# Fused model of the alternating spin chain from ABJM theory*

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**Abstract:** In this paper, we present an algebraic construction of the fused model for the ABJM spin chain by gluing two adjacent quantum spaces and two original auxiliary spaces. We prove the integrability of the fused model by demonstrating the validity of the Yang-Baxter equation. Owing to the regularity property of the fused R-matrix, we successfully construct the boost operator for the fused model and obtain the third-order charge accordingly. We also investigate the open spin chain Hamiltonian for the fused model and indicate the general common structures of the boundary terms which are further used to analyze the integrability of the flavored ABJM Hamiltonian.

**Keywords:** ABJM theory, integrable system, AdS/CFT

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## I. INTRODUCTION

ABJM theory is a three-dimensional superconformal Chern-Simons theory with gravity dual as the type IIA superstring theory on an $AdS_4\times\mathbb{CP}^3$ background; it was proposed by Aharony, Bergman, Jafferis, and Maldacena in 2008 [1]. Soon after the discovery, the integrability of ABJM theory was determined, where the anomalous dimension matrix of the single trace operator composed of bi-fundamentals was mapped to an integrable closed spin chain of alternating types at planar two-loop order in the $SU(4)$ subsector [2, 3]. The integrability was later extended to the complete $Osp(6|4)$ sector and to all-loop order [4−6].

Intensive studies on the integrable models have resulted from the original ABJM spin chain with various nontrivial boundaries. For instance, in the orbifold ABJM theory, we have an integrable closed spin chain with a twisted boundary condition [7]. In the study of determinant-like operator in ABJM theory, we will treat an open spin chain Hamiltonian [8], whose integrability is proved by determining a concrete projected K-matrices in the framework of algebraic Bethe ansatz (ABA) [9]. In the flavored ABJM theory [10−12], we can construct the gauge invariant operator using fundamental/anti-fundamental flavors at two ends without the trace, and such an operator will correspond to an open spin chain, which is argued to be integrable using a coordinate Bethe ansatz (CBA) [13].

In the constructions of integrable models, the so-called boost operator has become an important object that connects different conserved charges through a recursive relation [14−16]. The boost operator can be also used to generate an integrable long range spin chain from a nearest-neighbour spin chain [17]. For the integrable model with a regular $R$-matrix, the boost operator can be easily established [18, 19]. However for the non-regular $R$-matrix, such as two of the four $R$-matrices adopted in the original ABJM spin chains, the existence of the boost operator is yet unknown. One of the major motivations for the present work is to determine a suitable boost operator for the ABJM spin chain model.

Motivated by Reference [20], in which a general algebraic treatment for medium-range spin chain was proposed, in this paper, we reformulate the original ABJM spin chain model with a local three-site interacting Hamiltonian by combining two adjacent quantum spaces into a new single one and thus obtain the fused ABJM model with nearest-neighbour interactions. We demonstrate the integrability of the fused model by presenting the concrete R-matrix and checking the validity of the Yang-Baxter equation. Owing to the regularity of the fused R-matrix, we can obtain the boost operator for the fused model and then use it to analyze the structure of the higher charges. We also discuss the existence of the boost operators in two sub-chains of the original ABJM model.

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Finally, we investigate the fused model for an open spin chain and attempt to determine some common structures of the boundary terms through a careful calculation of the open spin chain Hamiltonian.

The remainder of this paper is organized as follows: In section II, we briefly review the ABJM spin chain model. In section III, we present the details of the construction of the fused model for the ABJM spin chain and discuss the boost operator in the fused and original ABJM models. In section IV, we study the fused model for the open spin chain and present the concrete spin chain Hamiltonian. We also analyze the structures of the boundary terms and discuss the integrability of the flavored ABJM spin chain from the algebraic aspects. In the final section, we present the conclusion and indicate some future research directions.

## II. REVIEW OF THE ABJM SPIN CHAIN

In this section, we review the spin chain model originating from ABJM theory [2, 3]. The quantum space in each site of the spin chain is a representation space of the $SU(4)$ group, alternating from fundamental representation "**4**" to an anti-fundamental one "$\bar{\mathbf{4}}$".

There are four types of $R$-matrices:

$$\begin{aligned} &R_{ab}(u)=u\mathbb{I}_{ab}+\mathbb{P}_{ab}, \quad R_{a\bar{b}}(u)=-(u+2)\mathbb{I}_{a\bar{b}}+\mathbb{K}_{a\bar{b}},\\ &R_{\bar{a}\bar{b}}(u)=u\mathbb{I}_{\bar{a}\bar{b}}+\mathbb{P}_{\bar{a}\bar{b}}, \quad R_{\bar{a}b}(u)=-(u+2)\mathbb{I}_{\bar{a}b}+\mathbb{K}_{\bar{a}b}, \end{aligned} \tag{1}$$

where the subscripts $a(\bar{a})$ or $b(\bar{b})$ indicate they belong to $\mathbf{4}(\bar{\mathbf{4}})$ representation spaces. $\mathbb{P}$ and $\mathbb{K}$ are permutation and trace operators defined using the standard basis matrices $\{e_{ij}, i,j=1,2,3,4\}$ as

$$\mathbb{P}=e_{ij}\otimes e_{ji}, \quad \mathbb{K}=e_{ij}\otimes e_{ij}, \tag{2}$$

where the repeated indices are summed implicitly. These $R$-matrices satisfy a total of eight Yang-Baxter equations, which are expressed concisely as

$$\begin{aligned} &R_{AB}(\lambda_A-\lambda_B)R_{AC}(\lambda_A-\lambda_C)R_{BC}(\lambda_B-\lambda_C)\\ &=R_{BC}(\lambda_B-\lambda_C)R_{AC}(\lambda_A-\lambda_C)R_{AB}(\lambda_A-\lambda_B), \end{aligned} \tag{3}$$

where $A=\{a,\bar{a}\}$, $B=\{b,\bar{b}\}$, $C=\{c,\bar{c}\}$.

For the closed alternating spin chain with $2L$ sites, we have the following two monodromy matrices:

$$\begin{aligned} &T_0(u)=R_{01}(u)R_{0\bar{2}}(u)\cdots R_{0,2L-1}(u)R_{0,\overline{2L}}(u),\\ &\bar{T}_{\bar{0}}(u)=R_{\bar{0}1}(u)R_{\bar{0}\bar{2}}(u)\cdots R_{\bar{0},2L-1}(u)R_{\bar{0},\overline{2L}}(u), \end{aligned} \tag{4}$$

where $V_0$ and $V_{\bar{0}}$ are auxiliary spaces, and the corresponding transfer matrices are $\tau(u)=\mathrm{Tr}_0 T_0(u)$ and $\bar{\tau}(u)=\mathrm{Tr}_{\bar{0}}\bar{T}_{\bar{0}}(u)$. Owing to the Yang-Baxter relations (3), the transfer matrices commute with each other for arbitrary spectral parameters:

$$\begin{aligned} &[\tau(u),\tau(v)]=0, \quad [\bar{\tau}(u),\bar{\tau}(v)]=0,\\ &[\tau(u),\bar{\tau}(v)]=0. \quad \forall u,v\in\mathbb{C}. \end{aligned} \tag{5}$$

The Hamiltonian of the ABJM spin chain model is obtained from $\tau(u)$ and $\bar{\tau}(u)$ as

$$H_{\text{ABJM}}=\frac{\mathrm{d}}{\mathrm{d}u}\log\tau(u)\bigg|_{u=0}+\frac{\mathrm{d}}{\mathrm{d}u}\log\bar{\tau}(u)\bigg|_{u=0}, \tag{6}$$

and a direct computation yields its concrete expression:

$$H_{\text{ABJM}}=\sum_{l=1}^{2L}\left(\mathbb{P}_{l,l+2}-\frac{1}{2}\mathbb{K}_{l,l+1}\mathbb{P}_{l,l+2}-\frac{1}{2}\mathbb{P}_{l,l+2}\mathbb{K}_{l,l+1}\right), \tag{7}$$

which, up to an overall prefactor and a constant term, is exactly the anomalous dimension matrix of the dilatational operator in ABJM theory. Furthermore, we observe that $H_{\text{ABJM}}$ is a three-site interacting model with next-to-nearest local Hamiltonian density,

$$h_{l,l+1,l+2}=\mathbb{P}_{l,l+2}-\frac{1}{2}\mathbb{K}_{l,l+1}\mathbb{P}_{l,l+2}-\frac{1}{2}\mathbb{P}_{l,l+2}\mathbb{K}_{l,l+1}. \tag{8}$$

## III. CONSTRUCTION OF THE FUSED MODEL

In this section, we present the algebraic construction of the fused model of the ABJM spin chain by introducing the Lax operator, $R$-matrix, and boost operator.

### A. Lax operator and *R*-matrix

In Eq. (6), the Hamiltonian of the ABJM spin chain is expressed as the sum of the conserved charges from two different transfer matrices. In the following, we construct a fused model for the ABJM spin chain in which the Hamiltonian $H_{\text{ABJM}}$ is generated from a single transfer matrix. First, we multiply the original two monodromy matrices $T_0(u)$ and $\bar{T}_{\bar{0}}(u)$ to obtain a new one $\mathbf{T}_{0\bar{0}}(u)=T_0(u)\bar{T}_{\bar{0}}(u)$. Subsequently, we rearrange the positions of R-matrices in $\mathbf{T}_{0\bar{0}}(u)$ to obtain:

$$\begin{aligned} \mathbf{T}_{0\bar{0}}(u)=&(R_{01}(u)R_{\bar{0}1}(u)R_{0\bar{2}}(u)R_{\bar{0}\bar{2}}(u))\cdots\\ &\times\left(R_{0,2L-1}(u)R_{\bar{0},2L-1}(u)R_{0,\overline{2L}}(u)R_{\bar{0},\overline{2L}}(u)\right). \end{aligned} \tag{9}$$

Thus, we find that if we treat the tensor product of two nearest quantum spaces as an enlarged new quantum space, such as $V_{2j-1,\overline{2j}}=V_{2j-1}\otimes V_{\overline{2j}}$, and introduce a new auxiliary space as $V_{0\bar{0}}=V_0\otimes V_{\bar{0}}$, we can define a new Lax operator on the tensor product space $V_{0\bar{0}}\otimes V_{2j-1,\overline{2j}}$ as

$$\mathcal{L}_{(0\bar{0}),(2j-1,\overline{2j})}(u)=R_{0,2j-1}(u)R_{0,\overline{2j}}(u)R_{\bar{0},2j-1}(u)R_{\bar{0},\overline{2j}}(u), \quad (10)$$

or in a more general form,

$$\mathcal{L}_{(a\bar{a}),(b\bar{b})}(u)=R_{ab}(u)R_{a\bar{b}}(u)R_{\bar{a}b}(u)R_{\bar{a}\bar{b}}(u), \quad (11)$$

and then $\mathbf{T}_{0\bar{0}}(u)$ becomes

$$\mathbf{T}_{0\bar{0}}(u)=\mathcal{L}_{(0\bar{0}),(1\bar{2})}(u)\mathcal{L}_{(0\bar{0}),(3\bar{4})}(u)\cdots\mathcal{L}_{(0\bar{0}),(2L-1,\overline{2L})}(u). \quad (12)$$

Hence, we can observe that $\mathbf{T}_{0\bar{0}}(u)$ represents a new spin chain of length $L$ with isomorphic auxiliary and quantum spaces in each site:

$$V_{0\bar{0}}\cong V_{2j-1,\overline{2j}}=\mathbf{4}\otimes\bar{\mathbf{4}}, \quad j=1,2,\cdots,L. \quad (13)$$

More importantly, the above spin chain is integrable because the following $R$-matrix

$$\mathcal{R}_{(a\bar{a}),(b\bar{b})}(u)=R_{a\bar{b}}(u)R_{ab}(u)R_{\bar{a}\bar{b}}(u)R_{\bar{a}b}(u) \quad (14)$$

makes the "RLL" exchange relation hold:

$$\begin{aligned}&\mathcal{R}_{(a\bar{a}),(b\bar{b})}(u-v)\mathcal{L}_{(a\bar{a}),(c\bar{c})}(u)\mathcal{L}_{(b\bar{b}),(c\bar{c})}(v)\\=&\mathcal{L}_{(b\bar{b}),(c\bar{c})}(v)\mathcal{L}_{(a\bar{a}),(c\bar{c})}(u)\mathcal{R}_{(a\bar{a}),(b\bar{b})}(u-v).\end{aligned} \quad (15)$$

As a consistency condition, the $R$-matrix (14) itself should satisfy the Yang-Baxter relation:

$$\begin{aligned}&\mathcal{R}_{(a\bar{a}),(b\bar{b})}(u-v)\mathcal{R}_{(a\bar{a}),(c\bar{c})}(u)\mathcal{R}_{(b\bar{b}),(c\bar{c})}(v)\\=&\mathcal{R}_{(b\bar{b}),(c\bar{c})}(v)\mathcal{R}_{(a\bar{a}),(c\bar{c})}(u)\mathcal{R}_{(a\bar{a}),(b\bar{b})}(u-v).\end{aligned} \quad (16)$$

Using the Yang-Baxter equations (3), the above RLL relation (15) and Yang-Baxter relation (16) can be verified straightforwardly. Notice that the Lax operator $\mathcal{L}_{(a\bar{a}),(b\bar{b})}(u)$, although considerably similar to $\mathcal{R}_{(a\bar{a}),(b\bar{b})}(u)$, does not obey the intertwining relation:

$$\begin{aligned}&\mathcal{L}_{(a\bar{a}),(b\bar{b})}(u-v)\mathcal{L}_{(a\bar{a}),(c\bar{c})}(u)\mathcal{L}_{(b\bar{b}),(c\bar{c})}(v)\\\neq&\mathcal{L}_{(b\bar{b}),(c\bar{c})}(v)\mathcal{L}_{(a\bar{a}),(c\bar{c})}(u)\mathcal{L}_{(a\bar{a}),(b\bar{b})}(u-v).\end{aligned} \quad (17)$$

We have thus far established a new integrable model, which will be called the fused model because both the auxiliary and quantum spaces are the fusion of two neighboring representation spaces. The transfer matrix of the fused model is simply the multiplication of two original ones,

$$t(u)=\mathrm{Tr}_{0\bar{0}}\mathbf{T}_{0\bar{0}}(u)=\tau(u)\bar{\tau}(u), \quad (18)$$

and the Hamiltonian generated from $t(u)$ is $H_{\mathrm{ABJM}}$. However, from the perspective of the fused model, $H_{\mathrm{ABJM}}$ becomes a nearest-neighbour interacting model:

$$H_{\mathrm{ABJM}}=\sum_{j=1}^{L}H_{(2j-1,\overline{2j}),(2j+1,\overline{2j+2})}, \quad (19)$$

where the local Hamiltonian density is

$$H_{(2j-1,\overline{2j}),(2j+1,\overline{2j+2})}=h_{2j-1,\overline{2j},2j+1}+h_{\overline{2j},2j+1,\overline{2j+2}}. \quad (20)$$

Now, let us switch back to the original spin chain, where the nearest two quantum spaces $\mathbf{4}$ and $\bar{\mathbf{4}}$ are separate sites. Subsequently, through construction, the transfer matrix $t(u)$ for fused model is two-site translation invariant. However, we can observe that the Lax operator has the factorized form

$$\mathcal{L}_{(0\bar{0}),(2j-1,\overline{2j})}(u)=\mathcal{L}_{(0\bar{0}),2j-1}(u)\mathcal{L}_{(0\bar{0}),\overline{2j}}(u), \quad (21)$$

where

$$\begin{aligned}\mathcal{L}_{(0\bar{0}),2j-1}(u)&=R_{0,2j-1}(u)R_{\bar{0},2j-1}(u),\\\mathcal{L}_{(0\bar{0}),\overline{2j}}(u)&=R_{0,\overline{2j}}(u)R_{\bar{0},\overline{2j}}(u),\end{aligned} \quad (22)$$

and thus the transfer matrix can be expressed as

$$t(u)=\mathrm{Tr}_{0\bar{0}}\mathcal{L}_{(0\bar{0}),1}(u)\mathcal{L}_{(0\bar{0}),\bar{2}}(u)\cdots\mathcal{L}_{(0\bar{0}),2L-1}(u)\mathcal{L}_{(0\bar{0}),\overline{2L}}(u), \quad (23)$$

which is clearly a one-site shift invariant.

We mention one last point: We can also use $\mathcal{R}_{(a\bar{a}),(b\bar{b})}(u)$ as the Lax operator to generate a new integrable spin chain, as shown below:

$$\tilde{t}(u)=\mathrm{Tr}_{0\bar{0}}\mathcal{R}_{(0\bar{0}),(1\bar{2})}(u)\mathcal{R}_{(0\bar{0}),(3\bar{4})}(u)\cdots\mathcal{R}_{(0\bar{0}),(2L-1,\overline{2L})}(u). \quad (24)$$

The relation between $t(u)$ and $\tilde{t}(u)$ can be determined as follows: Notice that

$$\mathcal{R}_{(0\bar{0}),(2j-1,\overline{2j})}(u)=R_{2j-1,\overline{2j}}(0)\mathcal{L}_{(0\bar{0}),(2j-1,\overline{2j})}(u)R^{-1}_{2j-1,\overline{2j}}(0), \quad (25)$$

where the similarity transformation matrix $R_{2j-1,\overline{2j}}(0)=-2+\mathbb{K}_{2j-1,\overline{2j}}$ has the properties

$$R_{2j-1,\overline{2j}}(0)^T=R_{2j-1,\overline{2j}}(0), \quad R_{2j-1,\overline{2j}}(0)^TR_{2j-1,\overline{2j}}(0)=4\mathbb{I}_{2j-1,\overline{2j}}, \quad (26)$$

and thus can be observed as a rotation in local quantum space $V_{2j-1,\overline{2j}}=V_{2j-1}\otimes V_{\overline{2j}}$. Subsequently, we observe that $t(u)$ and $\tilde{t}(u)$ are related by a global rotation in the entire

Hilbert space $\otimes_{j=1}^{L} V_{2j-1,\overline{2j}}$:

$$\tilde{t}(u) = \Lambda t(u)\Lambda^{-1}, \tag{27}$$

where

$$\Lambda = R_{1\bar{2}}(0)R_{3\bar{4}}(0)\cdots R_{2L-1,\overline{2L}}(0), \tag{28}$$

so is the Hamiltonian $\tilde{H}_{\rm ABJM}$ obtained from $\tilde{t}(u)$ and $H_{\rm ABJM}$:

$$\tilde{H}_{\rm ABJM} = \Lambda H_{\rm ABJM}\Lambda^{-1}. \tag{29}$$

### B. Boost operator

Now, we investigate the boost operator for the fused model, which can be used to generate higher conserved charges.

From the definition of the fused $R$-matrix in (14), we obtain

$$\mathcal{R}_{(a\bar{a}),(b\bar{b})}(0) = 4\mathbb{P}_{ab}\mathbb{P}_{\bar{a}\bar{b}}. \tag{30}$$

The $R$-matrix with the above condition is often called regular. For the integrable model with a regular $R$-matrix, the method of constructing the boost operator is well-known in the literature [18] and is applied to our fused model as follows: First, the Lax operator can be shown to be $P$-symmetric, that is

$$\mathcal{L}_{(a\bar{a}),(b\bar{b})}(u) = \mathcal{L}_{(b\bar{b}),(a\bar{a})}(u), \tag{31}$$

and thus the RLL relation in (15) can be expressed as

$$\begin{aligned}&\mathcal{R}_{(1\bar{2}),(3\bar{4})}(v)\mathcal{L}_{(0\bar{0}),(1\bar{2})}(u+v)\mathcal{L}_{(0\bar{0}),(3\bar{4})}(u)\\ &= \mathcal{L}_{(0\bar{0}),(3\bar{4})}(u)\mathcal{L}_{(0\bar{0}),(1\bar{2})}(u+v)\mathcal{R}_{(1\bar{2}),(3\bar{4})}(v).\end{aligned} \tag{32}$$

Subsequently, by taking the derivative with respect to $v$ on both sides and setting $v=0$ in the end, we obtain

$$\begin{aligned}&\dot{\mathcal{R}}_{(1\bar{2}),(3\bar{4})}(0)\mathcal{L}_{(0\bar{0}),(1\bar{2})}(u)\mathcal{L}_{(0\bar{0}),(3\bar{4})}(u)\\ &+\mathcal{R}_{(1\bar{2}),(3\bar{4})}(0)\dot{\mathcal{L}}_{(0\bar{0}),(1\bar{2})}(u)\mathcal{L}_{(0\bar{0}),(3\bar{4})}(u)\\ &= \mathcal{L}_{(0\bar{0}),(3\bar{4})}(u)\dot{\mathcal{L}}_{(0\bar{0}),(1\bar{2})}(u)\mathcal{R}_{(1\bar{2}),(3\bar{4})}(0)\\ &+\mathcal{L}_{(0\bar{0}),(3\bar{4})}(u)\mathcal{L}_{(0\bar{0}),(1\bar{2})}(u)\dot{\mathcal{R}}_{(1\bar{2}),(3\bar{4})}(0).\end{aligned} \tag{33}$$

Multiplying $\mathcal{R}_{(1\bar{2}),(3\bar{4})}(0)$ on both sides from the left and using the regularity condition $\mathcal{R}_{(1\bar{2}),(3\bar{4})}(0) = 4\mathbb{P}_{13}\mathbb{P}_{\bar{2}\bar{4}}$, we obtain

$$\begin{aligned}&\left[\frac{1}{16}\mathcal{R}_{(1\bar{2}),(3\bar{4})}(0)\dot{\mathcal{R}}_{(1\bar{2}),(3\bar{4})}(0), \mathcal{L}_{(0\bar{0}),(1\bar{2})}(u)\mathcal{L}_{(0\bar{0}),(3\bar{4})}(u)\right]\\ &= -\dot{\mathcal{L}}_{(0\bar{0}),(1\bar{2})}(u)\mathcal{L}_{(0\bar{0}),(3\bar{4})}(u) + \mathcal{L}_{(0\bar{0}),(1\bar{2})}(u)\dot{\mathcal{L}}_{(0\bar{0}),(3\bar{4})}(u),\end{aligned} \tag{34}$$

where $\frac{1}{16}\mathcal{R}_{(1\bar{2}),(3\bar{4})}(0)\dot{\mathcal{R}}_{1\bar{2},3\bar{4}}(0)$ is simply the local Hamiltonian $H_{(1\bar{2}),(3\bar{4})}$ plus an identity operator; thus, it can be replaced by $H_{(1\bar{2}),(3\bar{4})}$ in the commutation relation. Through substitution of the indices, $1\to 2k-1$, $2\to 2k$, $3\to 2k+1$, $4\to 2k+2$, we obtain

$$\begin{aligned}&\left[H_{(2k-1,\overline{2k}),(2k+1,\overline{2k+2})}, \mathcal{L}_{(0\bar{0}),(2k-1,\overline{2k})}(u)\mathcal{L}_{(0\bar{0}),(2k+1,\overline{2k+2})}(u)\right] =\\ &-\dot{\mathcal{L}}_{(0\bar{0}),(2k-1,\overline{2k})}(u)\mathcal{L}_{(0\bar{0}),(2k+1,\overline{2k+2})}(u)\\ &+\mathcal{L}_{(0\bar{0}),(2k-1,\overline{2k})}(u)\dot{\mathcal{L}}_{(0\bar{0}),(2k+1,\overline{2k+2})}(u).\end{aligned} \tag{35}$$

Subsequently, by multiplying $\prod_{j=1}^{k-1}\mathcal{L}_{(0\bar{0}),(2j-1,\overline{2j})}(u)$ on the left and $\prod_{j=k+2}^{L}\mathcal{L}_{(0\bar{0}),(2j-1,\overline{2j})}(u)$ on the right to both sides of the above equation, we obtain

$$\begin{aligned}&\left[H_{(2k-1,\overline{2k}),(2k+1,\overline{2k+2})}, \mathbf{T}_{0\bar{0}}(u)\right] =\\ &-\left[\prod_{j=1}^{k-1}\mathcal{L}_{(0\bar{0}),(2j-1,\overline{2j})}(u)\right]\dot{\mathcal{L}}_{(0\bar{0}),(2k-1,\overline{2k})}(u)\left[\prod_{j=k+1}^{L}\mathcal{L}_{(0\bar{0}),(2j-1,\overline{2j})}(u)\right]\\ &+\left[\prod_{j=1}^{k}\mathcal{L}_{(0\bar{0}),(2j-1,\overline{2j})}(u)\right]\dot{\mathcal{L}}_{(0\bar{0}),(2k+1,\overline{2k+2})}(u)\\ &\times\left[\prod_{j=k+2}^{L}\mathcal{L}_{(0\bar{0}),(2j-1,\overline{2j})}(u)\right].\end{aligned} \tag{36}$$

Finally, by summing up the above equation for each $k=1,2,\cdots,L-1$, we find

$$\begin{aligned}&\left[\sum_{k=1}^{L-1} kH_{(2k-1,\overline{2k}),(2k+1,\overline{2k+2})}, \mathbf{T}_{0\bar{0}}(u)\right] =\\ &-\frac{\mathrm{d}\mathbf{T}_{0\bar{0}}(u)}{\mathrm{d}u} + L\left[\prod_{j=1}^{L-1}\mathcal{L}_{0\bar{0},(2j-1,\overline{2j})}(u)\right]\dot{\mathcal{L}}_{0\bar{0},(2L-1,\overline{2L})}(u).\end{aligned} \tag{37}$$

For an infinite spin chain or closed spin chain, we obtain

$$\frac{\mathrm{d}t(u)}{\mathrm{d}u} = [\mathcal{B}, t(u)], \tag{38}$$

where $\mathcal{B}$ is the boost operator defined as

$$\mathcal{B} = -\sum_{k} kH_{(2k-1,\overline{2k}),(2k+1,\overline{2k+2})}. \tag{39}$$

The conserved charges are defined as the coefficients of the Taylor expansion of $\log t(u)$ at $u=0$,

$$\log t(u) = \mathrm{i}\sum_{n=0} Q_{n+1}u^n, \tag{40}$$

subsequently, the first charge $Q_1$ is simply $-\mathrm{i}\log t(0)$, which is non-local because

$$t(0) = 4^L \mathbb{P}_{2L-3,2L-1}\cdots\mathbb{P}_{13}\mathbb{P}_{\overline{2L-2},\overline{2L}}\cdots\mathbb{P}_{\bar{2}\bar{4}} \tag{41}$$

is the shift operator acting on the entire spin chain. The second charge $Q_2$ corresponds to the spin chain Hamiltonian:

$$\mathrm{i}Q_2 = \frac{\mathrm{d}}{\mathrm{d}u}\log t(u)\bigg|_{u=0} = H_{\mathrm{ABJM}}. \tag{42}$$

The remaining higher charges can be derived using the boost operator $\mathcal{B}$ from the relation (38) as follows:

$$Q_{n+1} = \frac{1}{n}[\mathcal{B}, Q_n], \quad n = 2,3,\cdots \tag{43}$$

Thus, the next charge $Q_3$ is determined to be

$$\begin{aligned} 2Q_3 =& -\mathrm{i}\sum_j \left[H_{(2j-1,\overline{2j}),(2j+1,\overline{2j+2})},\, H_{(2j+1,\overline{2j+2}),(2j+3,\overline{2j+4})}\right] \\ =& -\mathrm{i}\sum_j \Big\{ \left[h_{\overline{2j},2j+1,\overline{2j+2}},\, h_{2j+1,\overline{2j+2},2j+3} + h_{\overline{2j+2},2j+3,\overline{2j+4}}\right] \\ &+\Delta(j)\Big\}, \end{aligned} \tag{44}$$

where $\Delta(j)$ is a local operator with the interaction range over five sites:

$$\Delta(j) = \left[h_{2j-1,\overline{2j},2j+1},\, h_{2j+1,\overline{2j+2},2j+3}\right]. \tag{45}$$

The form of the third charge $Q_3$ shown above seemingly violates the generalized Reshetikhin condition proposed in [20] for the integrable three-site model, in which $\Delta(j)$ is conjectured to be a three-site operator. However, we would like to emphasize that our fused model is essentially a two-site model with auxiliary and quantum spaces isomorphic to the tensor product space $\mathbf{4}\otimes\bar{\mathbf{4}}$. Thus, even in its factorized form (23), we require two different reduced Lax operators $\mathcal{L}_{(a\bar{a}),b}(u)$ and $\mathcal{L}_{(a\bar{a}),\bar{b}}(u)$, which are also not regular:

$$\mathcal{L}_{(a\bar{a}),b}(0) = \mathbb{P}_{ab}(-2+\mathbb{K}_{\bar{a}b}), \quad \mathcal{L}_{(a\bar{a}),\bar{b}}(0) = (-2+\mathbb{K}_{a\bar{b}})\mathbb{P}_{\bar{a}\bar{b}}. \tag{46}$$

These major differences indicate that the fused model investigated in this work does not belong to the normal three-site interacting model considered in [20]; thus, it does not obey the conjecture.

As a final remark on the boost operator of ABJM spin chain, let us consider the possibility of the existence of boost operator in either of the two sub-chains, $\tau(u)$ or $\bar{\tau}(u)$. If the boost operator we are seeking is a "matrix" type operator and is composed of the local inhomogeneous density, we can express it in a very general form; for instance, the boost operator $\mathfrak{b}$ for $\tau(u)$ can be expressed as

$$\mathfrak{b} = \sum_j f(j)b_{j,j+1,\cdots,j+l-1}, \tag{47}$$

where $f(j)$ is a function of the site position representing the inhomogeneity of the operator, and $b_{j,j+1,\cdots,j+l-1}$ is the local density with the interaction range over $l$ sites starting at the $j$-th site. The boost operator $\mathfrak{b}$ should satisfy the condition

$$\frac{\mathrm{d}\tau(u)}{\mathrm{d}u} = [\mathfrak{b}, \tau(u)], \quad \forall u \in \mathbb{C}. \tag{48}$$

At a special point $u=0$, because $\left[\tau^{-1}(0), \dot{\tau}(0)\right] = 0$, the above equation will reduce to

$$\tau^{-1}(0)\mathfrak{b}\tau(0) + \tau(0)\mathfrak{b}\tau^{-1}(0) = 2\mathfrak{b}. \tag{49}$$

Note that $\tau(0)$ is no longer a shift operator but has the following form:

$$\tau(0) = (-2+\mathbb{K}_{1\bar{2}})\cdots\left(-2+\mathbb{K}_{2L-1,\overline{2L}}\right)\mathbb{P}_{2L-3,2L-1}\cdots\mathbb{P}_{35}\mathbb{P}_{13}, \tag{50}$$

thus, $\tau(0)b_{j,j+1,\cdots,j+l-1}\tau^{-1}(0)$ not only shifts the sites at which $b_{j,j+1,\cdots,j+l-1}$ acts, but also increases its interaction range. Consequently, the interaction range of the local density on both sides of (49) cannot be equal. Therefore, we conclude that no matrix type boost operator with inhomogeneous local density exists for integrable spin chain $\tau(u)$ or $\bar{\tau}(u)$.

## IV. OPEN SPIN CHAIN HAMILTONIAN FROM THE FUSED MODEL

In this section, we study the open spin chain Hamiltonian for the fused model. First, we review the construction of the Hamiltonian for the ordinary $2L$-sites alternating spin chain with open boundaries. We require the following two so-called double row transfer matrices [21]:

$$\tau_b(u) = \mathrm{Tr}_0 K_0^+(u) T_0(u) K_0^-(u) T_0^{-1}(-u),$$

$$\bar{\tau}_b(u) = \mathrm{Tr}_{\bar{0}} \bar{K}_{\bar{0}}^+(u) \bar{T}_{\bar{0}}(u) \bar{K}_{\bar{0}}^-(u) \bar{T}_{\bar{0}}^{-1}(-u), \quad (51)$$

where the symbol "$b$" is used to distinguish them from the closed spin chain transfer matrices defined in Eq. (4). $K_0^+(u)(\bar{K}_{\bar{0}}^+(u))$ and $K_0^-(u)(\bar{K}_{\bar{0}}^-(u))$ are reflection matrices accounting for the left and the right boundary local Hamiltonian, respectively. The two right boundary reflection matrices $K_0^-(u)$ and $\bar{K}_{\bar{0}}^-(u)$ satisfy four reflection equations (REs) given below:

$$\begin{aligned}
R_{ab}(u-v)K_a^-(u)R_{ba}(u+v)K_b^-(v) &= K_b^-(v)R_{ab}(u+v)K_a^-(u)R_{ba}(u-v),\\
R_{\bar{a}\bar{b}}(u-v)\bar{K}_{\bar{a}}^-(u)R_{\bar{b}\bar{a}}(u+v)\bar{K}_{\bar{b}}^-(v) &= \bar{K}_{\bar{b}}^-(v)R_{\bar{a}\bar{b}}(u+v)\bar{K}_{\bar{a}}^-(u)R_{\bar{b}\bar{a}}(u-v),\\
R_{a\bar{b}}(u-v)K_a^-(u)R_{\bar{b}a}(u+v)\bar{K}_{\bar{b}}^-(v) &= \bar{K}_{\bar{b}}^-(v)R_{a\bar{b}}(u+v)K_a^-(u)R_{\bar{b}a}(u-v),\\
R_{\bar{a}b}(u-v)\bar{K}_{\bar{a}}^-(u)R_{b\bar{a}}(u+v)K_b^-(v) &= K_b^-(v)R_{\bar{a}b}(u+v)\bar{K}_{\bar{a}}^-(u)R_{b\bar{a}}(u-v),
\end{aligned} \quad (52)$$

while the other two left boundary reflection matrices $K_0^+(u)$ and $\bar{K}_{\bar{0}}^+(u)$ satisfy similar dual REs and can be obtained from $K_0^-(u)$ and $\bar{K}_{\bar{0}}^-(u)$ using some isomorphism transformations. Owing to these reflection relations, the transfer matrices form the commutating class

$$[\tau_b(u), \tau_b(v)] = 0, \quad [\bar{\tau}_b(u), \bar{\tau}_b(v)] = 0,$$

$$[\tau_b(u), \bar{\tau}_b(v)] = 0, \quad \forall u, v \in \mathbb{C}. \quad (53)$$

revealing the integrability of the open spin chain model. Subsequently, the boundary Hamiltonian from the original two open spin chains is given as

$$H_b = \frac{\mathrm{d}}{\mathrm{d}u} \log\left(\tau_b(u)\bar{\tau}_b(u)\right)\Big|_{u=0}. \quad (54)$$

Now, we consider the open spin chain from the fused model. Because the quantum and auxiliary spaces are both $\mathbf{4}\otimes\bar{\mathbf{4}}$, we introduce two reflection matrices $K_{0\bar{0}}^+(u)$ and $K_{0\bar{0}}^-(u)$ defined on tensor product space $V_0 \otimes V_{\bar{0}}$, of which $K_{0\bar{0}}^-(u)$ satisfies the RE

$$\begin{aligned}
&\mathcal{R}_{(a\bar{a}),(b\bar{b})}(u-v)K_{a\bar{a}}^-(u)\mathcal{R}_{(b\bar{b}),(a\bar{a})}(u+v)K_{b\bar{b}}^-(v)\\
&= K_{b\bar{b}}^-(v)\mathcal{R}_{(a\bar{a}),(b\bar{b})}(u+v)K_{a\bar{a}}^-(u)\mathcal{R}_{(b\bar{b}),(a\bar{a})}(u-v),
\end{aligned} \quad (55)$$

and $K_{0\bar{0}}^+(u)$ satisfies a similar dual RE. The double row transfer matrix for our fused model is

$$t_b(u) = \mathrm{Tr}_{0\bar{0}} K_{0\bar{0}}^+(u)\mathbf{T}_{0\bar{0}}(u)K_{0\bar{0}}^-(u)\mathbf{T}_{0\bar{0}}^{-1}(-u), \quad (56)$$

and the open spin chain Hamiltonian is then obtained as

$$\tilde{H}_b = \frac{\mathrm{d}}{\mathrm{d}u}\log t_b(u)\Big|_{u=0}. \quad (57)$$

We can observe that, unlike the closed spin chain case, the fused Hamiltonian $\tilde{H}_b$ is different from the Hamiltonian $H_b$ composed of two original spin chains. Hence, it represents two different open spin chain models. Although in very rare cases, with highly constrained reflection matrices, the two open spin chain models could be equivalent. For instance, if we assume that $K_{0\bar{0}}^+(u) = K_0^+(u)\bar{K}_{\bar{0}}^+(u)$ and $K_{0\bar{0}}^-(u) = K_0^-(u)\bar{K}_{\bar{0}}^-(u)$, the transfer matrix reduces to

$$\begin{aligned}
t_b(u) &= \mathrm{Tr}_{0\bar{0}} K_0^+(u)\bar{K}_{\bar{0}}^+(u)T_0(u)\bar{T}_{\bar{0}}(u)K_0^-(u)\bar{K}_{\bar{0}}^-(u)\bar{T}_{\bar{0}}^{-1}(-u)T_0^{-1}(-u)\\
&= \mathrm{Tr}_{0\bar{0}} K_0^+(u)T_0(u)K_0^-(u)\left[\bar{K}_{\bar{0}}^+(u)\bar{T}_{\bar{0}}(u)\bar{K}_{\bar{0}}^-(u)\bar{T}_{\bar{0}}^{-1}(-u)\right]T_0^{-1}(-u).
\end{aligned} \quad (58)$$

Therefore, if we further impose the condition

$$\left[\bar{K}_{\bar{0}}^+(u)\bar{T}_{\bar{0}}(u)\bar{K}_{\bar{0}}^-(u)\bar{T}_{\bar{0}}^{-1}(-u),\, T_0^{-1}(-u)\right] = 0, \quad (59)$$

then we will have $t_b(u) = \tau_b(u)\bar{\tau}_b(u)$, which leads to the same Hamiltonian.

We also note that, to address the open spin chain model with degrees of freedom on the boundary, the reflection $K$-matrices will also act on an additional internal space. By tracing over the auxiliary spaces $V_0 \otimes V_{\bar{0}}$, we can achieve the interactions between the boundary and bulk. Hence, in the most general settings, the reflection matrices should be treated as an operator-valued matrix on auxiliary space.

Finally, we discuss the concrete open spin chain Hamiltonian derived from the fused model transfer matrix (57). Because the entire calculation is straightforward but quite tedious, here we directly present the final results. For an open spin chain of length $2L$ with internal degrees of freedom acting on the boundaries, the complete Hamiltonian $\tilde{H}_b$ consists of three parts: the left boundary term $H_l$, bulk Hamiltonian $H_{\rm in}$, and right boundary term $H_r$:

$$\tilde{H}_b = H_l + H_{\rm in} + H_r. \tag{60}$$

The bulk part is simply the ordinary closed spin chain Hamiltonian

$$H_{\rm in} = \sum_{l=1}^{2L-4} \left(2\mathbb{P}_{l,l+2} - \mathbb{P}_{l,l+2}\mathbb{K}_{l,l+1} - \mathbb{K}_{l,l+1}\mathbb{P}_{l,l+2}\right). \tag{61}$$

The left boundary term can be further organized into three parts:

$$H_l = H_{l1} + H_{l2} + H_{l3}. \tag{62}$$

Among them, $H_{l1}$ is a pure left boundary term, which acts trivially on the bulk Hilbert space, given as

$$\begin{aligned} H_{l1} =& \left[\mathrm{Tr}_{0\bar{0}} K^+_{0\bar{0}}(0)\right]^{-1} \left[\mathrm{Tr}_{0\bar{0}} \frac{\mathrm{d}K^+_{0\bar{0}}(u)}{\mathrm{d}u}\bigg|_{u=0}\right] \\ &- \frac{1}{2}\left[\mathrm{Tr}_{0\bar{0}} K^+_{0\bar{0}}(0)\right]^{-1} \left[\mathrm{Tr}_{0\bar{0}} K^+_{0\bar{0}}(0)(-2+\mathbb{K}_{0\bar{0}})\right]; \end{aligned} \tag{63}$$

$H_{l2}$ is in fact an bulk term acting on the leftmost two sites:

$$H_{l2} = -\frac{1}{2}(-2+\mathbb{K}_{12}); \tag{64}$$

$H_{l3}$, representing the true bulk-boundary interaction, has the following form:

$$\begin{aligned} H_{l3} =& 2\left[\mathrm{Tr}_{0\bar{0}} K^+_{0\bar{0}}(0)\right]^{-1} \left[\mathrm{Tr}_{0\bar{0}} K^+_{0\bar{0}}(0)\mathbb{P}_{01}\right] \\ &+ \frac{1}{8}\left[\mathrm{Tr}_{0\bar{0}} K^+_{0\bar{0}}(0)\right]^{-1} (-2+\mathbb{K}_{1\bar{2}}) \\ &\times \left[\mathrm{Tr}_{0\bar{0}} K^+_{0\bar{0}}(0)(-2+\mathbb{K}_{0\bar{0}})\mathbb{P}_{\bar{0}\bar{2}}(-2+\mathbb{K}_{0\bar{0}})\right] \\ &\times (-2+\mathbb{K}_{1\bar{2}}). \end{aligned} \tag{65}$$

The right boundary term can be divided into two parts:

$$H_r = H_{r1} + H_{r2}, \tag{66}$$

where $H_{r1}$ is the remaining part of the bulk Hamiltonian acting on the rightmost several sites, presented below:

$$\begin{aligned} H_{r1} =& \mathbb{P}_{\overline{2L-2},\overline{2L}} - \frac{1}{2}\mathbb{P}_{\overline{2L-2},\overline{2L}}\mathbb{K}_{2L-1,\overline{2L}} - \frac{1}{2}\mathbb{K}_{2L-1,\overline{2L}}\mathbb{P}_{\overline{2L-2},\overline{2L}} \\ &+ \mathbb{P}_{2L-3,2L-1} - \frac{1}{2}\mathbb{P}_{2L-3,2L-1}\mathbb{K}_{2L-3,\overline{2L-2}} \\ &- \frac{1}{2}\mathbb{K}_{2L-3,\overline{2L-2}}\mathbb{P}_{2L-3,2L-1} - \frac{1}{4}\mathbb{K}_{2L-3,\overline{2L-2}}. \end{aligned} \tag{67}$$

$H_{r2}$ includes the interaction between the bulk and right boundary internal degrees of freedom. Because the expression of $H_{r2}$ is lengthy, we first define the following quantity:

$$\Delta = \left(-2+\mathbb{K}_{2L-1,\overline{2L}}\right) K^-_{2L-1,\overline{2L}}(0) \left(-2+\mathbb{K}_{2L-1,\overline{2L}}\right). \tag{68}$$

Thus, $H_{r2}$ can be expressed in terms of following four parts:

$$H_{r2} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4, \tag{69}$$

where

$$\begin{aligned} \Delta_1 =& -\frac{1}{4}\Delta^{-1}\left(-2+\mathbb{K}_{2L-1,\overline{2L}}\right)\Delta, \\ \Delta_2 =& \left(-2+\mathbb{K}_{2L-1,\overline{2L}}\right)^{-1}\left[K^-_{2L-1,\overline{2L}}(0)\right]^{-1}\left[\frac{\mathrm{d}K^-_{2L-1,\overline{2L}}(0)}{\mathrm{d}u}\bigg|_{u=0}\right] \\ &\times \left(-2+\mathbb{K}_{2L-1,\overline{2L}}\right), \\ \Delta_3 =& \Delta^{-1}\bigg(\mathbb{P}_{2L-3,2L-1} - \frac{1}{2}\mathbb{P}_{2L-3,2L-1}\mathbb{K}_{2L-3,\overline{2L-2}} \\ &- \frac{1}{2}\mathbb{K}_{2L-3,\overline{2L-2}}\mathbb{P}_{2L-3,2L-1} + \frac{1}{4}\mathbb{K}_{2L-3,\overline{2L-2}}\bigg)\Delta, \\ \Delta_4 =& \Delta^{-1}\bigg(\mathbb{P}_{\overline{2L-2},\overline{2L}} - \frac{1}{2}\mathbb{P}_{\overline{2L-2},\overline{2L}}\mathbb{K}_{2L-1,\overline{2L}} - \frac{1}{2}\mathbb{K}_{2L-1,\overline{2L}}\mathbb{P}_{\overline{2L-2},\overline{2L}} \\ &+ \frac{1}{4}\mathbb{K}_{2L-1,\overline{2L}}\bigg)\Delta. \end{aligned} \tag{70}$$

As shown above, the boundary Hamiltonians have very complicated forms, and by selecting different reflection $K$-matrices, we obtain various boundary terms. However, we can determine some common structures by analyzing the indices of the components of the boundary Hamiltonian. To be concrete, we focus on the left boundary, particularly the nontrivial bulk-boundary interaction term $H_{l3}$, which acts on the boundary internal space $V_{\rm in}$ and two leftmost quantum spaces $V_1$ and $V_{\bar{2}}$:

$$H_{l3}\in \mathrm{End}\left(V_{\rm in}\otimes V_1\otimes V_{\bar{2}}\right). \tag{71}$$

Note that the internal space $V_{\rm in}$ needn't to be isomorphic to $V_1$ or $V_{\bar{2}}$ and thus can have different dimensions. Now, we examine the component of $H_{l3}$: $[H_{l3}]^{j,I_1,J_2}_{i,J_1,I_2}$, where $\{i,j\}\in V_{\rm in}, \{I_1,J_1\}\in V_1, \{I_2,J_2\}\in V_{\bar{2}}$. $H_{l3}$ has several terms, and we discuss them separately. First, for notational convenience, we express

$$\begin{aligned}
B &= \left[\mathrm{Tr}_{0\bar{0}}K^+_{0\bar{0}}(0)\right]^{-1}\in \mathrm{End}(V_{\rm in}),\\
M &= \mathrm{Tr}_{0\bar{0}}K^+_{0\bar{0}}(0)\mathbb{P}_{01}\in \mathrm{End}(V_{\rm in}\otimes V_1),\\
S &= \mathrm{Tr}_{0\bar{0}}K^+_{0\bar{0}}(0)\,(-2+\mathbb{K}_{0\bar{0}})\,\mathbb{P}_{\bar{0}\bar{2}}\,(-2+\mathbb{K}_{0\bar{0}})\in \mathrm{End}(V_{\rm in}\otimes V_{\bar{2}}),
\end{aligned} \tag{72}$$

then the component of each term in $H_{l3}$ becomes

$$\begin{aligned}
(BM)^{j,I_1,J_2}_{i,j_1,I_2} &= (BM)^{j,I_1}_{i,j_1}\cdot\delta^{J_2}_{I_2},\\
(BS)^{j,i_1,J_2}_{i,j_1,i_2} &= (BS)^{j,J_2}_{i,I_2}\cdot\delta^{I_1}_{J_1},\\
(B\mathbb{K}_{1\bar{2}}S)^{j,I_1,J_2}_{i,J_1,I_2} &= \left(B^k_i S^{j,J_2}_{k,J_1}\right)\cdot\delta^{I_1}_{I_2},\\
(BS\,\mathbb{K}_{1\bar{2}})^{j,I_1,J_2}_{i,J_1,I_2} &= \left(B^k_i S^{j,I_1}_{k,I_2}\right)\cdot\delta^{J_2}_{J_1},\\
(B\mathbb{K}_{1\bar{2}}S\,\mathbb{K}_{1\bar{2}})^{j,I_1,J_2}_{i,J_1,I_2} &= \left(B^k_i S^{j,L}_{k,L}\right)\cdot\delta^{I_1}_{I_2}\delta^{J_2}_{J_1},
\end{aligned} \tag{73}$$

where we have used the component forms of $\mathbb{K}$: $\mathbb{K}^{I_1,J_2}_{J_1,I_2}=\delta^{I_1}_{I_2}\delta^{J_2}_{J_1}$. For the other two left boundary terms, we can easily find

$$\begin{aligned}
(H_{l1})^{j,I_1,J_2}_{i,J_1,I_2} &= (H_{l1})^j_i\cdot\delta^{I_1}_{J_1}\delta^{J_2}_{I_2},\\
(H_{l2})^{j,I_1,J_2}_{i,J_1,I_2} &= (H_{l2})^{I_1,J_2}_{J_1,I_2}\cdot\delta^j_i.
\end{aligned} \tag{74}$$

Thus, we observe that each of the left boundary terms has a unique universal Kronecker delta factor, independent of the specific selection of the $K$-matrix. In other words, given an open spin chain Hamiltonian, if the components of the left boundary terms do not belong to the boundary types shown above, then such an open spin chain cannot be an integrable spin chain, at least not one originated from our fused model.

For a concrete example, we may consider the open spin chain Hamiltonian from the flavored ABJM theory [13]. Owing to the coupling between the bulk bi-fundamental fields and boundary fundamental flavors, the bulk $SU(4)$ $R$-symmetry will break into a remaining $SU(2)_R$ and diagonal subgroup $SU(2)_D$:

$$SU(4)_R\to SU(2)_R\times SU(2)_D \tag{75}$$

In this case, the boundary internal space $V_{\rm in}$ is the two dimensional fundamental representation space of $SU(2)_R$, i.e., $V_{\rm in}=\mathbf{2}\cong\bar{\mathbf{2}}$, $i,j\in\{1,2\}$; $V_1$ and $V_{\bar{2}}$ are the four-dimensional representation spaces $\mathbf{2}\times\mathbf{2}$ and $\bar{\mathbf{2}}\times\bar{\mathbf{2}}$ of $SU(2)_R\times SU(2)_D$, respectively, whose component indices can be formulated by a pair of $SU(2)$ indices, i.e., $I=iA, J=jB, i,j\in\{1,2\}, A,B\in\{1,2\}$, and thus the delta function is simply given as $\delta^I_J=\delta^i_j\delta^A_B$. Subsequently, it is a simple task to rewrite the boundary terms in (73) and (74) using the composite $SU(2)_R\times SU(2)_D$ indices to replace $SU(4)_R$ indices, i.e., $I\to(iA)$. The Hamiltonian of the flavored ABJM spin chain has the following three types of left boundary terms [13, 22]:

$$\begin{aligned}
&\text{type 1}: && \delta^{A_1}_{A_2}\delta^{B_2}_{B_1}\delta^{j_2}_{i}\delta^{i_1}_{j_1}\delta^{j}_{i_2},\\
&\text{type 2}: && \delta^{A_1}_{B_1}\delta^{B_2}_{A_2}\delta^{j_2}_{i}\delta^{i_1}_{j_1}\delta^{j}_{i_2},\\
&\text{type 3}: && \delta^{A_1}_{B_1}\delta^{B_2}_{A_2}\delta^{i_1}_{i}\delta^{j}_{j_1}\delta^{j_2}_{i_2}.
\end{aligned} \tag{76}$$

We can easily observe that the type 2 and 3 terms have the factors $\delta^{I_1}_{J_1}=\delta^{A_1}_{B_1}\delta^{i_1}_{j_1}$ and $\delta^{J_2}_{I_2}=\delta^{B_2}_{A_2}\delta^{j_2}_{i_2}$, respectively. Thus, they can be obtained from the $BM$ and $BS$ terms in (73). The type 1 term mixes the indices of all three spaces $V_{\rm in}\otimes V_1\otimes V_{\bar{2}}$ and does not belong to any type of boundary terms in (73) and (74). Therefore, we observe that the flavored ABJM spin chain cannot be generated from our fused model. Moreover, by the same argument, we observe that the flavored ABJM spin chain cannot also be obtained from another integrable boundary model $H_b$ (54).

## V. CONCLUSION AND DISCUSSION

In this paper, we construct the fused model of the ABJM alternating spin chain by gluing two adjacent quantum spaces and two original auxiliary spaces. For the closed spin chain, we prove the integrability of the fused model by constructing the R-matrix and demonstrating that the Yang-Baxter relation holds. We obtain the boost operator for the fused model based on the regularity condition of the fused $R$-matrix. We also argue that the usual matrix type boost operator with local densities cannot exist in either of two original ABJM spin chains. For the open spin chain, we have calculated the concrete Hamiltonian for general fused $K$-matrices satisfying the reflection and dual reflection equations. We then analyze the boundary terms of the Hamiltonian and determine some common structures of the component indices that are independent of the concrete choices of $K$-matrices. By comparing the boundary terms, we claim that the previously studied flavored ABJM spin chain Hamiltonian cannot originate from the fused model or simply the combination of two original sub-chains Hamiltonians. Thus, it is expected to be non-integrable from the perspective of the algebraic Bethe ansatz method.

Several interesting directions can be taken for future research. First, as we have mentioned in the main text, the ABJM spin chain can be considered a three-site interacting model with a homogeneous local Hamiltonian density $h_{j,j+1,j+2}$, in the sense that $h_{j,j+1,j+2}$ has the same expression whether the starting site $j$ is odd or even. However, to construct the transfer matrix of the fused model, we have used two different Lax operators, $\mathcal{L}_{0\bar{0},2j-1}(u)$ on an odd quantum site and $\mathcal{L}_{0\bar{0},2j}(u)$ on an even quantum site. This remarkable difference inspires us to search for a new construction of the ABJM spin chain using one single Lax operator, say $L_{a,b,c}(u,\xi)$ with possibly an additional inhomogeneity parameter $\xi$.

Second, we can continue to study the boost operator in the ABJM spin chain. Although we have excluded the existence of matrix-type boost operators in the original two spin chains $\tau(u)$ and $\bar{\tau}(u)$, possibilities exist to have other types of boost operators, such as a differential operator that occurs in one-dimensional Hubbard model [23], although the corresponding $R$-matrix is of non-difference form. For the fused model, because we have established the boost operator, the conserved charges can be related by the recursive relation $Q_{n+1} \sim [B, Q_n]$, and thus each charge will have a definite parity under spatial reflection transformation. Subsequently, we can investigate the so-called integrable initial state $|\Psi\rangle$ introduced in [24] for the ABJM fused spin chain, which is defined as a state annihilated by all the conserved charges with odd parities, i.e., $Q_{2n+1}|\Psi\rangle = 0$. For the original spin chain $\tau(u)$ or $\bar{\tau}(u)$, owing to the lack of boost operator, the integrable state is defined alternatively as $\tau(u)|\Psi\rangle = \Pi\bar{\tau}(u)\Pi|\Psi\rangle$, where $\Pi$ is the reflection operator of the spin chain [25−29], or through a more fundamental KT-relation [30, 31].

Finally, we could also consider some general problems addressed in the past studies of integrable models for our fused model. For instance, we can study the long range deformation of the fused model, where the boost of the conserved charge serves as an integrable deformation operator. Another challenging problem is to determine a systematic method of classifying the integrable alternating spin chain models, including the ABJM spin chain as a prototype.

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