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Schwinger 玻色子表示中转动算符的对易性质

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摘　　要

本文发展 Schwinger 玻色子表示中的角动量理论，其中包括用正规乘积内积分法导出转动算符对易子 $[e^{i\alpha J_x}, e^{i\beta J_y}]$ 的正规乘积表达式及不同次序的转动对相干态波函数的影响。还给出了 $e^{\sigma J_-}e^{\lambda J_+}$ 等算符的正规乘积表达式及其对原子相干态的应用.

一、引　言

量子论中绕不同轴的空间转动不对易，例如 $[e^{i\alpha J_x}, e^{i\beta J_y}] \neq 0$，其中 J_x、J_y 是角动量算符，α 与 β 是转角，通常只能算出转角为无穷小时的两个次序不同的转动所产生的位移差[1]．而当 α、β 有限时，上述对易子的解析式尚未见给出过．本文指出用正规乘积内积分法[2]和转动算符的 Schwinger 表示[3]可以算出在转角有限时不同次序转动的对易子的解析式（正规乘积形式），由此可立得相干态[4]波函数在不同转动次序下变化的差别．此外，还可求出 $e^{\sigma J_-}e^{\lambda J_+}$和 $e^{\lambda J_+}e^{\sigma J_-}$（$\lambda$、$\sigma$ 是参数，$J_\pm = J_x \pm iJ_y$）等指数算符乘积的正规乘积形式及其对易子，并且给出了它们在研究原子相干态[5]性质时的应用．以上讨论对 Schwinger 角动量理论的发展和扩大正规乘积内积分技术的应用范围颇有意义．

二、对易子 $[e^{i\alpha J_x}, e^{i\beta J_y}]$ 的导出

角动量算符的 Schwinger 玻色表示为

$$J_x = \frac{1}{2}(a^+b + b^+a),\ J_y = \frac{1}{2i}(a^+b - b^+a),\ J_z = \frac{1}{2}(a^+a - b^+b). \tag{1}$$

其中两维谐振子消灭算符 a、b 和产生算符 a^+、b^+ 满足

$$[a,a^+] = [b,b^+] = 1,[a,b] = 0,[a,b^+] = 0, a|00\rangle = b|00\rangle = 0. \tag{2}$$

其中 $|00\rangle$ 是两维谐振子真空态，按文献[2]它有性质

$$|00\rangle\langle 00| = :e^{-a^+a-b^+b}:,\ (::\text{指正规乘积}) \tag{3}$$

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引入具有非正交性和超完备性的 Glauber 相干态 $|z_1z_2\rangle$

$$|z_1z_2\rangle = \exp\left[-\frac{1}{2}(|z_1|^2 + |z_2|^2) + z_1a^+ + z_2b^+\right]|00\rangle, a|z_1z_2\rangle$$

$$= z_1|z_1z_2\rangle, b|z_1z_2\rangle = z_2|z_1z_2\rangle, \tag{4}$$

$$\langle z_1'z_2'|z_1z_2\rangle = \exp\left[-\frac{1}{2}(|z_1|^2 + |z_2|^2 + |z_1'|^2 + |z_2'|^2) + z_1'^*z_1 + z_2'^*z_2\right]. \tag{5}$$

$$\int\frac{d^2z_1d^2z_2}{\pi^2}|z_1z_2\rangle\langle z_1z_2| = 1. \tag{6}$$

由(1)(2)式易证得以下性质

$$e^{i\beta J_y}|00\rangle = e^{i\alpha J_x}|00\rangle = e^{i\gamma J_z}|00\rangle = |00\rangle, \tag{7}$$

$$e^{i\alpha J_x}a^+e^{-i\alpha J_x} = a^+\cos\frac{\alpha}{2} + ib^+\sin\frac{\alpha}{2},\ e^{i\alpha J_x}b^+e^{-i\alpha J_x} = b^+\cos\frac{\alpha}{2} + ia^+\sin\frac{\alpha}{2}, \tag{8}$$

$$e^{i\beta J_y}ae^{-i\beta J_y} = a\cos\frac{\beta}{2} - b\sin\frac{\beta}{2}, e^{i\beta J_y}be^{-i\beta J_y} = b\cos\frac{\beta}{2} + a\sin\frac{\beta}{2}. \tag{9}$$

$$e^{\mu J_z}a^+e^{-\mu J_z} = a^+e^{\frac{\mu}{2}},\ e^{\mu J_z}b^+e^{-\mu J_z} = b^+e^{-\frac{\mu}{2}}, (\mu\ \text{是参数}). \tag{9'}$$

类似于文献[7]中作者的做法，先把 $e^{i\alpha J_x}e^{i\beta J_y}$ 化为正规乘积，用正规乘积内积分法和(6)—(9)式和数学公式

$$\int\frac{d^2z}{\pi}e^{-\zeta|z|^2+\xi z+\eta z^*} = -\frac{1}{\zeta}\exp\left[-\frac{\xi\eta}{\zeta}\right],\ R_e\zeta < 0, \tag{10}$$

得到

$$e^{i\alpha J_x}e^{i\beta J_y} = \int\frac{d^2z_1d^2z_2}{\pi^2}e^{i\alpha J_x}|z_1z_2\rangle\langle z_1z_2|e^{i\beta J_y}$$

$$= :\int\frac{d^2z_1d^2z_2}{\pi^2}e^{-|z_1|^2-|z_2|^2+z_1\left(a^+\cos\frac{\alpha}{2}+ib^+\sin\frac{\alpha}{2}\right)+z_2\left(b^+\cos\frac{\alpha}{2}+ia^+\sin\frac{\alpha}{2}\right)}$$
$$e^{z_1^*\left(a\cos\frac{\beta}{2}+b\sin\frac{\beta}{2}\right)+z_2^*\left(b\cos\frac{\beta}{2}-a\sin\frac{\beta}{2}\right)-a^+a-b^+b}:$$

$$= :\exp\left[a^+a\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - i\sin\frac{\alpha}{2}\sin\frac{\beta}{2} - 1\right)\right.$$
$$+ b^+b\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + i\sin\frac{\alpha}{2}\sin\frac{\beta}{2} - 1\right)$$
$$+ b^+a\left(i\sin\frac{\alpha}{2}\cos\frac{\beta}{2} - \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right)$$
$$\left.+ a^+b\left(\cos\frac{\alpha}{2}\sin\frac{\beta}{2} + i\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\right)\right]: \tag{11}$$

$$= :\exp\left\{(a^+b^+)\begin{pmatrix}\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - i\sin\frac{\alpha}{2}\sin\frac{\beta}{2} - 1 & \cos\frac{\alpha}{2}\sin\frac{\beta}{2} + i\sin\frac{\alpha}{2}\cos\frac{\beta}{2} \\ i\sin\frac{\alpha}{2}\cos\frac{\beta}{2} - \cos\frac{\alpha}{2}\sin\frac{\beta}{2} & \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + i\sin\frac{\alpha}{2}\sin\frac{\beta}{2} - 1\end{pmatrix}\begin{pmatrix}a \\ b\end{pmatrix}\right\}: \tag{12}$$

$$= :\exp\left[\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-1\right)s+2i\sin\frac{\alpha}{2}\cos\frac{\beta}{2}J_x+2i\cos\frac{\alpha}{2}\sin\frac{\beta}{2}J_y\right.$$

$$\left.-2i\sin\frac{\alpha}{2}\sin\frac{\beta}{2}J_z\right]:, \tag{13}$$

其中 $s\equiv a^+a+b^+b$. 用同样步骤可以求出

$$e^{i\beta J_y}e^{i\alpha J_x}= :\exp\left[a^+a\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+i\sin\frac{\alpha}{2}\sin\frac{\beta}{2}-1\right)\right.$$

$$+b^+b\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-i\sin\frac{\alpha}{2}\sin\frac{\beta}{2}-1\right)$$

$$-b^+a\left(\cos\frac{\alpha}{2}\sin\frac{\beta}{2}-i\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\right)$$

$$\left.+a^+b\left(\cos\frac{\alpha}{2}\sin\frac{\beta}{2}+i\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\right)\right]: \tag{14}$$

$$= :\exp\left[\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-1\right)s+2i\sin\frac{\alpha}{2}\cos\frac{\beta}{2}J_x+2i\cos\frac{\alpha}{2}\sin\frac{\beta}{2}J_y\right.$$

$$\left.+2i\sin\frac{\alpha}{2}\sin\frac{\beta}{2}J_z\right]:. \tag{15}$$

注意到正规乘积具有以下性质[6]

$$:U\cdots V: + :W\cdots X: = :(U\cdots V+W\cdots X):,\ :U\cdots V\cdots X: + :U\cdots W\cdots X:$$

$$= :U\cdots(V+W)\cdots X:, \tag{16}$$

$$\langle z_1'|:f(a,a^+):|z_1\rangle = f(z_1,z_1'^*)\exp\left[-\frac{1}{2}(|z_1'|^2+|z_1|^2)+z_1'^*z_1\right]. \tag{17}$$

因此由 (13)(15) 式给出以下对易子

$$[e^{i\alpha J_x},e^{i\beta J_y}] = 2i:\exp\left[\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-1\right)s+2i\sin\frac{\alpha}{2}\cos\frac{\beta}{2}J_x\right.$$

$$\left.+2i\cos\frac{\alpha}{2}\sin\frac{\beta}{2}J_y\right]\sin\left(-2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}J_z\right):. \tag{18}$$

此即本文首次导出的转角有限时两种顺序不同的转动的差别的正规乘积表式. 当 α、β 是一级无穷小,分别把(18)左、右两边展到二级无穷小可得正确极限,即

$$[e^{i\alpha J_x},\ e^{i\beta J_y}] = -i\alpha\beta J_z,\ (\alpha,\ \beta\ 是一级无穷小). \tag{19}$$

作为(18)的应用,考虑 $|z_1z_2\rangle$ 分别经 $e^{i\alpha J_x}e^{i\beta J_y}$ 和经 $e^{i\beta J_y}e^{i\alpha J_x}$ 转动到 $|z_1'z_2'\rangle$ 的两个矩阵元之差,由 (4)、(17)、(18) 立刻得到

$$\langle z_1'z_2'|[e^{i\alpha J_x},\ e^{i\beta J_y}]|z_1z_2\rangle = 2i\exp\left[-\frac{1}{2}(|z_1|^2+|z_2|^2+|z_1'|^2+|z_2'|^2)\right.$$

$$+\cos\frac{\alpha}{2}\cos\frac{\beta}{2}(z_1'^*z_1+z_2'^*z_2)+i\sin\frac{\alpha}{2}\cos\frac{\beta}{2}(z_1'^*z_2+z_2'^*z_1)$$

$$\left.+\cos\frac{\alpha}{2}\sin\frac{\beta}{2}(z_1'^*z_2-z_2'^*z_1)\right]\sin\left\{-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}(z_1'^*z_1-z_2'^*z_2)\right\}. \tag{20}$$

三、$e^{\lambda J_+}e^{\sigma J_-}$ 的正规乘积形式的导出

用(1)式得 $J_+ = a^+b$, $J_- = b^+a$, 它们满足 $e^{\lambda J_+}|00\rangle = e^{\sigma J_-}|00\rangle = |00\rangle$ 及

$$e^{\lambda J_+}b^+e^{-\lambda J_+} = b^+ + \lambda a^+,\ e^{\sigma J_-}a^+e^{-\sigma J_-} = a^+ + \sigma b^+,\ e^{\lambda J_+}(a^+ + \sigma b^+)e^{-\lambda J_+} = a^+ + \sigma(b^+ + \lambda a^+). \tag{21}$$

用(6)、(10)、(21)式可得以下算符的正规乘积形式

$$e^{\lambda J_+}e^{\sigma J_-} = \int \frac{d^2z_1d^2z_2}{\pi^2} :e^{-|z_1|^2-|z_2|^2+z_1[a^+(1+\lambda\sigma)+\sigma b^+]+z_1^*a+z_2(b^++\lambda a^+)+z_2^*b-a^+a-b^+b}: = :\exp[\lambda\sigma a^+a + \sigma J_- + \lambda J_+]:, \tag{22}$$

$$e^{\sigma J_-}e^{\lambda J_+} = :\exp[\lambda J_+ + \sigma J_- + \lambda\sigma b^+b]:. \tag{23}$$

由此立刻得到

$$\langle z_1'z_2'|e^{\lambda J_+}e^{\sigma J_-}|z_1z_2\rangle = \exp\Big[-\frac{1}{2}(|z_1'|^2 + |z_2'|^2 + |z_1|^2 + |z_2|^2) + (\lambda\sigma + 1)z_1'^*z_1 + z_2'^*z_2 + \sigma z_2'^*z_1 + \lambda z_1'^*z_2\Big], \tag{24}$$

$$\langle z_1'z_2'|e^{\sigma J_-}e^{\lambda J_+}|z_1z_2\rangle = \exp\Big[-\frac{1}{2}(|z_1'|^2 + |z_2'|^2 + |z_1|^2 + |z_2|^2) + (\lambda\sigma + 1)z_2'^*z_2 + z_1'^*z_1 + \sigma z_2'^*z_1 + \lambda z_1'^*z_2\Big], \tag{25}$$

$$[e^{\lambda J_+},\ e^{\sigma J_-}] = :\exp[\lambda J_+ + \sigma J_-](e^{\lambda\sigma a^+a} - e^{\lambda\sigma b^+b}):. \tag{26}$$

作为(23)的应用,计算用下式定义的原子相干态的内积

$$|\theta\varphi\rangle = \Big(\frac{1}{1+|\tau|^2}\Big)^j e^{\tau J_+}|j,\ -j\rangle,\quad \tau = e^{-i\varphi}\mathrm{tg}\frac{\theta}{2}, \tag{27}$$

其中 $|j,-j\rangle$ 按 Schwinger 角动量态的定义 $|jm\rangle = [(j+m)(j-m)!]^{-1/2}a^{+j+m}b^{+j-m}|00\rangle$ 为

$$|j,\ -j\rangle = \frac{b^{+2j}}{\sqrt{(2j)!}}|00\rangle,\ \langle j,-j|z_1z_2\rangle = e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)}\frac{z_2^{2j}}{\sqrt{(2j)!}}. \tag{28}$$

于是 $|\theta\varphi\rangle$ 的内积由(23)(6)(28)可知为

$$\begin{aligned}\langle\theta'\varphi'|\theta\varphi\rangle &= \Big[\frac{1}{(1+|\tau|^2)(1+|\tau'|^2)}\Big]^j\langle j,-j|e^{\tau'^*J_-}e^{\tau J_+}|j,-j\rangle \\ &= \Big[\frac{1}{(1+|\tau|^2)(1+|\tau'|^2)}\Big]^j\int\frac{d^2z_1d^2z_2d^2z_1'd^2z_2'}{\pi^4} \\ &\quad\times e^{-|z_1|^2-|z_2|^2-|z_1'|^2-|z_2'|^2+\tau z_1'^*z_2+\tau'^*z_2'^*z_1+(1+\tau\tau'^*)z_2'^*z_2+z_1'^*z_1}\frac{(z_2'z_2^*)^{2j}}{(2j)!} \\ &= \Big[\frac{1}{(1+|\tau|^2)(1+|\tau'|^2)}\Big]^j(1+\tau\tau'^*)^{2j},\end{aligned} \tag{29}$$

其中利用了数学公式

$$\int\frac{d^2z}{\pi}f(z^*)e^{\zeta|z|^2+cz} = -\frac{1}{\zeta}f\Big(-\frac{c}{\zeta}\Big),\ \int\frac{d^2z}{\pi}z^mz^{*n}e^{\zeta|z|^2} = \delta_{mn}m!(-)^{m+1}\zeta^{-(m+1)},\ R_e\zeta < 0. \tag{30}$$

由此可见，指数算符 $e^{\sigma J_-}e^{\lambda J_+}$等的正规乘积形式的优点是极易求出其相干态矩阵元，从而容易过渡到求其他态的矩阵元。用(9′)及类似方法可得算符恒等式

$$e^{\mu J_z}e^{\sigma J_-} = :\exp[(e^{\frac{\mu}{2}} - 1)a^+a + e^{-\frac{\mu}{2}}\sigma J_- + (e^{-\frac{\mu}{2}} - 1)b^+b]:, \tag{31}$$

$$e^{\mu J_z}e^{\lambda J_+} = :\exp[(e^{\frac{\mu}{2}} - 1)a^+a + (e^{-\frac{\mu}{2}} - 1)b^+b + \lambda J_+e^{\frac{\mu}{2}}]:, \tag{32}$$

$$e^{\sigma J_-}e^{\mu J_z} = :\exp[(e^{\frac{\mu}{2}} - 1)a^+a + e^{\frac{\mu}{2}}\sigma J_- + (e^{-\frac{\mu}{2}} - 1)b^+b]:, \tag{33}$$

$$e^{\lambda J_+}e^{\mu J_z} = :\exp[(e^{\frac{\mu}{2}} - 1)a^+a + (e^{-\frac{\mu}{2}} - 1)b^+b + e^{-\frac{\mu}{2}}\lambda J_+]:. \tag{34}$$

由(31)(33)式导出对易子

$$[e^{\sigma J_-}, e^{\mu J_z}] = :\exp[(e^{\frac{\mu}{2}} - 1)a^+a + (e^{-\frac{\mu}{2}} - 1)b^+b]\{\exp[e^{\frac{\mu}{2}}\sigma J_-] - \exp[e^{-\frac{\mu}{2}}\sigma J_-]\}: \tag{35}$$

此式在 $\sigma, \mu \to 0$ 时给出 $[J_-, J_z] = J_-$。 上述方法还可以推广到求多个角动量指数算符的乘积的正规乘积形式。

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COMMUTATIVE PROPERTIES OF OPERATORS IN SCHWINGER BOSON REPRESENTATION

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Abstract

The angular momentum theory in the schwinger boson representation is developed to derive a new normal product expression for the commutator $[e^{i\alpha J_x}, e^{i\beta J_y}]$ by means of the integral technique within normal product, and the effect on the coherent state caused by two rotations which are in different seqences. Some new normal product expressions for exponential operators, such as $e^{\sigma J_-}e^{\lambda J_+}$ etc, are also derived and their applications in atomic coherent states are presented.