certain cases. For example, if the additional parameters in the Johannsen metric introduce modifications to the gravitational field that counteract or weaken the frame-dragging effect caused by the black hole's rotation, purely radial motion could be potentially allowed. This could occur if the additional parameters change the geometry of spacetime in such a way that the angular momentum component induced by the rotation becomes negligible or is canceled out. The characterization of Φ_{ij}^{m} will be influenced by the geodesic parameters, including energy and angular momentum. Next, we consider the scenario of radial propagation with $L=K=0$, where

$$\mathcal{S}=-Et+\int\frac{\sqrt{R_*(r)}}{\Delta}\mathrm{d}r, \tag{55}$$

and

$$R_*(r)=\frac{(r^2+a^2)^2A_1^2(r)E^2+m^2\Delta\left(r^2+f(r)\right)}{A_5(r)}. \tag{56}$$

By expanding the action to its second order in m^2, we obtain

$$\mathcal{S}_*^{(1)}(r_D-r_S)=\frac{1}{2E}\int_{r_S}^{r_D}\frac{\left(r^2+f(r)\right)\sqrt{A_5(r)}}{A_1(r)(r^2+a^2)}\mathrm{d}r, \tag{57}$$

and

$$\mathcal{S}_*^{(2)}(r_D-r_S)=\frac{1}{4E^3}\int_{r_S}^{r_D}\frac{\Delta\left(r^2+f(r)\right)^2A_5^{3/2}(r)}{A_1^3(r)(r^2+a^2)^3}\mathrm{d}r. \tag{58}$$

From Eq. (41), the probability expression for radial propagation adopts the following form

$$\mathcal{P}_{\beta\alpha}=\begin{cases}\sin^2\Theta\sin^2\left[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2+\frac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\right], & \beta\neq\alpha\\ 1-\sin^2\Theta\sin^2\left[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2+\frac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\right], & \beta=\alpha\end{cases} \tag{59}$$

Consequently, from Eq. (59), it becomes evident that in the case of radial propagation with $L=K=0$, the contribution of ϕ to the phase of oscillations is not discernible. This implies that the phase of oscillations remains independent of ϕ. However, it is important to stress that the specific implications of the Johannsen metric on test particle dynamics and the conditions under which purely radial motion might be allowed depend on the precise form of the metric and the values of the additional parameters. The exploration of these effects and conditions requires detailed analysis and numerical simulations specific to the Johannsen metric [52]. Numerical simulations are used to study the behavior of the Johannsen metric and its effect on various physical phenomena. By simulating the motion of test particles, electromagnetic fields, or other relevant quantities in the Johannsen spacetime, the results from observations or theoretical predictions can be compared [53]. These simulations can help constrain the values of the additional parameters that best match the observed data or desired physical behavior.

Subsequently, conducting a comparative analysis of the outcomes obtained from the Schwarzschild, Kerr, and Johannsen spacetimes is of great interest. The Johannsen metric is determined by the mass (M) and spin (a) of the black hole, along with four independent functions that account for potential deviations from the Kerr solution. When $\alpha_{1k}=\alpha_{3k}=\alpha_{5k}=\epsilon_k=0$, the metric simplifies to the Kerr solution [50]. In this paper, for the sake of simplicity, we specifically examine two scenarios: one where only α_{52} is non-zero, and another where only ϵ_3 is set to zero. Consequently, we can derive the expressions for $\mathcal{S}_*^{(1)}(r_D-r_S)$ and $\mathcal{S}_*^{(2)}(r_D-r_S)$ as follows:

$$\begin{aligned}\mathcal{S}_*^{(1)}(r_D-r_S)=&\frac{1}{2E}\cdot\frac{1}{a^3\cdot M^2\cdot r_D^2}\cdot\left[a\cdot M^2\cdot\sqrt{\alpha_{52}\cdot M^2+r_D^2}\cdot\left(a^2\cdot r_D-\epsilon_3\cdot M^3\right)+\epsilon_3\cdot M^5\cdot r_D\cdot\sqrt{\alpha_{52}M^2-a^2}\cdot\cot^{-1}\right.\\&\left.\left(\frac{a\cdot\sqrt{\alpha_{52}\cdot M^2-a^2}}{a^2-r_D\cdot\sqrt{\alpha_{52}\cdot M^2+r_D^2}+r_D^2}\right)\right]-\frac{1}{2E}\cdot\frac{1}{a^3\cdot M^2\cdot r_S^2}\cdot\left[a\cdot M^2\cdot\sqrt{\alpha_{52}\cdot M^2+r_S^2}\cdot\left(a^2\cdot r_S-\epsilon_3\cdot M^3\right)\right.\\&\left.+\epsilon_3\cdot M^5\cdot r_S\cdot\sqrt{\alpha_{52}M^2-a^2}\cdot\cot^{-1}\left(\frac{a\cdot\sqrt{\alpha_{52}\cdot M^2-a^2}}{a^2-r_S\cdot\sqrt{\alpha_{52}\cdot M^2+r_S^2}+r_S^2}\right)\right].\end{aligned} \tag{60}$$

and

$$
\begin{aligned}
\mathcal{S}_*^{(2)}(r_D-r_S) = &\frac{1}{4E^3}\cdot\frac{r_D^2\sqrt{(\alpha_{52}\cdot M^2+r_D^2)}}{8\left(\epsilon_3\cdot M^3+r_D^3\right)^2}\cdot\left(\frac{\epsilon_3\cdot M^3}{r_D}+r_D^2\right)^2\cdot\left[8+\frac{12\left(a^2+3Mr_D-\epsilon_3M^2\right))}{3\cdot\left(a^2+r_D^2\right)\cdot}\right.\\
&-\frac{M^3\cdot(\alpha_{52}\cdot r_D+2\epsilon_3\cdot(M-2r_D))}{3\cdot a^2\cdot\left(a^2+r_D^2\right)}-\frac{2\cdot\epsilon_3\cdot M^5\cdot(2\alpha_{52}\cdot(2M+r_D)+\epsilon_3\cdot M)}{3\cdot a^4\cdot\left(a^2+r_D^2\right)}\\
&+\frac{2a^2\epsilon_3^2M^7(\alpha_{52}M+3r_D)-11\alpha_{52}\epsilon_3^2M^9r_D}{3a^8\left(a^2+r_D^2\right)}-\frac{16M\log\left(\sqrt{\alpha_{52}M^2+r_D^2}+r_D\right)}{\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}+\frac{1}{a^9\cdot\sqrt{a^2-\alpha_{52}\cdot M^2}\cdot\left(\alpha_{52}\cdot M^2+r_D^2\right)^{1/2}}\\
&+\frac{\epsilon_3\cdot M^5\cdot\log(r_D)\cdot\left(3\epsilon_3a^4-8\alpha_{52}a^2M^2(4\alpha_{52}+3\epsilon_3)+24\alpha_{52}^2\epsilon_3M^4\right)}{a^8\cdot\sqrt{\alpha_{52}}\cdot\left(a_{52}\alpha_{52}M^2+r_D^2\right)^{1/2}}\\
&+\frac{\epsilon_3\cdot M^5\cdot\left(8a^2\alpha_{52}M^2(4\alpha_{52}+3\epsilon_3)-3\epsilon_3a^4-24\alpha_{52}^2\epsilon_3M^4\right)\cdot\log\left(\sqrt{\alpha_{52}}\cdot\sqrt{\alpha_{52}M^2+r_D^2}+a_{52}\cdot M\right)}{a^8\sqrt{\alpha_{52}}\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}\\
&-\frac{12M\left(a^2-\alpha_{52}M^2\right)\left(a^6r_D-2a^4\epsilon_3M^3-\epsilon_3^2M^6r_D\right)}{\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}-\frac{16\epsilon_3M^5\left(3a^4\alpha_{52}-4a^2\epsilon_3M^2+9\alpha_{52}\epsilon_3M^4\right)}{a^8r_D\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}\\
&\left.+\frac{3\epsilon_3^2M^6\left(8\alpha_{52}M^2-5a^2\right)}{a^6r_D^2\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}-\frac{6\alpha_{52}\epsilon_3^2M^8}{a^4r_D^4\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}+\frac{16\alpha_{52}\epsilon_3^2M^9}{a^6r_D^3\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}\right]\\
&-\frac{1}{4E^3}\cdot\frac{r_S^2\sqrt{(\alpha_{52}\cdot M^2+r_S^2)}}{8\left(\epsilon_3\cdot M^3+r_S^3\right)^2}\cdot\cdot\left(\frac{\epsilon_3\cdot M^3}{r_S}+r_S^2\right)^2\cdot\left[8+\frac{12\left(a^2+3Mr_S-\epsilon_3M^2\right))}{3\cdot\left(a^2+r_S^2\right)\cdot}\right.\\
&+\frac{2a^2\epsilon_3^2M^7(\alpha_{52}M+3r_S)-11\alpha_{52}\epsilon_3^2M^9r_S}{3a^8\left(a^2+r_S^2\right)}-\frac{M^3\cdot(\alpha_{52}\cdot r_S+2\epsilon_3\cdot(M-2r_S))}{3\cdot a^2\cdot\left(a^2+r_S^2\right)}\\
&-\frac{2\cdot\epsilon_3\cdot M^5\cdot(2\alpha_{52}\cdot(2M+r_S)+\epsilon_3\cdot M)}{3\cdot a^4\cdot\left(a^2+r_S^2\right)}-\frac{16M\log\left(\sqrt{\alpha_{52}M^2+r_S^2}+r_S\right)}{\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}+\frac{1}{a^9\cdot\sqrt{a^2-\alpha_{52}\cdot M^2}\cdot\left(\alpha_{52}\cdot M^2+r_S^2\right)^{1/2}}\\
&+\frac{\epsilon_3\cdot M^5\cdot\log(r_S)\cdot\left(3\epsilon_3a^4-8\alpha_{52}a^2M^2(4\alpha_{52}+3\epsilon_3)+24\alpha_{52}^2\epsilon_3M^4\right)}{a^8\cdot\sqrt{\alpha_{52}}\cdot\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}\\
&+\frac{\epsilon_3\cdot M^5\cdot\left(8a^2\alpha_{52}M^2(4\alpha_{52}+3\epsilon_3)-3\epsilon_3a^4-24\alpha_{52}^2\epsilon_3M^4\right)\cdot\log\left(\sqrt{\alpha_{52}}\cdot\sqrt{\alpha_{52}M^2+r_S^2}+\alpha_{52}\cdot M\right)}{a^8\sqrt{\alpha_{52}}\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}\\
&-\frac{12M\left(a^2-\alpha_{52}M^2\right)\left(a^6r_S-2a^4\epsilon_3M^3-\epsilon_3^2M^6r_S\right)}{\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}-\frac{16\epsilon_3M^5\left(3a^4a_{52}-4a^2\epsilon_3M^2+9\alpha_{52}\epsilon_3M^4\right)}{a^8r_S\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}\\
&\left.+\frac{3\epsilon_3^2M^6\left(8\alpha_{52}M^2-5a^2\right)}{a^6r^2\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}-\frac{6\alpha_{52}\epsilon_3^2M^8}{a^4r_S^4\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}+\frac{16\alpha_{52}\epsilon_3^2M^9}{a^6r_S^3\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}\right].
\end{aligned}
\tag{61}
$$

As a result, the probability expression for radial propagation in Johannsen spacetime adopts the following form:

$$
\mathcal{P}_{\beta\alpha}^{\text{Johannsen}}=\begin{cases}\sin^2\Theta\sin^2\left[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2\right.\\ \quad\left.+\dfrac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\right], & \beta\neq\alpha\\ 1-\sin^2\Theta\sin^2\left[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2\right.\\ \quad\left.+\dfrac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\right], & \beta=\alpha\end{cases}
\tag{62}
$$

When $a_{52}=\epsilon_3=0$, the metric expressed by Eq. (42) is simplified and reduces to the Kerr solution. Consequently, the expressions for $\mathcal{S}*^{(1)}(r_D-r_S)$ and $\mathcal{S}*^{(2)}(r_D-r_S)$ also reduce in a manner similar to that observed in Kerr spacetime [36, 18, 29]:

$$
\mathcal{S}_*^{(1)}(r_D-r_S)=\frac{1}{2E}\cdot\left[r_D-r_S-a\left(\arctan\frac{r_D}{a}-\arctan\frac{r_S}{a}\right)\right].
\tag{63}
$$

and

$$
\mathcal{S}_*^{(2)}(r_D-r_S)=\frac{1}{8E^3}\cdot\left[2(r_D-r_S)-3a\left(\arctan\frac{r_D}{a}-\arctan\frac{r_S}{a}\right)\right.
$$

$$+\frac{a^2(r_D-4M)}{a^2+r_D^2}-\frac{a^2(r_S-4M)}{a^2+r_S^2} -2M\ln\left(\frac{a^2+r_D^2}{a^2+r_S^2}\right)+\frac{a^4M}{\left(a^2+r_D^2\right)^2}-\frac{a^4M}{\left(a^2+r_S^2\right)^2}\Bigg]. \quad (64)$$

As a result, the probability expression for radial propagation in Kerr spacetime adopts the following form:

$$\mathcal{P}^{Kerr}_{\beta\alpha}=\begin{cases}\sin^2\Theta\sin^2\Big[\mathcal{S}^{(1)}_*(r_D-r_S)\Delta m^2_{12}+\dfrac{1}{2}\mathcal{S}^{(2)}_*(r_D-r_S)\Delta m^4_{12}\Big], & \beta\neq\alpha\\ 1-\sin^2\Theta\sin^2\Big[\mathcal{S}^{(1)}_*(r_D-r_S)\Delta m^2_{12}+\dfrac{1}{2}\mathcal{S}^{(2)}_*(r_D-r_S)\Delta m^4_{12}\Big], & \beta=\alpha\end{cases} \quad (65)$$

If we set a_{52}, ϵ_3, and a to zero, the metric given by Eq. (42) is simplified and we obtain the Schwarzschild solution. Consequently, the expressions for $\mathcal{S}^{(1)}_*(r_D-r_S)$ and $\mathcal{S}^{(2)}_*(r_D-r_S)$ also simplify, yielding the same result as in Schwarzschild spacetime [16, 11]:

$$\mathcal{S}^{(1)}_*(r_D-r_S)=\frac{1}{2E}\cdot(r_D-r_S)\ . \quad (66)$$

and

$$\mathcal{S}^{(2)}_*(r_D-r_S)=\frac{1}{4E^3}\cdot\left[r_D-r_S-2M\cdot\ln\frac{r_D}{r_S}\right]. \quad (67)$$

As a result, the probability expression for radial propagation in Schwarzschild spacetime adopts the following form:

$$\mathcal{P}^{\text{Schwrzd}}_{\beta\alpha}=\begin{cases}\sin^2\Theta\sin^2\Big[\mathcal{S}^{(1)}_*(r_D-r_S)\Delta m^2_{12}+\dfrac{1}{2}\mathcal{S}^{(2)}_*(r_D-r_S)\Delta m^4_{12}\Big], & \beta\neq\alpha\\ 1-\sin^2\Theta\sin^2\Big[\mathcal{S}^{(1)}_*(r_D-r_S)\Delta m^2_{12}+\dfrac{1}{2}\mathcal{S}^{(2)}_*(r_D-r_S)\Delta m^4_{12}\Big], & \beta=\alpha\end{cases} \quad (68)$$

VII. PROPER DICTANCE

The neutrino propagates across its proper distance, while dr is simply a coordinate. The proper distance can be expressed as [14]

$$\mathrm{d}L_p=\sqrt{\left(\frac{g_{0\nu}g_{0\mu}}{g_{00}}-g_{\nu\mu}\right)\mathrm{d}x^\nu\mathrm{d}x^\mu}. \quad (69)$$

In the context of Johannsen spacetime, there exists

$$\mathrm{d}L_p=\sqrt{-g_{rr}\mathrm{d}r^2+\left(\frac{g^2_{t\phi}}{g_{tt}}-g_{\phi\phi}\right)\mathrm{d}\phi^2}. \quad (70)$$

The following expression is obtained by multiplying by dr^2

$$L_p=\int_{r_S}^{r_D}\sqrt{-g_{rr}+\left(\frac{g^2_{t\phi}}{g_{tt}}-g_{\phi\phi}\right)\frac{\dot\phi^2}{\dot r^2}}\mathrm{d}r. \quad (71)$$

With the use of four-velocity normalization, $g_{\mu\nu}\dot x^\mu\dot x^\nu=-1$, and considering the aforementioned expressions, $\dot r$ and $\dot\phi$ can be expressed as

$$\dot r=\frac{A_5(r)}{m\tilde\Sigma}\sqrt{R(r)}=\frac{\sqrt{A_5(r)}}{m\tilde\Sigma}\times\sqrt{\left[(r^2+a^2)A_1(r)E-aA_2(r)L\right]^2-\Delta\left[K-m^2(r^2+f(r))\right]}$$
$$\dot\phi=\frac{L}{\tilde\Sigma\sin^2\theta}\frac{\left[A_1(r)(r^2+a^2)-A_2(r)a^2\sin^2\theta\right]^2}{\left[(r^2+a^2)^2A_1^2(r)-a^2\Delta\sin^2\theta\right]} \quad (72)$$

It is apparent that performing an analytical calculation of Eq. (71) is more complex.

VIII. FINDINGS AND PROSPECTS FOR THE FUTURE

In this study, we delved into the mathematical aspects of the Dirac equation within a curved spacetime and investigated its application in analyzing neutrino oscillations. To achieve this objective, we used the WKB approximation. In particular, we devised a technique for determining the phase shift in flavor neutrino oscillations by employing a Taylor series expansion of the action, considering contributions up to fourth order in Δm^4. In Sec. IV, we examined the intricate dynamics of transition probabilities within our framework, revealing their intricate nature despite fluctuations in mass representation. Furthermore, this method has been employed to evaluate the variation in the phase difference of neutrino mass eigenstates caused by the gravitational field described by the Johannsen spacetime.

It is well known that the phenomenon of neutrino oscillation, which is a quantum phenomenon, takes place in both flat and curved spacetime metrics. It is anticipated that the presence of massive objects such as stars and black holes, with their gravitational fields, can impact the propagation of neutrinos and alter their oscillation patterns. Research on neutrino oscillation in curved spacetime is being actively pursued, and the resulting findings hold significant implications for the fields of astrophysics and cosmology. Furthermore, it is widely acknowledged that the application of gravitational lensing methods offers compelling proof of the presence of Dark Matter. According to the principles of general relativity, the trajectory of the light can be bent when encountering massive objects or gravitational fields. This deflection is closely linked to the mass of the object and can be likened to the focusing effect of a lens. In upcoming research, we will investigate the gravitational lensing effect on neutrino oscillations to provide evidence of the existence of dark matter.

We would like to emphasize the significant progress made in a previous study[38] regarding the intricacies of neutrino spin oscillations within a curved spacetime, particularly in the presence of background matter and an external electromagnetic field. The authors successfully derived the Dirac equation that governs neutrino oscillations under these external field conditions. They also conducted numerical solutions to explore the behavior of these oscillations. Numerical simulations and analysis of experimental data are crucial in elucidating the behavior of neutrinos in realistic scenarios. Neutrino oscillation experiments, such as those conducted at particle accelerators or involving atmospheric and solar neutrinos, provide invaluable data for testing theoretical predictions and gaining insights into the spin oscillations of massive Dirac neutrinos in diverse physical environments. This research requires a significant amount of time and extensive numerical calculations owing to the complexity of studying spin oscillations of massive Dirac neutrinos in the presence of background matter, electromagnetic fields, and gravitational fields. In future research, our objective is to explore and uncover the potential for spin precession in the presence of background matter and an external electromagnetic field within a curved spacetime. The derivation of the neutrino spin evolution equation provided in this context relies on the general spin evolution equation in the Heisenberg representation. Adopting this approach will allow for careful analysis of the contributions of various external fields previously mentioned to the evolution of neutrino spin.

References

[1] B. Pontecorvo, Soviet J. Exp. Theor. Phys. **26**, 984 (1968)
[2] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962)
[3] V. Bertone, R. Gauld, and J. Rojo, JHEP **01**, 217 (2019)
[4] S. R. Mohanty, S. Ghosh, and B. Kumar, arXiv: 2304.02439
[5] G. Lambiase, L. Mastrototaro, and L. Visinelli, arXiv:2306.16977
[6] Y. Fukuda *et al*., Phys. Lett. B **433**, 9 (1998)
[7] Q. R. Ahmad *et al*., Phys. Rev. Lett. **89**, 011301 (2002)
[8] Y. Fukuda *et al*., Phys. Rev. Lett. **81**, 1562 (1998)
[9] D. G. Michael *et al*., Phys. Rev. Lett. **97**, 191801 (2006)
[10] P. Adamson *et al*., Phys. Rev. D **73**, 072002 (2006)
[11] H. Swami, K. Lochan, and K. M. Patel, Phys. Rev. D **102**, 024043 (2020)
[12] H. Duan, G. M. Fuller, and Y.-Z. Qian, Ann. Rev. Nucl. Part. Sci. **60**, 569 (2010)
[13] J. Wudka, Mod. Phys. Lett. A **6**, 3291 (1991)
[14] C. Y. Cardall, G. M. Fuller, Phys. Rev. D **55**, 7960 (1997)
[15] N. Fornengo, C. Giunti, C. W. Kim *et al*., Nucl. Phys. B Proceedings Supplements **70**, 264 (1999)
[16] N. Fornengo, C. Giunti, C. W. Kim *et al*., Phys. Rev. D **56**, 1895 (1997)
[17] S. Mandal, Nucl. Phys. B **965**, 115338 (2021)
[18] H. Swami, Eur. Phys. J. C **82**, 974 (2022)
[19] S. Abe *et al*., Phys. Rev. Lett. **100**, 221803 (2008)
[20] T. Bhattacharya, S. Habib, and E. Mottola, Phys. Rev. D **59**, 067301 (1999)
[21] F. Hammad, P. Sadeghi, and N. Fleury, Phys. Rev. D **107**, 104015 (2023)
[22] H. Chakrabarty, D. Borah, A. Abdujabbarov *et al*., Eur. Phys. J. C **82**, 24 (2022)
[23] P. Collas and D. Klein, arXiv:2306.16977
[24] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley and Sons, New York, 1972)
[25] K. Konno and M. Kasai, Prog. Theor. Phys. **100**, 1145 (1998)
[26] I. Furuoya, Prog. Theor. Phys. **58**, 334 (1977)
[27] H. Green and Dirac matrices, Nucl. Phys. **7**, 373 (1958)
[28] N. J. Poplawski, *Covariant differentiation of spinors for a general affine connection*, 2007
[29] L. Visinelli, Gen. Rel. Gravit. **47**, 62 (2015)
[30] J. W. Van Orden, S. Jeschonnek, and J. Tjon, Phys. Rev. D **72**, 054020 (2005)
[31] L. Stodolsky, Gen. Rel. Grav. **11**, 391 (1979)
[32] J. Silk and L. Stodolsky, Phys. Lett. B **639**, 14 (2006)
[33] S. I. Godunov and G. S. Pastukhov, Phys. Atom. Nucl. **74**, 302 (2011)
[34] H. A. Weldon, Phys. Rev. D **63**, 104010 (2001)
[35] L. D. Landau and E. M. Lifshitz, T*he Classical Theory of Fields*, Course of Theoretical Physics, Volume 2 (Elsevier Butterworth-Heinemann, Oxford, 2004)
[36] J. Wudka, Phys. Rev. D **64**, 065009 (2001)
[37] H. Maiwa and S. Naka, *Neutrino oscillations in gravitational fields*, 2004
[38] M. Dvornikov, Phys. Rev. D **99**, 116021 (2019)
[39] M. Dvornikov, A. Studenikin, JHEP **09**, 016 (2002)
[40] E. K. Akhmedov and A. Wilhelm, JHEP **2013**, 165 (2013)
[41] D. V. Ahluwalia, Gen. Rel. Grav. **29**, 1491 (1997)
[42] D. V. Ahluwalia, C. Burgard, Gen. Rel. Grav. **28**, 1161 (1996)
[43] L. Johns, H. Nagakura, G. M. Fuller *et al*., Phys. Rev. D **101**, 043009 (2020)
[44] L. Buoninfante, G. Gaetano Luciano, L. Petruzziello *et al*., arXiv:2306.14023
[45] H. Chakrabarty, A. Chatrabhuti, D. Malafarina *et al*., JCAP **2023**, 018 (2023)
[46] E. Aydiner, *Anomalous Cyclic in the Neutrino Oscillations*, 2022
[47] P. Fisher, B. Kayser, and K. S. McFarland, Annu. Rev. Nucl. Part. Sci. **49**, 481 (1999)
[48] S. Bilenky, Nucl. Phys. B **908**, 2 (2016)
[49] G. Lambiase, L. Mastrototaro, and L. Visinelli, JCAP **2023**, 011 (2023)
[50] T. Johannsen, Phys. Rev. D **88**, 044002 (2013)
[51] C. Bambi, *Black Holes: A Laboratory for Testing Strong Gravity*, 2017
[52] A. B. Abdikamalov, D. Ayzenberg, C. Bambi *et al*., Astrophys. J. **878**, 91 (2019)
[53] A. B. Abdikamalov, D. Ayzenberg, C. Bambi *et al*., Astrophys. J. **899**, 80 (2020)
[54] S. Chandrasekhar, *The mathematical theory of black holes*, 1983