Spatially covariant gravity with two degrees of freedom in the presence of an auxiliary scalar field: Hamiltonian analysis*

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Abstract: A class of gravity theories respecting spatial covariance and in the presence of non-dynamical auxiliary scalar fields with only spatial derivatives is investigated. Generally, without higher temporal derivatives in the metric sector, there are 3 degrees of freedom (DOFs) propagating due to the breaking of general covariance. Through a Hamiltonian constraint analysis, we examine the conditions to eliminate the scalar DOF such that only 2 DOFs, which correspond the tensorial gravitational waves in a homogeneous and isotropic background, are propagating. We find that two conditions are needed, each of which can eliminate half degree of freedom. The second condition can be further classified into two cases according to its effect on the Dirac matrix. We also apply the formal conditions to a polynomial-type Lagrangian as a concrete example, in which all the monomials are spatially covariant scalars containing two derivatives. Our results are consistent with the previous analysis based on the perturbative method.

Keywords: modified gravity, spatially covariant gravity, Hamiltonian analysis, gravitational theories

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I. INTRODUCTION

In light of the detection of gravitational waves, gravitational waves provide a novel tool for testing theories of gravity. General relativity possesses only two propagating degrees of freedom (DOFs), which manifests as tensorial gravitational waves (GWs) in a homogeneous and isotropic background. The detection of GWs raises a fundamental question: Is general relativity the only theory of gravity that propagates GWs with precisely two degrees of freedom (2DOFs)? This question is partially answered by Lovelock's theorem [1, 2], which states that in four dimensions, requiring spacetime covariance and second-order equations of motion, general relativity is indeed the unique theory propagating gravitational waves with two degrees of freedom. This also implies that, by violating the assumptions of Lovelock's theorem, other theories of gravity propagating 2DOFs can, in principle, exist. From the theoretical point of view, a Lorentz covariant massless spin-2 particle possesses two physical polarization states. If one constructs a Lorentz invariant theory for such a massless spin-2 particle and further demands that it can be consistently coupled to matter fields, the resulting structure is necessarily that of general relativity. In this sense, general relativity as a theory of gravity is unique since its form is inextricably related to the masslessness of the graviton and the Lorentz symmetry of spacetime. Consequently, alternative theories of gravity that also propagate only 2DOFs provide a valuable testing ground for the foundation principles of general relativity, such as Lorentz invariance, diffeomorphism invariance, and the masslessness of graviton, etc.

Gravitational theories propagating 2DOFs can be traced back to the Cuscuton theory [3], which is introduced as a scalar field model with a non-canonical kinetic term $\propto \sqrt{|-\partial_\mu \phi \partial^\mu \phi|}$. The cuscuton theory represents the incompressible (infinite speed of sound) limit of k-essence, in which the scalar perturbation propagates with an infinite speed of sound in a cosmological background. Although linear perturbations exhibit superluminal propagation, the local phase space degenerates, implying no independent dynamical DOFs and thus preserving causality. As a result, the cuscuton theory modifies gravitational dynamics without introducing additional local DOFs. The Cuscuton theory was extensively studied [4−16] and was extended in the framework of scalar-tensor theory with higher order derivatives [17, 18]. Such kind of theory has also been discussed as a special case of Hořava gravity [19−22]. The relation between the 2DOFs and the space-

time symmetry has been investigated in [23−26]. Another class of of theories propagating 2DOFs was proposed in [27], which is dubbed the minimally modified gravity (MMG)[1]. As the name suggests, MMG theories modify generally relativity "minimally" in the sense that they modify the gravitational sector without introducing extra DOFs beyond the standard 2DOFs. MMG theories extend general relativity by breaking time diffeomorphism invariance while preserving spatial diffeomorphism invariance, and whose action is linear in the lapse function. A key result is the derivation of a self-consistency condition, which ensures the theory possesses no more than two local physical DOFs. Examples of MMG include general relativity, square-root gravity, exponential gravity, and theories with lapse-independent terms, all of which exhibit a constrained phase space structure that eliminates the scalar graviton through first- or second-class constraints. The so-called type-II MMG [34, 35] includes the original Cuscuton theory as a special case. The MMG has been studied extensively [36−51]. It has also been generalized in [52, 53] by introducing auxiliary constraints in the phase space.

A general framework of spatially covariant gravity (SCG) theories respecting only the spatial covariance was proposed in [54, 55]. Due to the violation of general covariance, SCG generally propagates a scalar degree of freedom (besides the 2DOFs corresponding to the gravitational waves) and thus can be viewed as an alternative approach to constructing the scalar-tensor theories [56−59]. The SCG includes the Horava gravity [60−62], effective field theory of inflation/dark energy [63−67] as special cases. It was further generalized with a dynamical lapse function [68, 69], with nonmetricity [70], with parity violation [71] and with multiple scalar modes [72]. The SCG has been applied in the study of cosmology and gravitational waves [73−77]. Subclasses of SCG propagating only 2DOFs were explored in [78, 79] and with a dynamical lapse function in [80]. The concrete Lagrangian found in [78] has been applied in the study of cosmology [10, 81−84].

In this work, we will generalize the SCG in the presence of an auxiliary scalar field, and determine the conditions such that the theory propagates only 2DOFs. The idea of spatially covariant gravity with an auxiliary scalar field was firstly introduced in [85], which was originally motivated by the generally covariant scalar-tensor theory when the scalar field possess a spacelike gradient[2]. The presence of an auxiliary scalar field gives us more and novel possibilities to build the theory propagating two degrees of freedom.

In [86], by performing a perturbative analysis similar to [17, 69, 79], necessary conditions such that no scalar mode propagates at linear order in perturbations around a cosmological background were determined. Nevertheless a Hamiltonian constraint analysis is still needed in order to determine the necessary and sufficient conditions for the theory to propagate only 2DOFs at the nonlinear level. This paper is thus devoted to this issue.

This paper is organized as follows. In Sec. II, we describe our model of spatially covariant gravity with an auxiliary scalar field. It was found that new terms in the Lagrangian are allowed thanks to the introduction of the auxiliary scalar field. In Sec. III, we derive the Hamiltonian formalism and show that the theory generally propagates a scalar mode if no further constraint is imposed. In Sec. IV, we make the degeneracy analysis and derive the conditions such that the theory propagate only 2DOFs. In Sec. V, we apply our formal analysis to a concrete model, of which the Lagrangian is built of SCG monomials of $d = 2$. In Sec. VI, we summarize our results.

II. SPATIALLY COVARIANT GRAVITY WITH AN AUXILIARY SCALAR FIELD

Our starting point is the action

$$S = \int \mathrm{d}t \mathrm{d}^3 x N \sqrt{h} \mathcal{L} \left(\phi, N, h_{ij}, {}^3R_{ij}; \mathrm{D}_i, \pounds_{\boldsymbol{n}} \right), \qquad (1)$$

where N is the usual lapse function, h_{ij} is the spatial metric on the spacelike hypersurfaces, ${}^3R_{ij}$ is the spatial Ricci tensor, D_i is the spatially covariant derivative compatible with h_{ij}, and $\pounds_{\boldsymbol{n}}$ is the Lie derivative with respect to the normal vector n^a of the hypersurfaces. By definition, the action (1) respects only the 3-dimensional spatial invariance. We require that the Lie derivative $\pounds_{\boldsymbol{n}}$ acts only on h_{ij}, which yields the extrinsic curvature K_{ij} defined by

$$\pounds_{\boldsymbol{n}} h_{ij} = 2K_{ij} = \frac{1}{N} \left(\dot{h}_{ij} - \pounds_{\vec{N}} h_{ij} \right). \qquad (2)$$

As a result, both the lapse function N and the scalar field ϕ play the role of an auxiliary variables. Note the shift vector N^i should not appear explicitly in the Lagrangian, which is actually the gauge field of the spatial covariance.

In this work, we restrict our attention to the first-order Lie derivative to prevent higher-order time derivatives from appearing in the equations of motion. Moreover, since the Lie derivative is assumed to act solely on h_{ij}, it enters the Lagrangian exclusively through

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1) See also the minimal theory of massive gravity studied in [28-33].

2) However, we would like to point out that they are completely different theories. Please refer to Appendix A for details.

the extrinsic curvature (2). Therefore, we simply introduce K_{ij} to replace the Lie derivative $£_n$ in (1), and consider the following action

$$\tilde{S} = \int \mathrm{d}t \mathrm{d}^3 x \, N \sqrt{h} \mathcal{L} \left(\phi, N, h_{ij}, {}^3R_{ij}, K_{ij}; \mathrm{D}_i\right). \tag{3}$$

According to the Hamiltonian analysis in the next section, generally the action (3) has 3 DOFs. This is simply because ϕ has no dynamics, the number of DOFs of (3) is the same as the spatially covariant gravity without the scalar field [54, 55]. In particular, since the extrinsic curvature is linear in the Lie derivative (and thus in time derivative), the resulting equations of motion from the action (3) contain at most second-order time derivatives. As a result, the theory avoids the Ostrogradsky ghost instability.

In principle, Lie derivatives of the lapse function $£_n N$ and the extrinsic curvature $£_n K_{ij}$ can be considered, which will introduced more degrees of freedom. It was shown that extra conditions must be put in order to evade these unwanted degrees of freedom. For example, the theory will propagate an extra DOF (i.e., in total 4 DOFs) in the case of a dynamical lapse function. Two conditions are needed in order to evade such an extra DOF [68]. Similar analysis was considered in order to fully eliminate the scalar DOFs [80].

Generally, the Lagrangian in (3) may be nonlinear in K_{ij}, which makes the explicit reversion of velocity $\dot{h}_{ij}$ in terms of the momentum π^{ij} impossible. This problem can be solved by rewriting (3) in an equivalent form

$$\begin{aligned}\tilde{S} &= \int \mathrm{d}t \mathrm{d}^3 x \, N \sqrt{h} \tilde{\mathcal{L}} \left(\phi, N, h_{ij}, {}^3R_{ij}, K_{ij}; \mathrm{D}_i\right)\\ &= S \left(\phi, N, h_{ij}, {}^3R_{ij}, B_{ij}; \mathrm{D}_i\right) + \int \mathrm{d}t \mathrm{d}^3 x \, \sqrt{h} \Lambda^{ij} \left(K_{ij} - B_{ij}\right),\end{aligned} \tag{4}$$

where the action $S\left(\phi, N, h_{ij}, {}^3R_{ij}, B_{ij}; \mathrm{D}_i\right)$ is the same action as (3) with K_{ij} replaced by B_{ij}. This strategy of using the Lagrange multiplier to linearize the velocities has been used in [68, 78]. The equations of motion for Λ^{ij} ensure the auxiliary field B_{ij} to be exactly identical to K_{ij}. Thus the two actions are equivalent, at least at the classical level. By making use of the equation of motion for the auxiliary field B_{ij}, we can fix the Lagrangian multiplier Λ^{ij} as

$$\Lambda^{ij} = \frac{1}{N\sqrt{h}} \frac{\delta S}{\delta B_{ij}}. \tag{5}$$

Finally, by plugging the solution of Λ^{ij}, the action (4) can be recast in the form

$$\tilde{S} = S \left(\phi, N, h_{ij}, {}^3R_{ij}, B_{ij}; \mathrm{D}_i\right) + \int \mathrm{d}t \mathrm{d}^3 x \frac{\delta S}{\delta B_{ij}} \left(K_{ij} - B_{ij}\right). \tag{6}$$

In the following, we will use (6) as our starting point for the Hamiltonian analysis.

III. HAMILTONIAN FORMALISM

A. Hamiltonian and primary constraints

In the action (6), the set of variables is

$$\Phi^I := \left\{ N^i, \phi, B_{ij}, N, h_{ij}\right\}, \tag{7}$$

and the set of their conjugate momenta is denoted by

$$\Pi_I := \left\{\pi_i, p, p^{ij}, \pi, \pi^{ij}\right\}. \tag{8}$$

The conjugate momenta are defined as

$$\Pi_I := \frac{\delta \tilde{S}}{\delta \dot{\Phi}^I}. \tag{9}$$

Since all the velocities $\dot{\Phi}^I$ in the Lagrangian (6) are linearized, the conjugate momenta automatically correspond to constraints among phase space variables. Indeed, the conjugate momenta are

$$\begin{aligned}\left\{\pi_i, p, p^{ij}, \pi, \pi^{ij}\right\} &= \left\{\frac{\delta \tilde{S}}{\delta \dot{N}^i}, \frac{\delta \tilde{S}}{\delta \dot{\phi}}, \frac{\delta \tilde{S}}{\delta \dot{B}_{ij}}, \frac{\delta \tilde{S}}{\delta \dot{N}}, \frac{\delta \tilde{S}}{\delta \dot{h}_{ij}}\right\}\\ &= \left\{0^i, 0, 0_{ij}, 0, \frac{1}{2N}\frac{\delta S}{\delta B_{ij}}\right\},\end{aligned} \tag{10}$$

which result in the primary constraints

$$\pi_i \approx 0, \quad p \approx 0, \quad p^{ij} \approx 0, \quad \pi \approx 0, \quad \tilde{\pi}^{ij} \equiv \pi^{ij} - \frac{1}{2N}\frac{\delta S}{\delta B_{ij}} \approx 0. \tag{11}$$

Here and throughout this work "$\approx$" represents "weak equality", i.e., an equality that holds only when the constraints are satisfied. On the other hand, the strong equality "=" holds no matter the constraints are satisfied or not.

We denote the set of primary constraints as

$$\varphi^I := \left\{\pi_i, p, p^{ij}, \pi, \tilde{\pi}^{ij}\right\}, \tag{12}$$

and the subspace of the phase space when the primary constraints are satisfied as Γ_P, which can be viewed as a hypersurface in the original phase space $\left\{\Phi^I, \Pi_I\right\}$.

The canonical Hamiltonian is obtained by performing the Legendre transformation

$$H_C = \int \mathrm{d}^3 x \left(\sum_I \Pi_I \dot{\Phi}^I - N \sqrt{h} \tilde{\mathcal{L}} \right) \approx H_C|_{\Gamma_P}, \qquad (13)$$

where

$$H_C|_{\Gamma_P} = \int \mathrm{d}^3 x (NC) + X[\vec{N}], \qquad (14)$$

with

$$C = 2\pi^{ij} B_{ij} - \sqrt{h}\mathcal{L}. \qquad (15)$$

For a general "spatial" vector field $\vec{\xi}$, $X[\vec{\xi}]$ is defined as [68]

$$X[\vec{\xi}] := \int \mathrm{d}^3 x \, \Pi_I \pounds_{\vec{\xi}} \Phi^I = \int \mathrm{d}^3 x \left(\pi_i \pounds_{\vec{\xi}} N^i \right.$$
$$\left. + p \pounds_{\vec{\xi}} \phi + p^{ij} \pounds_{\vec{\xi}} B_{ij} + \pi \pounds_{\vec{\xi}} N + \pi^{ij} \pounds_{\vec{\xi}} h_{ij} \right), \qquad (16)$$

In the following we briefly mention the properties of $X[\vec{\xi}]$ following [68]. Supposing $Q(\vec{x})$ is a general spatial tensor field or density (with spatial indices suppressed for briefness) built of phase space variables $\{\Phi^I, \Pi_I\}$, and $\mathcal{F}$ is a scalar functional defined by

$$\mathcal{F}\left[f, \Phi^I, \Pi_I\right] = \int \mathrm{d}^3 x f(\vec{x}) Q(\vec{x}), \qquad (17)$$

with $f(\vec{x})$ is a smoothing function. Note $f(\vec{x})$ also constains appropriate spatial indices (suppressed as well) such that $\mathcal{F}$ is invariant under time-independent spatial diffeomorphism. The Poisson bracket of $X[\vec{\xi}]$ and $\mathcal{F}$ are

$$[X[\vec{\xi}], \mathcal{F}] = X\left[[\vec{\xi}, \mathcal{F}]\right] + \int \mathrm{d}^3 x \frac{\delta \mathcal{F}}{\delta f} \pounds_{\vec{\xi}} f = \int \mathrm{d}^3 x \, Q(\vec{x}) \pounds_{\vec{\xi}} f(\vec{x}). \qquad (18)$$

In (18), the Poisson bracket $[\mathcal{F}, \mathcal{G}]$ is defined by

$$[\mathcal{F}, \mathcal{G}] := \int \mathrm{d}^3 z \left(\frac{\delta \mathcal{F}}{\delta \Phi_I(\vec{z})} \frac{\delta \mathcal{G}}{\delta \Pi^I(\vec{z})} - \frac{\delta \mathcal{F}}{\delta \Pi^I(\vec{z})} \frac{\delta \mathcal{G}}{\delta \Phi_I(\vec{z})} \right). \qquad (19)$$

The key point is that the right side of the second line of the equation vanishes on the constraint surface if $Q(\vec{x})$ is a constraint.

Through integrations by parts, $X[\vec{\xi}]$ can be written in a more convenient form

$$X[\vec{\xi}] = \int \mathrm{d}^3 x \xi^i C_i, \qquad (20)$$

where

$$C_i = \pi \mathrm{D}_i N - 2\sqrt{h} \mathrm{D}_j \left(\frac{1}{\sqrt{h}} \pi_i^j \right) + p \mathrm{D}_i \phi + p^{kl} \mathrm{D}_i B_{kl}$$
$$- 2\sqrt{h} \mathrm{D}_j \left(\frac{p^{jk}}{\sqrt{h}} B_{ik} \right) + \pi_j \mathrm{D}_i N^j + \sqrt{h} \mathrm{D}_j \left(\frac{1}{\sqrt{h}} \pi_i N^j \right). \qquad (21)$$

As we shall see later, $C_i \approx 0$ are actually the secondary constraints associated with the primary constraints $\pi_i \approx 0$. Moreover, the Poisson bracket of C_i with any constraint Q can be obtained by

$$\left[X[\vec{\xi}], \mathcal{F}\right] = \int \mathrm{d}^3 x \mathrm{d}^3 y \, \xi^i(x) f(y) \left[C_i(\vec{x}), Q(\vec{y})\right]. \qquad (22)$$

It immediately follows from (18) that

$$\left[C_i(\vec{x}), Q(\vec{y})\right] \approx 0, \quad \text{for any} \quad Q \approx 0. \qquad (23)$$

Due to the presence of primary constraints, the time evolution is determined by the total Hamiltonian defined by

$$H_T := H_C + \int \mathrm{d}^3 y \sum_I \lambda_I(\vec{y}) \varphi^I(\vec{y}), \qquad (24)$$

where $\lambda_I := \left\{v^i, v, v_{ij}, \lambda, \lambda_{ij}\right\}$ are the undetermined Lagrange multipliers. Since the canonical Hamiltonian H_C is well-defined only on the subspace defined by the primary constraints Γ_P, we can directly use $H_C|_{\Gamma_P}$ given in (14) instead of H_C in (13) in the subsequent calculations.

B. Consistency condition

Constraints must be preserved in time evolution. For the primary constraints, we must require that

$$\int \mathrm{d}^3 y \left[\varphi^I(\vec{x}), \varphi^J(\vec{y})\right] \lambda_J(\vec{y}) + \left[\varphi^I(\vec{x}), H_C\right] \approx 0, \qquad (25)$$

which are the so-called consistency conditions for the primary constraints. These conditions may yields further constraints.

With no further requirement on the structure of the Lagrangian in (6), the Poisson brackets between primary constraints have been calculated in detail in [85], which are summarized in Appendix B. For later convenience, we write (25) in matrix form

$$\int \mathrm{d}^3 y \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [p(\vec{x}),\tilde{\pi}^{kl}(\vec{y})] \\ 0 & 0 & 0 & 0 & [p^{ij}(\vec{x}),\tilde{\pi}^{kl}(\vec{y})] \\ 0 & 0 & 0 & 0 & [\pi(\vec{x}),\tilde{\pi}^{kl}(\vec{y})] \\ 0 & [\tilde{\pi}^{ij}(\vec{x}),p(\vec{y})] & [\tilde{\pi}^{ij}(\vec{x}),p^{kl}(\vec{y})] & [\tilde{\pi}^{ij}(\vec{x}),\pi(\vec{y})] & [\tilde{\pi}^{ij}(\vec{x}),\tilde{\pi}^{kl}(\vec{y})] \end{pmatrix} \begin{pmatrix} v^i \\ v \\ v_{ij} \\ \lambda \\ \lambda_{ij} \end{pmatrix} \approx \begin{pmatrix} -C_i(\vec{x}) \\ \dfrac{\delta S}{\delta\phi(\vec{x})} \\ 0 \\ -C(\vec{x}) \\ [\tilde{\pi}^{ij}(\vec{x}),H_C] \end{pmatrix}, \quad (26)$$

where C_i is defined in (22). Note C_i must have vanishing Poisson brackets with any other constraints due to the property (22). In the fourth line of (26), C is defined by

$$C = 2\pi^{ij}(\vec{x})B_{ij}(\vec{x}) - \frac{\delta S}{\delta N(\vec{x})}, \quad (27)$$

and $[\tilde{\pi}^{ij}(\vec{x}),H_C]$ in the fifth line is given by

$$[\tilde{\pi}^{ij}(\vec{x}),H_C] = \frac{\delta S}{\delta h_{ij}(\vec{x})} - \frac{1}{N(\vec{x})}\int \mathrm{d}^3 z \frac{\delta^2 S}{\delta B_{ij}(\vec{x})\delta h_{kl}(\vec{z})}N(\vec{z})B_{kl}(\vec{z}). \quad (28)$$

According to (125), $[p^{ij}(\vec{x}),\tilde{\pi}^{kl}(\vec{y})]$ in the third line in (26) is proportional to the kinetic matrix

$$\frac{\delta^2 S_B}{\delta B_{ij}(\vec{x})\delta B_{kl}(\vec{y})}, \quad (29)$$

which we assume to be non-degenerate. This is because B_{ij} is the auxiliary field corresponding to K_{ij}, The non-degeneracy of (29) implies that the kinetic term for $\dot{h}_{ij}$ is not degenerate, which guarantees the theory includes General Relativity as its limit, and in particular, the existence of gravitational waves.

As a result, the Lagrange multiplies λ_{ij} can be fixed from the third line in (26) to be

$$\lambda_{ij} = 0. \quad (30)$$

Then the second and the fourth line of (26) yield another two secondary constraints

$$\frac{\delta S}{\delta\phi(\vec{x})} \approx 0, \quad C(\vec{x}) \approx 0. \quad (31)$$

Constraints $\dfrac{\delta S}{\delta\phi(\vec{x})}$ and $C(\vec{x})$ do not generate additional constraints due to consistency conditions, which will be clear in the next section.

After finding all the primary and secondary constraints, we are now ready to classify all the constraints. According to the terminology of Dirac-Bergmann algorithm, a constraint that has vanishing Poisson brackets with all constraints (including itself) is dubbed a first-class constraint, otherwise a second-class constraint. The difference is that each first-class constraint eliminates a single degree of freedom, while a second-class constraint, which reduce one phase space dimension, eliminate half degree of freedom.

According to the above discussion on C_i and constraint algebra in Appendix B, the Poisson brackets among all the constraints can be summarized in the so-called Dirac matrix:

$[\cdot,\cdot]$	$\pi_i(\vec{y})$	$C_i(\vec{y})$	$p(\vec{y})$	$\pi(\vec{y})$	$p^{kl}(\vec{y})$	$\tilde{\pi}^{kl}(\vec{y})$	$C(\vec{y})$	$\dfrac{\delta S}{\delta\phi(\vec{y})}$
$\pi_i(\vec{x})$	0	0	0	0	0	0	0	0
$C_i(\vec{x})$	0	0	0	0	0	0	0	0
$p(\vec{x})$	0	0	0	0	0	X	X	X
$\pi(\vec{x})$	0	0	0	0	0	X	X	X
$p^{ij}(\vec{x})$	0	0	0	0	0	X	X	X
$\tilde{\pi}^{ij}(\vec{x})$	0	0	X	X	X	X	X	X
$C(\vec{x})$	0	0	X	X	X	X	X	X
$\dfrac{\delta S}{\delta\phi(\vec{x})}$	0	0	X	X	X	X	X	0

$$(32)$$

where a "X" stands for generally non-vanishing Poisson brackets. In total there are 22 constraints which can be divided into two classes:

$$6 \text{ first} - \text{class}: \quad \pi_i, \quad C_i,$$
$$16 \text{ second} - \text{class}: \quad p, \quad p^{ij}, \quad \pi, \quad \tilde{\pi}^{ij}, \quad \frac{\delta S}{\delta\phi}, \quad C.$$

The number of DOF is thus

$$\#_{\text{DOF}} = \frac{1}{2}(2\times\#_{\text{var}} - 2\times\#_{\text{1st}} - \#_{\text{2nd}}) = \frac{1}{2}(2\times 17$$
$$-2\times 6 - 16) = 3, \quad (33)$$

which explicit shows that spatially covariant gravity with temporal derivative arises only through the extrinsic curvature, has 3 propagating degrees of freedom, no matter with or without extra auxiliary field(s).

IV. DEGENERACY ANALYSIS

To reduce the number of DOFs, additional conditions must be put on the Lagrangian (6). As we will see in this section, two conditions are needed in order to make the theory propagate only two DOFs.

A. The first condition

As being the first-class constraints, π_i and C_i correspond to the spatial invariance. Without breaking the spatial invariance, π_i and C_i will always be kept as the first-class constraints, which have vanishing Poisson brackets with any other constraints. For the sake of simplicity, we are allowed to omit these two first-class constraints in the subsequent degeneracy analysis. We thus define a simplified Dirac matrix by omitting the two columns and rows corresponding to π_i and C_i:

$$\mathcal{M}^{ab}(\vec{x},\vec{y}) = \begin{array}{c|cccccc} [\cdot,\cdot] & p(\vec{y}) & \pi(\vec{y}) & p^{kl}(\vec{y}) & \tilde{\pi}^{kl}(\vec{y}) & C(\vec{y}) & \dfrac{\delta S}{\delta\phi(\vec{y})} \\ \hline p(\vec{x}) & 0 & 0 & 0 & X & X & X \\ \pi(\vec{x}) & 0 & 0 & 0 & X & X & X \\ p^{ij}(\vec{x}) & 0 & 0 & 0 & X & X & X \\ \hline \tilde{\pi}^{ij}(\vec{x}) & X & X & X & X & X & X \\ C(\vec{x}) & X & X & X & X & X & X \\ \dfrac{\delta S}{\delta\phi(\vec{x})} & X & X & X & X & X & 0 \end{array}. \tag{34}$$

Our purpose is to find conditions to eliminate one scalar type degree of freedom. Generally we have two choices. One is to introduce extra conditions to change the second-class constraints into the first-class constraints, the other is to generate new secondary constraints. In both cases, a necessary conditions is that the simplified Dirac matrix $\mathcal{M}^{ab}$ must be degenerate. The de-

terminant of the simplified Dirac matrix $\mathcal{M}^{ab}$ is

$$\det\mathcal{M}^{ab}(\vec{x},\vec{y}) = \det\mathcal{B}^{ab}(\vec{x},\vec{y}) \times \det\mathcal{B}^{ba}(\vec{y},\vec{x}), \tag{35}$$

where the matrix $\mathcal{B}^{ab}$ is the lower left corner of $\mathcal{M}^{ab}$. The degeneracy of $\mathcal{M}^{ab}$, i.e., $\det\mathcal{M}^{ab}=0$, thus implies that the sub-matrix $\mathcal{B}^{ab}$ of $\mathcal{M}^{ab}$ is degenerate, i.e., $\det\mathcal{B}^{ab}=0$.

Apparently, the determinant of matrix $\mathcal{B}^{ab}$ is the matrix form of the first 2DOF condition. This is not the case. Because the 2DOF condition should be an equation that the action satisfies, and should not contain conjugate momentum. Otherwise, we get the relationship between the action and the conjugate momentum, which is actually a constraint. By careful observation, all elements of the matrix $\mathcal{B}^{ab}$ are independent of conjugate momentum, except of $\mathcal{B}^{21}$ given by

$$\begin{aligned} \mathcal{B}^{21}(\vec{x},\vec{y}) &= \left[C(\vec{x}), p^{kl}(\vec{y}) \right] \\ &= 2\delta^3(\vec{x}-\vec{y})\pi^{kl}(\vec{x}) - \frac{\delta^2 S}{\delta N(\vec{x})\delta B_{kl}(\vec{y})}, \end{aligned} \tag{36}$$

which can be transformed into

$$\mathcal{B}^{21}(\vec{x},\vec{y}) \approx -\delta^3(\vec{x}-\vec{y})\frac{1}{N(\vec{y})}\frac{\delta S}{\delta B_{ij}(\vec{y})} + \frac{\delta^2 S}{\delta B_{ij}(\vec{x})\delta N(\vec{y})} \tag{37}$$

on the constraint surface of $\tilde{\pi}^{ij}$. In subsequent calculations, we always use (37) as the value of $\mathcal{B}^{21}$ instead of (36). As a result, the matrix form of the first 2DOF condition is

$$\mathcal{S}_1(\vec{x},\vec{y}) = \det\mathcal{B}^{ab}(\vec{x},\vec{y}) \approx 0, \tag{38}$$

where

$$\mathcal{B}^{ab} = \begin{array}{c|ccc} [\cdot,\cdot] & p(\vec{y}) & \pi(\vec{y}) & p^{kl}(\vec{y}) \\ \hline -2N(\vec{x})\tilde{\pi}^{ij}(\vec{x}) & \dfrac{\delta^2 S}{\delta B_{ij}(\vec{x})\delta\phi(\vec{y})} & N(\vec{x})\dfrac{\delta}{\delta N(\vec{y})}\left(\dfrac{1}{N(\vec{x})}\dfrac{\delta S}{\delta B_{ij}(\vec{x})}\right) & \dfrac{\delta^2 S}{\delta B_{ij}(\vec{x})\delta B_{kl}(\vec{y})} \\ -C(\vec{x}) & \dfrac{\delta^2 S}{\delta N(\vec{x})\delta\phi(\vec{y})} & \dfrac{\delta^2 S}{\delta N(\vec{x})\delta N(\vec{y})} & N(\vec{y})\dfrac{\delta}{\delta N(\vec{x})}\left(\dfrac{1}{N(\vec{y})}\dfrac{\delta S}{\delta B_{kl}(\vec{y})}\right) \\ \dfrac{\delta S}{\delta\phi(\vec{x})} & \dfrac{\delta^2 S}{\delta\phi(\vec{x})\delta\phi(\vec{y})} & \dfrac{\delta^2 S}{\delta\phi(\vec{x})\delta N(\vec{y})} & \dfrac{\delta^2 S}{\delta\phi(\vec{x})\delta B_{kl}(\vec{y})} \end{array}. \tag{39}$$

In the above, we have adjusted the coefficients of $\tilde{\pi}^{ij}$ and C in the matrix $\mathcal{B}^{ab}$ to make the it more symmetric. Since $\mathcal{B}^{ab}$ is a degenerate matrix, there must be a null eigenvector $\mathcal{V}_a$ satisfying

$$\int \mathrm{d}^3 y\, \mathcal{B}^{ab}(\vec{x},\vec{y})\mathcal{V}_b(\vec{y}) \approx 0^a. \tag{40}$$

Generally, $\mathcal{V}_a$ takes the form

$$\mathcal{V}_a(\vec{x}) = \left(\begin{array}{ccc} (\mathcal{V}_1)_{ij} & \mathcal{V}_2 & \mathcal{V}_3 \end{array} \right)(\vec{x}) = \int \mathrm{d}^3 y\, U_a(\vec{x},\vec{y}) V(\vec{y}). \tag{41}$$

In (41), $V(\vec{x})$ is an arbitrary function of spatial coordinates and $U_a(\vec{x},\vec{y})$ depends on the phase space variables. One method to get the explicit form of the null eigenvector $\mathcal{V}_a$ is to calculate the adjoint matrix of matrix $\mathcal{B}^{ab}$, which can be found in Appendix C.

Assuming that $\mathcal{B}_2^*$ does not vanish, we can use the linear combination of the null eigenvector $\mathcal{V}_a$ and constraint $\tilde{\pi}^{ij}$, C, $\dfrac{\delta S}{\delta \phi}$ to define a new constraint C',

$$\int \mathrm{d}^3 x\, C'(\vec{x}) V(\vec{x}) = \int \mathrm{d}^3 x \left(\tilde{\pi}^{ij} C \frac{\delta S}{\delta \phi} \right)$$
$$(\vec{x}) \left(\begin{array}{ccc} (\mathcal{V}_1)_{ij} & \mathcal{V}_2 & \mathcal{V}_3 \end{array} \right)(\vec{x}) \approx 0. \tag{42}$$

In (42), $V(\vec{x})$ is the arbitrary function in (41).

The merit of introducing the new constraint C' is that the Poisson brackets between C' and constraints p, π, p^{kl} are all vanishing

$$\left[C'(\vec{x}), p(\vec{y}) \right] \approx \left[C'(\vec{x}), \pi(\vec{y}) \right] \approx \left[C'(\vec{x}), p^{kl}(\vec{y}) \right] \approx 0. \tag{43}$$

With this new set of primary constraints, the consistency conditions of the new constraint C' reduce to

$$\frac{\mathrm{d} C'(\vec{x})}{\mathrm{d} t} = \left[C'(\vec{x})(\vec{x}), H_{\mathrm{T}} \right]$$
$$= \int \mathrm{d}^3 y\, \left[C'(\vec{x}), \tilde{\pi}^{kl}(\vec{y}) \right] \lambda_{kl}(\vec{y}) + \left[C'(\vec{x}), H_{\mathrm{C}} \right] \approx 0 \tag{44}$$

where λ_{ij} has been fixed to be zero in (26). If $\left[C'(\vec{x}), H_{\mathrm{C}} \right] \neq 0$, (44) yields a new secondary constraint $\Phi(\vec{y})$:

$$\Phi(\vec{y}) = \left[C'(\vec{y}), H_C \right] \approx 0. \tag{45}$$

Otherwise, there is no secondary constraint, but we can always find a new first class constraint through linear combination of constraints p, π and p^{ij}. Because the necessary and sufficient condition for generating secondary constraints is that the consistency condition cannot be automatically satisfied, rather than requiring the simplified Dirac matrix to be degenerate. Therefore, after requiring the consistency condition, two cases may arise:

1. The consistency condition is not automatically satisfied, so that an additional secondary constraint is required to ensure that the consistency condition is satisfied;

2. The consistency condition is automatically satisfied, and a first class constraint is obtained by combining the existing constraints.

When we require a specific 2DOF condition, we should check whether this condition violates the consistency condition and generates additional secondary constraints. We should consider this newly generated secondary constraint in the subsequent discussion.

B.　The second condition

If $\left[C'(\vec{x}), H_{\mathrm{C}} \right] \neq 0$ in the previous section, we introduce a new constraint π' in the same way as C',

$$\int \mathrm{d}^3 x\, \pi'(\vec{x}) V(\vec{x}) \equiv \int \mathrm{d}^3 x \left(\begin{array}{ccc} p^{ij} & \pi & p \end{array} \right)$$
$$(\vec{x}) \left(\begin{array}{ccc} (\mathcal{V}'_1)_{ij} & \mathcal{V}'_2 & \mathcal{V}'_3 \end{array} \right)(\vec{x}) \approx 0, \tag{46}$$

where again $V(\vec{x})$ is the arbitrary function in (41). Similar to constraint C', the new constraint π' has the vanishing Poisson brackets

$$\left[\tilde{\pi}^{ij}(\vec{x}), \pi'(\vec{y}) \right] \approx \left[C'(\vec{x}), \pi'(\vec{y}) \right] \approx \left[\frac{\delta S}{\delta \phi(\vec{x})}, \pi'(\vec{y}) \right] \approx 0. \tag{47}$$

By making use of the constraints π', C' and Φ, the simplified Dirac matrix becomes

$$\mathcal{M}'^{ab} = \begin{array}{|c||c|c|c|c|c|} \hline [\cdot,\cdot] & \pi'(\vec{y}) & \{p(\vec{y}),p^{kl}(\vec{y})\} & C'(\vec{y}) & \left\{\tilde{\pi}^{kl}(\vec{y}),\dfrac{\delta S}{\delta\phi(\vec{y})}\right\} & \Phi(\vec{x}) \\ \hline\hline \pi'(\vec{x}) & 0 & 0 & 0 & 0 & X \\ \hline \{p(\vec{x}),p^{kl}(\vec{x})\} & 0 & 0 & 0 & -C^{ba}(\vec{y},\vec{x}) & X \\ \hline C'(\vec{x}) & 0 & 0 & X & X & X \\ \hline \left\{\tilde{\pi}^{kl}(\vec{x}),\dfrac{\delta S}{\delta\phi(\vec{x})}\right\} & 0 & C^{ab}(\vec{x},\vec{y}) & X & X & X \\ \hline \Phi(\vec{x}) & X & X & X & X & X \\ \hline \end{array} \tag{48}$$

Since the first 2DOF condition $\mathcal{S}_1(\vec{x}, \vec{y})$ has been imposed, which yields a secondary constraint to reduced the dimension of the phase space by one and thus reduce a half degree of freedom. We need to require an additional 2DOF condition to reduce the number of degrees of freedom to 2, which means that the simplified Dirac matrix is required to be degenerate again. The determinant of the matrix is proportional to $\left[C'(\vec{x}), C'(\vec{y})\right]$, $\left[\Phi(\vec{x}), \pi'(\vec{y})\right]$ and $\det C^{ab}(\vec{x}, \vec{y})$. If the determinant of this matrix is required to be vanishing, two types of 2DOF conditions can be obtained:

$$\mathcal{S}_2^{(1)}(\vec{x}, \vec{y}) = \det C^{ab}(\vec{x}, \vec{y}) \times \left[\Phi(\vec{x}), \pi'(\vec{y})\right] = \det \mathcal{D}^{ab}(\vec{x}, \vec{y}) \approx 0, \tag{49}$$

$$\mathcal{S}_2^{(2)}(\vec{x}, \vec{y}) = \left[C'(\vec{x}), C'(\vec{y})\right] \approx 0, \tag{50}$$

where

$$\mathcal{D}^{ab}(\vec{x}, \vec{y}) = \begin{array}{c|ccc} [\cdot, \cdot] & p(\vec{y}) & \pi(\vec{y}) & p^{kl}(\vec{y}) \\ \hline \tilde{\pi}^{ij}(\vec{x}) & X & X & X \\ \dfrac{\delta S}{\delta \phi(\vec{x})} & X & X & X \\ \Phi(\vec{x}) & X & X & X \end{array}. \tag{51}$$

The 2DOF condition $\mathcal{S}_2^{(1)}$ will generate a secondary constraint, if the consistency condition is not automatically satisfied. On the other hand, the 2DOF condition $\mathcal{S}_2^{(2)}$ clearly generates a first-class constraint.

If $\left[C'(\vec{x}), H_C\right] = 0$ in the previous section, the second 2DOF conditions will become

$$\mathcal{S}_2'^{(1)}(\vec{x}, \vec{y}) = \det C^{ab}(\vec{x}, \vec{y}) \approx 0, \tag{52}$$

$$\mathcal{S}_2'^{(2)}(\vec{x}, \vec{y}) = \left[C'(\vec{x}), C'(\vec{y})\right] \approx 0, \tag{53}$$

which is covered by TTDOF conditions $\mathcal{S}_2^{(1)}$ and $\mathcal{S}_2^{(2)}$.

The space of theories satisfying the 2DOF conditions $\mathcal{S}_1$, $\mathcal{S}_2^{(1)}$ and $\mathcal{S}_2^{(2)}$ can be divided into three branches:

1. Linear constraints branch: If the secondary constraints are linearly dependent, 2DOF conditions will be strongly vanishing. However, in this case the number of degrees of freedom will not change. Due to the linear dependence, the number of secondary constraints will be reduced by one, which will offset the half degree of freedom reduction caused by the 2DOF conditions. For example, if constraints $\tilde{\pi}^{ij}$, C and $\frac{\delta S}{\delta \phi}$ are linearly dependent, 2DOF condition $\mathcal{S}_1$ is automatically satisfied. However,

in this case the number of degrees of freedom remains unchanged.

2. Non-physical branch: Some 2DOF conditions may make the theory be trival. For example, if the secondary constraints generated by the consistency condition still cannot satisfy the consistency condition, additional secondary constraints will be generated. However, these additional secondary constraints may reduce the number of degrees of freedom to be less than 2, which is an unphysical situation.

3. Physical branch: In this branch, we get a scalar-tensor theory with only two tensorial degrees of freedom.

In the following when we mention 2DOF conditions, we always refer to the third physical branch.

V. A CONCRETE MODEL OF $d = 2$

A. The Lagrangian

In [57, 58] (see also [74]), monomials of SCG have been constructed and classified according to the total number of derivatives d in each monomial. As an example of our formalism, in this section we will apply the two conditions to the concrete model of $d = 2$. The action of is given by

$$\tilde{S}_2\left(\phi, N, h_{ij}, {}^3R_{ij}, K_{ij}; D_i\right) = S_2\left(\phi, N, h_{ij}, {}^3R_{ij}, B_{ij}; D_i\right) + \int dt d^3 x \frac{\delta S_2}{\delta B_{ij}}\left(K_{ij} - B_{ij}\right). \tag{54}$$

As being discussed before, we introduce the variable B_{ij} in order to make the explicit reversion of velocity $\dot{h}_{ij}$ in terms of the momentum possible. The equivalent action S_2 can be written as

$$S_2 = \int dt d^3 x N \sqrt{h} \mathcal{L}_2, \tag{55}$$

where

$$\mathcal{L}_2 = c_1 B_{ij} B^{ij} + c_2 B^2 + c_3 R + c_4 a_i a^i + d_1 D_i \phi D^i \phi + d_2 a_i D^i \phi. \tag{56}$$

The coefficients c_i and d_i are general functions of N and ϕ. The acceleration a_i is defined as

$$a_i = \frac{1}{N} D_i N. \tag{57}$$

For later convenience, we can evaluate some of the elements of $\mathcal{B}^{ab}$ as

$$\frac{\delta^2 S}{\delta B_{ij}(\vec{x})\delta B_{mn}(\vec{y})} = 2N\sqrt{h}\left(\frac{1}{2}c_1(\delta^{im}\delta^{jn}+\delta^{in}\delta^{jm})\right.$$
$$\left. +c_2\delta^{ij}\delta^{mn}\right)(\vec{x})\delta^3(\vec{x}-\vec{y}), \tag{58}$$

$$N(\vec{y})\frac{\delta}{\delta N(\vec{x})}\left(\frac{1}{N(\vec{y})}\frac{\delta S_B}{\delta B_{ij}(\vec{y})}\right) = 2N\sqrt{h}\left(\frac{\partial c_1}{\partial N}B^{ij}+\frac{\partial c_2}{\partial N}B\delta^{ij}\right)$$
$$(\vec{x})\delta^3(\vec{x}-\vec{y}), \tag{59}$$

$$\frac{\delta^2 S}{\delta B_{ij}(\vec{x})\delta\phi(\vec{y})} = 2N\sqrt{h}\left(\frac{\delta c_1}{\delta\phi}B^{ij}+\frac{\delta c_2}{\delta\phi}B\delta^{ij}\right)(\vec{x})\delta^3(\vec{x}-\vec{y}). \tag{60}$$

In order to prevent the dynamical terms of the theory from degenerating, we assume

$$\det\frac{\delta^2 S}{\delta B_{ij}(\vec{x})\delta B_{mn}(\vec{y})} = \left(2N\sqrt{h}\delta^3(\vec{x}-\vec{y})\right)^6 c_1^6(c_1+3c_2)\neq 0, \tag{61}$$

which implies

$$c_1\neq 0, \qquad c_1+3c_2\neq 0. \tag{62}$$

As long as (62) is satisfied, the non-physical branch of the 2DOF conditions is naturally removed.

Other elements of matrix $\mathcal{B}^{ab}$ are evaluated to be

$$\frac{\delta^2 S}{\delta\phi(\vec{x})\delta\phi(\vec{y})} = N\sqrt{h}\left[\left(\frac{\delta^2 S}{\delta\phi\delta\phi}\right)_0 + \left(-2\frac{\partial d_1}{\partial\phi}D^i\phi + \left(-2N\frac{\partial d_1}{\partial N}-2d_1\right)a^i\right)D_i - 2d_1 D^i D_i\right](\vec{x})\delta^3(\vec{x}-\vec{y}), \tag{63}$$

$$\frac{\delta^2 S}{\delta N(\vec{x})\delta N(\vec{y})} = N\sqrt{h}\left[\left(\frac{\delta^2 S}{\delta N\delta N}\right)_0 ++\left(-\frac{2}{N^2}\frac{\partial c_4}{\partial\phi}D^i\phi + \left(\frac{2c_4}{N^2}-\frac{2}{N}\frac{\partial c_4}{\partial N}\right)a^i\right)D_i - \frac{2c_4}{N^2}D^i D_i\right](\vec{x})\delta^3(\vec{x}-\vec{y}), \tag{64}$$

$$\frac{\delta}{\delta\phi(\vec{y})}\left(\frac{\delta S}{\delta N(\vec{x})}\right) = N\sqrt{h}\left[\left(\frac{\delta^2 S}{\delta\phi\delta N}\right)_0 + \left(\left(\frac{2d_1}{N}+2\frac{\partial d_1}{\partial N}-\frac{2}{N}\frac{\partial d_2}{\partial\phi}\right)D^i\phi - \frac{2}{N}\frac{\partial c_4}{\partial\phi}a^i\right)D_i - \frac{d_2}{N}D^i D_i\right](\vec{x})\delta^3(\vec{x}-\vec{y}), \tag{65}$$

$$\frac{\delta}{\delta N(\vec{y})}\left(\frac{\delta S}{\delta\phi(\vec{x})}\right) = N\sqrt{h}\left[\left(\frac{\delta^2 S}{\delta N\delta\phi}\right)_0 + \left(\left(-2\frac{\partial d_1}{\partial N}-\frac{2d_1}{N}\right)D^i\phi + \left(\frac{2}{N}\frac{\partial c_4}{\partial\phi}-2\frac{\partial d_2}{\partial N}\right)a^i\right)D_i - \frac{d_2}{N}D^i D_i\right](\vec{x})\delta^3(\vec{x}-\vec{y}), \tag{66}$$

where $\left(\frac{\delta^2 S}{\delta\phi\delta\phi}\right)_0$ represents terms in $\frac{\delta^2 S}{\delta\phi(\vec{x})\delta\phi(\vec{y})}$ that are independent of derivatives of the delta function. We use this notation because their specific expressions are very complicated and only the coefficients of derivatives of the delta function are relevant to the subsequent analysis.

B. The first condition

We have already obtained the first 2DOF condition in (38). To get the specific form of the 2DOF conditions for the this model, a straightforward approach is to calculate the determinant of the matrix $\mathcal{B}^{ab}$, which is however is very complicated and not illuminating. Alternatively, we can choose an equivalent approach by assuming that a null eigenvector $\mathcal{V}_a(\vec{x})$ exists and find the expression of this null eigenvector. In this approach, we can find which conditions are required to make the Dirac matrix $\mathcal{B}^{ab}$ degenerate.

We assume $U_a(\vec{x},\vec{y})$ in (41) is independent of the derivative of the delta function. As a result, different orders of derivative of the delta function will be separated in the matrix $\mathcal{B}^{ab}$,

$$\mathcal{B}^{ab}(\vec{x},\vec{y}) = \mathcal{B}_{(0)}^{ab}(\vec{x},\vec{y})+\mathcal{B}_{(1)}^{ab}(\vec{x},\vec{y})+\mathcal{B}_{(2)}^{ab}(\vec{x},\vec{y}), \tag{67}$$

in which terms with different order (of derivative of the delta function) in the matrix $\mathcal{B}^{ab}$ separately satisfies

$$\int d^3 x\, \mathcal{V}_a(\vec{x})\mathcal{B}_{(i)}^{ab}(\vec{x},\vec{y}) \approx 0^b, \quad i=0,1,2. \tag{68}$$

Note by observing the form of the secondary constraint C and $\frac{\delta S}{\delta\phi}$, (68) can only be weakly satisfied when $i=0$.

In matrix $\mathcal{B}_{(1)}^{ab}(\vec{x},\vec{y})$ and $\mathcal{B}_{(2)}^{ab}(\vec{x},\vec{y})$, only the 2×2 submatrices in the lower left corner are not vanishing, which are

$$\begin{pmatrix} 2\left(\frac{d_1}{N}+\frac{\partial d_1}{\partial N}-\frac{1}{N}\frac{\partial d_2}{\partial\phi}\right)D^i\phi - \frac{2}{N}\frac{\partial c_4}{\partial\phi}a^i & -\frac{2}{N^2}\frac{\partial c_4}{\partial\phi}D^i\phi + 2\left(\frac{c_4}{N^2}-\frac{1}{N}\frac{\partial c_4}{\partial N}\right)a^i \\ -2\frac{\partial d_1}{\partial\phi}D^i\phi - 2\left(N\frac{\partial d_1}{\partial N}+d_1\right)a^i & -2\left(\frac{\partial d_1}{\partial N}+\frac{d_1}{N}\right)D^i\phi + \left(\frac{2}{N}\frac{\partial c_4}{\partial\phi}-2\frac{\partial d_2}{\partial N}\right)a^i \end{pmatrix}_x D_{x^i}\delta^3\left(\vec{x}-\vec{y}\right) \tag{69}$$

and

$$
\begin{pmatrix} -\dfrac{d_2}{N} & -\dfrac{2c_4}{N^2} \\ -2d_1 & -\dfrac{d_2}{N} \end{pmatrix}_x D^{x^i} D_{x_i} \delta^3 \left(\vec{x} - \vec{y} \right), \tag{70}
$$

respectively. In the above a subscript "x" denotes functional dependence on $\vec{x}$. In the subsequent discussion, we omit $\vec{x}$ for brevity when the dependence on $\vec{x}$ is clear from the context.

According to (70), for the null eigenvalue equation of $\mathcal{B}_{(2)}^{ab}$, the part of matrix $\mathcal{B}^{ab}$ is proportional to $D^{x^i} D_{x_i} \delta^3 (\vec{x} - \vec{y})$, which yields

$$
d_2 \mathcal{V}_2 + 2N d_1 \mathcal{V}_3 = 0, \tag{71}
$$

$$
2c_4 \mathcal{V}_2 + N d_2 \mathcal{V}_3 = 0. \tag{72}
$$

As for the null eigenvalue equation of $\mathcal{B}_{(1)}^{ab}$, the part of matrix $\mathcal{B}^{ab}$ only related to $D_{x_i} \delta^3 (\vec{x} - \vec{y})$, which yields

$$
\left(\frac{2d_1}{N} + 2 \frac{\partial d_1}{\partial N} - \frac{2}{N} \frac{\partial d_2}{\partial \phi} \right) \mathcal{V}_2 - 2 \frac{\partial d_1}{\partial \phi} \mathcal{V}_3 = 0, \tag{73}
$$

$$
\frac{2}{N} \frac{\partial c_4}{\partial \phi} \mathcal{V}_2 - \left(2N \frac{\partial d_1}{\partial N} + 2d_1 \right) \mathcal{V}_3 = 0, \tag{74}
$$

$$
\frac{2}{N^2} \frac{\partial c_4}{\partial \phi} \mathcal{V}_2 - \left(2 \frac{\partial d_1}{\partial N} + \frac{2d_1}{N} \right) \mathcal{V}_3 = 0, \tag{75}
$$

$$
\left(\frac{2c_4}{N^2} - \frac{2}{N} \frac{\partial c_4}{\partial N} \right) \mathcal{V}_2 - \left(\frac{2}{N} \frac{\partial c_4}{\partial \phi} - 2 \frac{\partial d_2}{\partial N} \right) \mathcal{V}_3 = 0. \tag{76}
$$

There are four equations since the coefficients of $D^i \phi$ and a^i need to be proportional to the null eigenvector. We assume that d_1, d_2 and c_4 are not vanishing. We can simply choose

$$
\mathcal{V}_2 = 2N d_1, \tag{77}
$$

$$
\mathcal{V}_3 = -d_2, \tag{78}
$$

and we can get two independent equations

$$
d_2^2 - 4d_1 c_4 = 0, \tag{79}
$$

$$
\left(\frac{d_1}{N} + 2 \frac{\partial d_1}{\partial N} - \frac{2}{N} \frac{\partial d_2}{\partial \phi} \right) 2N d_1 + \frac{\partial d_1}{\partial \phi} d_2 = 0. \tag{80}
$$

The other two equations are not independent and can be obtained by combining these two equations.

We have a set of special solutions

$$
d_1 = \frac{A}{2N}, \quad d_2 = B, \quad c_4 = \frac{8NB^2}{A}, \tag{81}
$$

where A and B are constants. As we will see, this set of special solutions will simplify the calculations in the subsequent discussion.

When we discuss the degeneracy of the matrix $\mathcal{B}_{(0)}^{ab}$, we need to be careful to identify which parts weakly equal to zero. To make this clear, we should first discuss how secondary constraints participate in the weakly equal equation.

Since in the Lagrangian 54 the the velocity term $\dot{h}_{ij}$ (encoded in K_{ij}) has been linearized, each canonical momentum will correspond to a primary constraint. Then all the secondary constraints can be combined into a form that are independent of canonical momenta. For example, we make a linear combination of the constraints $\tilde{\pi}^{ij}$ and C,

$$
C_1 = 2\tilde{\pi}^{ij} B_{ij} - C = \frac{\delta S}{\delta N} - \frac{B_{ij}}{N} \frac{\delta S}{\delta B_{ij}}. \tag{82}
$$

By combining constraints C_1 and $\frac{\delta S}{\delta \phi}$, we can obtain a constraint C_2,

$$
\begin{aligned}
C_2 = d_2 \frac{\delta S}{\delta \phi} - 2d_1 N C_1 = N \sqrt{h} \Bigg[& \left(B \frac{\partial c_1}{\partial \phi} - A \left(-\frac{c_1}{N} \right. \right. \\
& \left. \left. + \frac{\partial c_1}{\partial N} \right) \right) B_{ij} B^{ij} + \left(B \frac{\partial c_2}{\partial \phi} - A \left(-\frac{c_2}{N} + \frac{\partial c_2}{\partial N} \right) \right) B^2 \\
& + \left(B \frac{\partial c_3}{\partial \phi} - A \left(\frac{c_3}{N} + \frac{\partial c_3}{\partial N} \right) \right) R \Bigg],
\end{aligned} \tag{83}
$$

which does not depend on the special solution (81) we chose before.

Since we have replaced $\mathcal{B}^{21}$ with a term which is independent of the canonical momentum, (38) can be equivalently written as

$$
\det \mathcal{B}^{ab}(\vec{x}, \vec{y}) = \alpha(\vec{x}, \vec{y}) C_2(\vec{x}) + \beta(\vec{x}, \vec{y}) \frac{\delta S}{\delta \phi(\vec{x})}. \tag{84}
$$

By solving equations

$$
\int \mathrm{d}^3 x \, \mathcal{V}_a(\vec{x}) \mathcal{B}_{(0)}^{ab}(\vec{x}, \vec{y}) \approx 0^b, \qquad b = 3, 4, ..., 8, \tag{85}
$$

we obtain

$$
(\mathcal{V}_1)_{ij} = -2d_1 N \mathcal{V}_{ij}^{(N)} + d_2 \mathcal{V}_{ij}^{(\phi)}, \tag{86}
$$

where

$$\mathcal{V}_{ij}^{(N)} = \frac{1}{c_1}\frac{\partial c_1}{\partial N}B_{ij} + \left(\frac{1}{c_1+3c_2}\frac{\partial c_2}{\partial N} - \frac{c_2}{(c_1+3c_2)c_1}\frac{\partial c_1}{\partial N}\right)B\delta_{ij},$$

(87)

$$\mathcal{V}_{ij}^{(\phi)} = \frac{1}{c_1}\frac{\partial c_1}{\partial \phi}B_{ij} + \left(\frac{1}{c_1+3c_2}\frac{\partial c_2}{\partial \phi} - \frac{c_2}{(c_1+3c_2)c_1}\frac{\partial c_1}{\partial \phi}\right)B\delta_{ij}.$$

(88)

Since we have previously chosen a set of special solutions for d_1, d_2 and c_4, equation (68) with $i = 0$ can be written as

$$\int d^3x\,\mathcal{V}_a(\vec{x})\mathcal{B}_{(0)}^{ab}(\vec{x},\vec{y}) = D_1^b(\vec{y})B_{ij}(\vec{y})B^{ij}(\vec{y}) + D_2^b(\vec{y})B^2(\vec{y}) + D_3^b(\vec{y})R(\vec{y}), \qquad b = 1,2. \quad (89)$$

Since there are no terms such as $a_i a^i$ in equation (89), the coefficient $\beta(\vec{x},\vec{y})$ is fixed to zero. The above formula can be equivalently written as

$$\int d^3x\,\mathcal{V}_a(\vec{x})\mathcal{B}_{(0)}^{ab}(\vec{x},\vec{y}) = \alpha^b(\vec{y})C_2(\vec{y}), \qquad b = 1,2. \quad (90)$$

In order to make (90) have a solution, the coefficients of $B_{ij}B^{ij}$, B^2 and R need to be vanishing. There will be 6 equations for c_1, c_2 and c_3. We assume that

$$\frac{c_1}{c_2} = \text{const.} \quad (91)$$

This assumption will simplify the expression of D_2^b.

By fixing the coefficients of $B_{ij}B^{ij}$ and R in (90), we obtain 4 differential equations

$$A\left[\frac{1}{N}\frac{\partial c_1}{\partial \phi} + \frac{\partial^2 c_1}{\partial N\partial \phi} - \frac{2}{c_1}\frac{\partial c_1}{\partial N}\frac{\partial c_1}{\partial \phi} - \alpha^1\left(-\frac{c_1}{N} + \frac{\partial c_1}{\partial N}\right)\right] - B\left(\frac{\partial^2 c_1}{\partial \phi^2} - \frac{2}{c_1}\left(\frac{\partial c_1}{\partial \phi}\right)^2 - \alpha^1\frac{\partial c_1}{\partial \phi}\right) = 0,$$

(92)

$$A\left[\frac{2}{N}\frac{\partial c_1}{\partial N} + \frac{\partial^2 c_1}{\partial N^2} - \frac{2}{c_1}\frac{\partial c_1}{\partial N}\frac{\partial c_1}{\partial N} - \alpha^2\left(-\frac{c_1}{N} + \frac{\partial c_1}{\partial N}\right)\right] - B\left(\frac{1}{N}\frac{\partial c_1}{\partial \phi} + \frac{\partial^2 c_1}{\partial N\partial \phi} - \frac{2}{c_1}\frac{\partial c_1}{\partial N}\frac{\partial c_1}{\partial \phi} - \alpha^2\frac{\partial c_1}{\partial \phi}\right) = 0,$$

$$A\left[\frac{1}{N}\frac{\partial c_3}{\partial \phi} + \frac{\partial^2 c_3}{\partial N\partial \phi} - \alpha^1\left(\frac{1}{N}\frac{\partial c_3}{\partial \phi} + \frac{\partial^2 c_3}{\partial N\partial \phi}\right)\right] - B\left(\frac{\partial^2 c_3}{\partial \phi^2} - \alpha^1\frac{\partial c_3}{\partial \phi}\right) = 0,$$

$$A\left[\frac{2}{N}\frac{\partial c_3}{\partial N} + \frac{\partial^2 c_3}{\partial N^2} - \alpha^2\left(\frac{1}{N}\frac{\partial c_3}{\partial \phi} + \frac{\partial^2 c_3}{\partial N\partial \phi}\right)\right] - B\left(\frac{1}{N}\frac{\partial c_3}{\partial \phi} + \frac{\partial^2 c_3}{\partial N\partial \phi} - \alpha^2\frac{\partial c_3}{\partial \phi}\right) = 0.$$

(93)

As long as equations (91) and (92)-(93) are satisfied, the coefficient of B^2 will be automatically fixed to be zero. Therefore, we do not have to discuss the differential equation corresponding to the coefficient of B^2 here.

We also choose a set of special solutions

$$c_1 = Nf(\phi), \quad (94)$$

$$c_2 = CNf(\phi), \quad (95)$$

$$c_3 = \frac{D}{N}g(\phi), \quad (96)$$

where the coefficients C and D are both constant.

By plugging the expressions of c_1 and c_3 in (92)-(93), we can simplify the four differential equations

$$\frac{\partial^2 c_1}{\partial \phi^2} - \frac{2}{c_1}\left(\frac{\partial c_1}{\partial \phi}\right)^2 - \alpha^1\frac{\partial c_1}{\partial \phi} = 0, \quad (97)$$

$$\alpha^2\frac{\partial c_1}{\partial \phi} = 0, \quad (98)$$

$$\frac{\partial^2 c_3}{\partial \phi^2} - \alpha^1\frac{\partial c_3}{\partial \phi} = 0, \quad (99)$$

$$\alpha^2\frac{\partial c_3}{\partial \phi} = 0. \quad (100)$$

Finally, constraint C_2 becomes

$$C_2 = BN\sqrt{h}\left(\frac{\partial c_1}{\partial \phi}B_{ij}B^{ij} + \frac{\partial c_2}{\partial \phi}B^2 + \frac{\partial c_3}{\partial \phi}R\right). \quad (101)$$

As we can see, if both c_1 and c_3 are independent of ϕ, constraint C_2 will do not exist. The number of second-class constraints will be reduced by one, and the number of degrees of freedom will be increased by 1/2. This is the linear constraints branch of 2DOF condition we mentioned in the previous section, which is not the case we are interested in. Therefore we assume that c_1 and c_3 cannot be independent of ϕ simultaneously. As a result, the coefficient α^2 has to be fixed to be zero.

C. The second condition

According to the above discussion, after requiring the first 2DOF condition $\mathcal{S}_1$, we can obtain constraint C' by the null eigenvector $\mathcal{V}_a(\vec{x})$,

$$\int d^3x\,C'(\vec{x})V(\vec{x}) \equiv \int d^3x\left(-2N\tilde{\pi}^{ij} - C\frac{\delta S}{\delta \phi}\right)(\vec{x})\left((\mathcal{V}_1)_{ij}\,\mathcal{V}_2\,\mathcal{V}_3\right)(\vec{x}), \quad (102)$$

which yields

$$C' = BN \left[\sqrt{h} \left(\frac{\partial c_1}{\partial \phi} B^{ij} B_{ij} + \left(\frac{2c_2}{c_1} \frac{\partial c_1}{\partial \phi} - \frac{\partial c_2}{\partial \phi} \right) B^2 \right. \right.$$
$$\left. \left. - \frac{\partial c_3}{\partial \phi} R \right) - \frac{2}{c_1} \frac{\partial c_1}{\partial \phi} \pi^{ij} B_{ij} \right]. \tag{103}$$

Poisson brackets between the constraint C' and constraints p, π, p^{kl} are

$$[C'(\vec{x}), p(\vec{y})] = \left[-BN \frac{\partial}{\partial \phi} \left(\frac{2}{c_1} \frac{\partial c_1}{\partial \phi} \right) \tilde{\pi}^{ij} B_{ij} - \frac{\partial c_3}{\partial \phi \partial \phi} \right.$$
$$\left. \left(\frac{\partial c_3}{\partial \phi} \right)^{-1} C_2(\vec{x}) \delta^3(\vec{x} - \vec{y}) \approx 0, \tag{104}$$

$$[C'(\vec{x}), \pi(\vec{y})] = \frac{B}{N(\vec{x})} \left(C'(\vec{x}) + C_2(\vec{x}) \right) \delta^3(\vec{x} - \vec{y}) \approx 0, \tag{105}$$

$$[C'(\vec{x}), p^{kl}(\vec{y})] = -\frac{BN(\vec{x})}{c_1(\vec{x})} \frac{\partial c_1(\vec{x})}{\partial \phi(\vec{x})} \tilde{\pi}^{kl}(\vec{x}) \delta^3(\vec{x} - \vec{y}) \approx 0, \tag{106}$$

which all vanish on the constraint surface. Poisson bracket between constraint C' and itself is generally nonzero, unless

$$\frac{\partial c_1}{\partial \phi} = 0, \quad \text{or} \quad \frac{\partial c_3}{\partial \phi} = 0. \tag{107}$$

As we have mentioned above, c_1 and c_3 cannot be independent of ϕ simultaneously, otherwise the number of degrees of freedom will increased by $1/2$. If we require the second 2DOF condition $\mathcal{S}_2^{(2)}$ to be valid, there will be two cases.

In the first case, if we choose $\frac{\partial c_1}{\partial \phi} = 0$, constraint C_2 will become

$$C_2 = BN \sqrt{h} \frac{\partial c_3}{\partial \phi} R \approx 0. \tag{108}$$

This case is clearly non-physical as it fixes R to be vanishing.

On the other hand, if we choose $\frac{\partial c_3}{\partial \phi} = 0$, constraint C_2 will become

$$C_2 = BN \sqrt{h} \left(\frac{\partial c_1}{\partial \phi} B_{ij} B^{ij} + \frac{\partial c_2}{\partial \phi} B^2 \right) \approx 0. \tag{109}$$

In this case, we can solve all the coefficients

$$c_1 = Nf(\phi), \quad c_2 = CNf(\phi), \quad c_3 = \frac{D}{N},$$
$$c_4 = \frac{8NB^2}{A}, \quad d_1 = \frac{A}{2N}, \quad d_2 = B. \tag{110}$$

The resulting theory thus corresponds to a SCG with an auxiliary scalar field of $d = 2$ that propagates two degrees of freedom. The constraint coefficients α^i in the previous differential equation are also uniquely determined

$$\alpha^1 = -\left(\frac{\partial f(\phi)}{\partial \phi} \right)^{-1} \left(\frac{2}{f(\phi)} \left(\frac{\partial f(\phi)}{\partial \phi} \right)^2 - \frac{\partial^2 f(\phi)}{\partial \phi^2} \right), \quad \alpha^2 = 0. \tag{111}$$

Now let us consider the sub-matrix $\mathcal{E}^{ab}$ of the simplified Dirac matrix $\mathcal{M}'^{ab}$,

$$\mathcal{E}^{ab}(\vec{x}, \vec{y}) = \begin{array}{c|cccc} [\cdot, \cdot] & p(\vec{y}) & \pi(\vec{y}) & p^{kl}(\vec{y}) & C'(\vec{y}) \\ \hline C'(\vec{x}) & 0 & 0 & 0 & 0 \\ \tilde{\pi}^{ij}(\vec{x}) & X & X & X & X \\ \dfrac{\delta S}{\delta \phi(\vec{x})} & X & X & X & X \\ \Phi(\vec{x}) & X & X & X & X \end{array}, \tag{112}$$

whose determinant is obviously zero. Based on the discussion on the null eigenvectors in Appendix (C), we can find a null eigenvector $\mathcal{V}_a^{(\mathcal{E})}$ to build a new first-class constraint $\tilde{C}$,

$$\int \mathrm{d}^3 x \, \tilde{C}(\vec{x}) V(\vec{x}) = \int \mathrm{d}^3 x \begin{pmatrix} p & \pi & p^{kl} & C' \end{pmatrix} (\vec{x})$$
$$\begin{pmatrix} (\mathcal{V}_1)_{ij}^{(\mathcal{E})} & \mathcal{V}_2^{(\mathcal{E})} & \mathcal{V}_3^{(\mathcal{E})} \end{pmatrix} (\vec{x}) \approx 0. \tag{113}$$

To summarize, there are 24 constraints which can be divided into two classes:

$$7 \text{ first} - \text{class}: \quad \pi_i, \quad C_i, \quad \tilde{C},$$
$$16 \text{ second} - \text{class}: \quad p, \quad p^{ij}, \quad \pi, \quad \tilde{\pi}^{ij}, \quad \frac{\delta S}{\delta \phi}, \quad \Phi.$$

The number of DOFs is thus

$$\#_{\text{dof}} = \frac{1}{2} (2 \times \#_{\text{var}} - 2 \times \#_{\text{1st}} - \#_{\text{2nd}}) = \frac{1}{2} (2 \times 17 - 2 \times 7 - 16) = 2. \tag{114}$$

Finally let us discuss another 2DOF condition $\mathcal{S}_2^{(1)}$. By applying the consistency condition of constraint C_2, we can obtain the constraint Φ,

$$\Phi(\vec{x}) = [C'(\vec{x}), H_C] = -B \frac{2N(\vec{x})}{c_1(\vec{x})} \frac{\partial c_1(\vec{x})}{\partial \phi(\vec{x})} B_{ij}(\vec{x}) \frac{\delta S}{\delta h_{ij}(\vec{x})}$$
$$- \int d^3 z BN(\vec{z}) \sqrt{h} \frac{\delta}{\delta h_{ij}(\vec{z})} \left(\frac{\partial c_1}{\partial \phi} B^{ij} B_{ij} + \left(\frac{2c_2}{c_1} \frac{\partial c_1}{\partial \phi} \right. \right.$$
$$\left. \left. - \frac{\partial c_2}{\partial \phi} \right) B^2 - \frac{\partial c_3}{\partial \phi} R(\vec{x}) 2N(\vec{z}) B_{ij}(\vec{z}). \tag{115}$$

Since the second-order derivative of the delta function arises when calculating the integral in the last line, the constraint Φ contains $D^i D_j B_{kl}$, which is proportional to

$$\frac{\partial c_3}{\partial \phi} D^i D_j B_{kl}. \tag{116}$$

So when we require the 2DOF condition $\mathcal{S}_2^{(1)} =$

$\det \mathcal{D}^{ab} \approx 0$, it will also imply that $\frac{\partial c_3}{\partial \phi}$ may be vanishing as one of the conditions for the matrix $\mathcal{D}^{ab}$ to be degenerate, which is sufficient to satisfy 2DOF condition $\mathcal{S}_2^{(2)}$.

VI. CONCLUSION

In this work, we investigated a class of spatially covariant gravity theories with a non-dynamical scalar field. We briefly describe our model in Sec. II. The general Lagrangian is given by (3), which has been proved to have two tensorial and one scalar DOFs [85]. The purpose of this work is to determine the conditions on the Lagrangian, which we dub the 2DOF conditions, under which only two degrees of freedom are propagating.

A perturbative approach has been taken in [86] to derive conditions such that the scalar mode is eliminated at linear order in perturbations. In this work we employ the strict Hamiltonian constraint analysis to derive the conditions such that that only two DOFs are propagating in the nonperturbative sense. In Sec. III, we describe the Hamiltonian formalism for the primary constraints and their consistency conditions. As expected, there are 3 DOFs in the theory if no further conditions are imposed.

In Sec. IV, by requiring the degeneracy of the Dirac matrix, we find that two conditions are required to make the theory propagate only 2DOFs. The first condition (38) states that the matrix $\mathcal{B}^{ab}$ given in (39) must be degenerate, which will result in a secondary constraint. In order to fully eliminate the scalar mode, a second condition is necessary. The second condition can be divided into two categories according to their different effects on the Dirac matrix. In the first case, the condition (49) will generally turn a second-class constraint into a first-class constraint. In the second case, the condition (50) will generate another secondary constraint Φ.

We would like to emphasize that even the Lagrangian satisfies the both two 2DOF conditions, the number of DOFs do not necessarily decrease. A special case is that the constraints are linearly dependent, which implies that the number of DOFs may remain unchanged. Another special case, which we refer to as the non-physical branch of the 2DOF conditions, will make the theory trivial or physically unacceptable (e.g., without tensor modes). One thus must be careful when dealing with the 2DOF conditions in order to pick up the physical case, i.e., theories with precisely two degrees of freedom.

In Sec. V, based on the spatially covariant monomials classified in [57, 58], we consider a concrete Lagrangian (56) built of monomials of $d = 2$ (i.e., with two derivatives) as an illustration of our formal analysis. Since the Dirac matrix of this model contains derivatives of the delta function, it is complicated to calculate the matrix form of the 2DOF conditions of this concrete model. Instead, we choose an equivalent but more effi-cient way to obtain the 2DOF conditions, which is to find out the null eigenvector $\mathcal{V}_a$ satisfing (40). The resulting 2DOF conditions are given in (62) and (107). In particular, for this concrete model, we are able to eliminate the non-physical branch and the linear constraints branch of the 2DOF conditions. Finally, we find a set of coefficients (110), and the resulting Lagrangian corresponds to a spatially covariant gravity theory with an auxiliary scalar field which propagates only two DOFs.

APPENDIX A: COMPARISON WITH SCALAR-TENSOR THEORY IN THE SPATIAL GAUGE

The spatially covariant gravity the an auxiliary scalar field was firstly proposed in [85], in which the Hamiltonian analysis was performed in order to show that the theory propagates 3 degrees of freedom. This idea was originally motivated by generally covariant scalar-tensor theory when the scalar field possesses a spacelike gradient. After choosing the so-called "spatial gauge" as in [85], the resulting action takes the form that is similar to (1). However, we should notice that "spatially covariant gravity the an auxiliary scalar field" and "scalar-tensor theory in the spatial gauge" are completely different theories.

First we will show the difference between (1) and the scalar-tensor theory in the spatial gauge. Let us consider a general action of scalar-tensor theory

$$S_{\text{GST}} = \int d^4 x \sqrt{-g} \mathcal{L} \left(\phi; g_{ab}, \varepsilon_{abcd}, {}^4R_{abcd}; \nabla_a \right), \quad \text{(A1)}$$

where the Lagrangian involves a scalar field ϕ, spacetime metric g_{ab}, the spacetime curvature tensor ${}^4R_{abcd}$ as well as their covariant derivatives. The 4-dimension Levi-Civita tensor ε_{abcd} encodes possible parity violation effects.

In cosmological context, the scalar field ϕ is assumed to possesses a timelike gradient so that the so-called unitary gauge with $\phi = t$ can be chosen. Here we consider the contrary situation by assuming that the scalar field possesses a spacelike gradient[1]. Contrary to the unitary gauge which is defined by requiring $D_a\phi = 0$, we now can choose a gauge in which

$$\pounds_n \phi = 0, \quad \text{(A2)}$$

where n^a is the normal vector to the hypersurfaces. This can be understood that we choose hypersurfaces such that the normal vector n^a exactly lie on the the constant ϕ hypersurfaces. As a result, the value of ϕ does not change

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1) Similar ideas appear in "elastic inflation" [87] and "solid inflation" [88] where it is the Nambu-Goldstone boson breaking spatial diffeomorphism that plays the role of the scalar field.

when being transported along the normal vector. This choice of spatial hypersurfaces is referred as "spatial gauge" in [85].

The normal vector n^a is normalized by $n^a n_a = -1$. The induced metric on the spatial hypersurfaces is as usual

$$h_{ab} = g_{ab} + n_a n_b. \tag{A3}$$

We can then split all the 4-dimensional objects into their temporal and spatial parts. For example, the decomposition of $\nabla_a \phi$ is

$$\nabla_a \phi = -n_a \pounds_{\boldsymbol{n}} \phi + \mathrm{D}_a \phi \xrightarrow{\text{spatial gauge}} \mathrm{D}_a \phi, \tag{A4}$$

where D_a is the covariant derivative compatible with induced metric h_{ab}. In the last equality of (120) we have used the fact that $\pounds_{\boldsymbol{n}} \phi = 0$ in the spatial gauge. This is exactly contrary to what in the unitary gauge where $\mathrm{D}_a \phi = 0$ and thus $\nabla_a \phi \to -n_a \pounds_{\boldsymbol{n}} \phi$.

After making the 3+1 decomposition and choosing the spatial gauge, the action (117) can written in the form

$$S_{\mathrm{GST}}^{(\text{s.g.})} = \int \mathrm{d}t \mathrm{d}^3 x N \sqrt{h} \mathcal{L} \left(\phi, N, h_{ij}, {}^3 R_{ij}; \mathrm{D}_i, \pounds_{\boldsymbol{n}} \right), \tag{A5}$$

where "s.g." stands for the spatial gauge and we have fixed the spatial coordinates adapted to the spacelike hypersurfaces. At the first glance, the action (121) takes the same form as (1), however, the crucial difference is that in the spatial gauge, the scalar field ϕ can be viewed as a time-independent but space-dependent field, which breaks spatial diffeomorphism. In other words, $\phi = \phi(\boldsymbol{x})$ in (121) cannot be viewed as a dynamical nor auxiliary variable, which is actually a "function" of space coordinates with fixed values. This is completely different from (1) which has spatial covariance, in which ϕ plays the role of an auxiliary field.

Then we show that the generally covariant version (i.e., correspondence) of (1) is nothing but a "two-field" scalar-tensor theory. Let us take $\mathrm{D}_i \phi \mathrm{D}^i \phi$ as an example. The covariant version is

$$\mathrm{D}_i \phi \mathrm{D}^i \phi = h^{ij} \mathrm{D}_i \phi \mathrm{D}_j \phi \to \left(g^{ab} + u^a u^b \right) \nabla_a \phi \nabla_b \phi, \tag{A6}$$

where $u_a = -N \nabla_a \Phi$ with Φ is the scalar field defining the spacelike hypersurfaces. In other words, the spacelike hypersurfaces are defined by hypersurfaces with $\Phi = \text{const.}$. In (122), $N = 1/\sqrt{-(\nabla \Phi)^2}$, which reduces to the lapse function when fixing the so-called unitary gauge with $\Phi = t$. By expanding (122) explicitly, we get

$$\mathrm{D}_i \phi \mathrm{D}^i \phi \to \left(g^{ab} - \frac{\nabla^a \Phi N \nabla^b \Phi}{(\nabla \Phi)^2} \right) \nabla_a \phi \nabla_b \phi, \tag{A7}$$

which is clearly a two-field scalar-tensor theory term.

APPENDIX B: CONSTRAINT ALGEBRA

In this appendix, we show the explicit expression for the Poisson brackets.

The Poisson brackets among constraints are

$$\left[p(\vec{x}), \tilde{\pi}^{kl}(\vec{y}) \right] = \frac{1}{2N(\vec{y})} \frac{\delta^2 S}{\delta \phi(\vec{x}) \delta B_{kl}(\vec{y})}, \tag{B1}$$

$$\left[p^{ij}(\vec{x}), \tilde{\pi}^{kl}(\vec{y}) \right] = \frac{1}{2} \frac{1}{N(\vec{y})} \frac{\delta^2 S}{\delta B_{ij}(\vec{x}) \delta B_{kl}(\vec{y})}, \tag{B2}$$

$$\left[\pi(\vec{x}), \tilde{\pi}^{ij}(\vec{y}) \right] = -\frac{1}{2} \delta^3 (\vec{x} - \vec{y}) \frac{1}{N^2(\vec{y})} \frac{\delta S}{\delta B_{ij}(\vec{y})} + \frac{1}{2} \frac{1}{N(\vec{y})} \frac{\delta^2 S}{\delta N(\vec{x}) \delta B_{ij}(\vec{y})}, \tag{B3}$$

$$\left[\tilde{\pi}^{ij}(\vec{x}), \tilde{\pi}^{kl}(\vec{y}) \right] = \frac{1}{2N(\vec{y})} \frac{\delta^2 S}{\delta h_{ij}(\vec{x}) \delta B_{kl}(\vec{y})} - \frac{1}{2N(\vec{x})} \frac{\delta^2 S}{\delta B_{ij}(\vec{x}) \delta h_{kl}(\vec{y})}, \tag{B4}$$

$$[\pi(\vec{x}), C(\vec{y})] = \frac{\delta^2 S}{\delta N(\vec{x}) \delta N(\vec{y})}, \tag{B5}$$

$$[p(\vec{x}), C(\vec{y})] = \frac{\delta^2 S}{\delta \phi(\vec{x}) \delta N(\vec{y})}, \tag{B6}$$

$$\left[p^{ij}(\vec{x}), C(\vec{y}) \right] = -2\delta^3 (\vec{x} - \vec{y}) \pi^{ij}(\vec{y}) + \frac{\delta^2 S}{\delta B_{ij}(\vec{x}) \delta N(\vec{y})}, \tag{B7}$$

$$\left[\tilde{\pi}^{ij}(\vec{x}), C(\vec{y}) \right] = \frac{\delta^2 S}{\delta h_{ij}(\vec{x}) \delta N(\vec{y})} - \frac{1}{N(\vec{x})} \frac{\delta^2 S}{\delta B_{ij}(\vec{x}) \delta h_{kl}(\vec{y})} B_{kl}(\vec{y}), \tag{B8}$$

$$\left[p(\vec{x}), \frac{\delta S}{\delta \phi(\vec{y})} \right] = -\frac{\delta^2 S}{\delta \phi(\vec{x}) \delta \phi(\vec{y})}, \tag{B9}$$

$$\left[p^{ij}(\vec{x}), \frac{\delta S}{\delta \phi(\vec{y})} \right] = -\frac{\delta^2 S}{\delta B_{ij}(\vec{x}) \delta \phi(\vec{y})}, \tag{B10}$$

$$\left[\pi(\vec{x}), \frac{\delta S}{\delta \phi(\vec{y})} \right] = -\frac{\delta^2 S}{\delta N(\vec{x}) \delta \phi(\vec{y})}, \tag{B11}$$

$$\left[\tilde{\pi}^{ij}(\vec{x}), \frac{\delta S}{\delta \phi(\vec{y})} \right] = -\frac{\delta^2 S}{\delta h_{ij}(\vec{x}) \delta \phi(\vec{y})}, \tag{B12}$$

$$[C(\vec{x}), C(\vec{y})] = -\frac{\delta^2 S}{\delta h_{ij}(\vec{y}) \delta N(\vec{x})} 2B_{ij}(\vec{y}) + 2B_{ij}(\vec{x}) \frac{\delta^2 S}{\delta h_{ij}(\vec{x}) \delta N(\vec{y})}, \tag{B13}$$

$$\left[\frac{\delta S}{\delta \phi(\vec{x})}, C(\vec{y}) \right] = \frac{\delta^2 S}{\delta \phi(\vec{x}) \delta h_{ij}(\vec{y})} 2B_{ij}(\vec{y}). \tag{B14}$$

The Poisson brackets involving the canonical Hamiltonian are

$$\left[\tilde{\pi}^{ij}(\vec{x}), H_C\right] = \frac{\delta S}{\delta h_{ij}(\vec{x})} - \frac{1}{N(\vec{x})}$$

$$\int d^3z \frac{\delta^2 S}{\delta B_{ij}(\vec{x})\delta h_{kl}(\vec{z})} N(\vec{z})B_{kl}(\vec{z}), \qquad \text{(B15)}$$

$$\left[C(\vec{x}), H_C\right] = 2B_{ij}(\vec{x})\frac{\delta S}{\delta h_{ij}(\vec{x})}$$

$$- \int d^3z \frac{\delta S}{\delta N(\vec{x})\delta h_{ij}(\vec{z})} N(\vec{z})2B_{ij}(\vec{z}), \qquad \text{(B16)}$$

$$\left[\frac{\delta S}{\delta \phi(\vec{x})}, H_C\right] = \int d^3z \frac{\delta^2 S}{\delta \phi(\vec{x})\delta h_{ij}(\vec{z})} N(\vec{z})2B_{ij}(\vec{z}). \qquad \text{(B17)}$$

APPENDIX C: ADJOINT MATRIX FORM OF NULL EIGENVECTOR

If we wish to find the null eigenvector of the degenerate matrix $\mathcal{B}^{ab}$, one method is to calculate the non-zero rows (or columns) of its corresponding adjoint matrix $\mathcal{B}^*_{ab}$, and to multiply the corresponding elements by $(-1)^{a+b}$ to build the null eigenvector. For simplicity, we assume that $\mathcal{B}^{ab}$ is a simple 3×3 matrix and the rows and columns correspond to constraints ϕ_1, ϕ_2 and ϕ_3.

To verify that this form of null eigenvector is feasible, we assume that the matrix $\mathcal{B}^{ab}$ is degenerate and without loss of generality we take the first row of its adjoint matrix $\mathcal{B}^*_{ab}$ to form a null eigenvector

$$\mathcal{V}_a = \begin{pmatrix} \mathcal{B}^*_{11} & -\mathcal{B}^*_{21} & \mathcal{B}^*_{31} \end{pmatrix}. \qquad \text{(C1)}$$

We then multiply the null eigenvector $\mathcal{V}_a$ by the matrix $\mathcal{B}^*_{ab}$,

$$\mathcal{V}_a\mathcal{B}^{ab} = \begin{pmatrix} \mathcal{B}^*_{11} & -\mathcal{B}^*_{21} & \mathcal{B}^*_{31} \end{pmatrix}\begin{pmatrix} \mathcal{B}^{1b} & \mathcal{B}^{2b} & \mathcal{B}^{3b} \end{pmatrix}^T, \qquad \text{(C2)}$$

which yields

$$\mathcal{V}_a\mathcal{B}^{a1} = \begin{pmatrix} \mathcal{B}^*_{11} & -\mathcal{B}^*_{21} & \mathcal{B}^*_{31} \end{pmatrix}\begin{pmatrix} \mathcal{B}^{11} & \mathcal{B}^{21} & \mathcal{B}^{31} \end{pmatrix}^T$$

$$= \det \mathcal{B}^{ab} = 0, \qquad \text{(C3)}$$

$$\mathcal{V}_a\mathcal{B}^{a2} = \begin{pmatrix} \mathcal{B}^*_{11} & -\mathcal{B}^*_{21} & \mathcal{B}^*_{31} \end{pmatrix}\begin{pmatrix} \mathcal{B}^{12} & \mathcal{B}^{22} & \mathcal{B}^{32} \end{pmatrix}^T$$

$$= \det\begin{pmatrix} \mathcal{B}^{12} & \mathcal{B}^{12} & \mathcal{B}^{13} \\ \mathcal{B}^{22} & \mathcal{B}^{22} & \mathcal{B}^{23} \\ \mathcal{B}^{32} & \mathcal{B}^{32} & \mathcal{B}^{33} \end{pmatrix} = 0, \qquad \text{(C4)}$$

$$\mathcal{V}_a\mathcal{B}^{a3} = \begin{pmatrix} \mathcal{B}^*_{11} & -\mathcal{B}^*_{21} & \mathcal{B}^*_{31} \end{pmatrix}\begin{pmatrix} \mathcal{B}^{13} & \mathcal{B}^{23} & \mathcal{B}^{33} \end{pmatrix}^T$$

$$= \det\begin{pmatrix} \mathcal{B}^{13} & \mathcal{B}^{12} & \mathcal{B}^{13} \\ \mathcal{B}^{23} & \mathcal{B}^{22} & \mathcal{B}^{23} \\ \mathcal{B}^{33} & \mathcal{B}^{32} & \mathcal{B}^{33} \end{pmatrix} = 0. \qquad \text{(C5)}$$

We can combine constraints ϕ_i with the null eigenvectors to make a new constraint

$$\phi'(\vec{y}) = \int d^3x \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \end{pmatrix}(\vec{x})$$

$$\begin{pmatrix} \mathcal{B}^*_{11} & -\mathcal{B}^*_{21} & \mathcal{B}^*_{31} \end{pmatrix}^T(\vec{x},\vec{y}). \qquad \text{(C6)}$$

The Poisson brackets of ϕ' with constraints ϕ_i are all vanishing,

$$\left[\phi_i(\vec{x}), \phi'(\vec{y})\right] \approx \int d^3z \left[\phi_i(\vec{x}), \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \end{pmatrix}(\vec{z})\right]$$

$$\begin{pmatrix} \mathcal{B}^*_{11} & -\mathcal{B}^*_{21} & \mathcal{B}^*_{31} \end{pmatrix}^T(\vec{z},\vec{y})$$

$$= \int d^3z \begin{pmatrix} \mathcal{B}^{1i} & \mathcal{B}^{2i} & \mathcal{B}^{3i} \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{B}^*_{11} & -\mathcal{B}^*_{21} & \mathcal{B}^*_{31} \end{pmatrix}^T(\vec{z},\vec{y}) = 0. \qquad \text{(C7)}$$

In practical calculations, in order to avoid integrating the delta function of higher order, we need to multiply the null eigenvector by a greatest common divisor, i.e.,

$$\mathcal{V}_a = \begin{pmatrix} \mathcal{V}_1 & \mathcal{V}_2 & \mathcal{V}_3 \end{pmatrix} = \frac{1}{\gcd(\mathcal{B}^*_{1i}, \mathcal{B}^*_{2i}, \mathcal{B}^*_{3i})}\begin{pmatrix} \mathcal{B}^*_{1i} & -\mathcal{B}^*_{2i} & \mathcal{B}^*_{3i} \end{pmatrix}. \qquad \text{(C8)}$$

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