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New Realization of $N=2$ Supersymmetric Quantum Mechanics and Shape Invariance*

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Abstract The new supercharges are constructed and the weight function is defined to study the $N=2$ one-dimensional supersymmetric quantum mechanics. Several examples are discussed in the new realization.

Key words supersymmetric quantum mechanics, Hamiltonian hierarchy, Schrödinger equation, shape invariance

1 Introduction

While studying the dynamical breaking of supersymmetry, in 1981 Witten[1] constructed a simple, but not trivial model——supersymmetric quantum mechanics (SSQM)[2], namely a system of N Hermitian supercharges $Q_i(i=1,2,\cdots,N)$ and supersymmetric Hamiltonian H, satisfying the following relations:

$$\{Q_i,Q_j\}=2H\delta_{ij},\qquad [Q_i,H]=0,\qquad Q_i^{\dagger}=Q_i\qquad (i=1,2,\cdots,N).\tag{1.1}$$

For the $N=2$ case, Eq. (1.1) can be expressed alternatively,

$$\{Q_+,Q_-\}=2H,\qquad Q_{\pm}^2=0,\qquad [Q_{\pm},H]=0,\qquad Q_+^{\dagger}=Q_-,\tag{1.2}$$

by introducing

$$Q_{\pm}=\frac{1}{\sqrt{2}}(Q_1\pm \mathrm{i}Q_2).\tag{1.3}$$

The problem of constructing the realization is an important one in study of SSQM. In one-dimensional and $N=2$ cases, the realization of SSQM in common use is given by the following form of supercharges:

$$Q_{\pm}=\left[\pm\frac{\mathrm{d}}{\mathrm{d}x}+W(x)\right]\sigma^{\pm},\tag{1.4}$$

and the supersymmetric Hamiltonian is

$$H^{\mathrm{SS}}=\frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+W\right)I+\frac{W'}{2}\sigma_3,\tag{1.5}$$

where $W(x)$ is the superpotential and the Pauli matrices are given by

$$\sigma^{\pm}=\frac{1}{2}(\sigma_1\pm \mathrm{i}\sigma_2),\qquad \sigma^{+}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad \sigma^{-}=\begin{pmatrix}0&0\\1&0\end{pmatrix}.\tag{1.6}$$

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Here we will study a new kind of realization, which will include more general cases of one-dimensional nonrelativistic quantum mechanical systems in the framework of $N=2$ SSQM (section 2). We will study the relevant Hamiltonian hierarchy and shape invariance (section 3) and some examples (section 4).

2 New Realization

Recently we studied on a new realization on SSQM[3]. Starting from the generalized supercharges

$$Q_1 = (M_1\sigma_1 + N\sigma_2)\frac{d}{dx} + K\sigma_1 + L\sigma_2, \quad Q_2 = (R_1\sigma_1 + S\sigma_2)\frac{d}{dx} + T\sigma_1 + V\sigma_2, \tag{2.1}$$

where M, N, K, L, R, S, T and V are (complex in general) functions of x. From the algebraic structure of SSQM, namely $Q_1^2 = Q_2^2 = H$ and $\{Q_1, Q_2\} = 0$, we can obtain the relations among those functions:

$$\begin{array}{ll} \text{Case(a)} & S = M,\ R = -N,\ T = -L,\ V = K; \\ \text{Case(b)} & S = -M,\ R = N,\ T = L,\ V = -K; \\ \text{Case(c)} & N = iM = \pm iS = \mp R,\ T = \mp L,\ V = \pm K; \\ \text{Case(d)} & N = -iM = \pm iS = \pm R,\ T = \pm L,\ V = \mp K. \end{array} \tag{2.2}$$

It is easy to see that, the Case(b) is physically equivalent to the Case(a); and the Cases(c) and (d) would lead to the H^{SS} excluding the $\frac{d^2}{dx^2}$ term, consequently are not physically acceptable. We, therefore, could adopt the following choice of the Case(a) and then the supercharges can be written as

$$Q_1 = (M\sigma_1 + N\sigma_2)\frac{d}{dx} + K\sigma_1 + L\sigma_2, \quad Q_2 = (-N\sigma_1 + M\sigma_2)\frac{d}{dx} - L\sigma_1 + K\sigma_2, \tag{2.3}$$

or alternatively,

$$Q_+ = \sqrt{2}\left[(M - iN)\frac{d}{dx} + K - iL\right]\sigma^+, \quad Q_- = \sqrt{2}\left[(M + iN)\frac{d}{dx} + K + iL\right]\sigma^-. \tag{2.4}$$

The following equation of the inner product should be satisfied due to the Hermiticity of supercharges, $Q_i^\dagger = Q_i$, or

$$(\psi, Q_i\phi) = (Q_i\psi, \phi) \qquad (i = 1,2). \tag{2.5}$$

Where, the inner product is defined as

$$(\xi, \eta) = \int \xi^*(x)\rho(x)\eta(x)dx \tag{2.6}$$

and the real function $\rho(x)$ is the weight function, which is to be chosen together with M, N, K and L, satisfying

$$\begin{array}{cc} M + M^* = 0, & N + N^* = 0, \\ (K^* - K - M^{*\prime})\rho - M^*\rho' = 0, & (L^* - L - N^{*\prime})\rho - N^*\rho' = 0. \end{array} \tag{2.7}$$

Eq. (2.7) are derived from Eqs. (2.3), (2.5) and (2.6). It is easy to see from the above equation that M and N are purely imaginary, satisfying

$$N^*(K^* - K - M^{*\prime}) = M^*(L^* - L - N^{*\prime}), \tag{2.8}$$

and ρ could be solved

$$\rho = \exp\left(\int \frac{K^* - K - M^{*\prime}}{M^*}dx\right) = \exp\left(\int \frac{L^* - L - N^{*\prime}}{N^*}dx\right). \tag{2.9}$$

If $M=0$, $N=\mathrm{i}\frac{C}{\sqrt{2}}$, $K=\frac{(B+D)}{2\sqrt{2}}$, $L=\mathrm{i}\frac{(B-D)}{2\sqrt{2}}$, where B, C and D are real functions of x, then we have

$$Q_+ = \left(C\frac{\mathrm{d}}{\mathrm{d}x}+B\right)\sigma^+ \equiv A^+\sigma^+, \quad Q_- = \left(-C\frac{\mathrm{d}}{\mathrm{d}x}+D\right)\sigma^- \equiv A^-\sigma^- \tag{2.10}$$

In this case,

$$\rho = \tag{2.11}$$

and the supersymmetric Hamiltonian

$$H^{\mathrm{SS}} = \frac{1}{\sqrt{2}}\left[-C^2\frac{\mathrm{d}^2}{\mathrm{d}^2x}+ \qquad BD+\frac{C(D'-B')}{2}\right]I+ \frac{C(D'+B')}{2\sqrt{2}}\sigma_3 \equiv \begin{pmatrix} H_+ \\ 0 \end{pmatrix} \tag{2.12}$$

This is the new realization discussed in this work. Obviously, supercharges Eq. (2.10) will be reduced to Eq. (1.4) if $C=1$ and $B=D\equiv W$.

3 New Hamiltonian Hierarchy and Shape Invariance

Using Sukumar's method[4], we can construct a Hamiltonian hierarchy $\{H_n \mid n=0,1,2,\cdots\}$ where the H_n can be represented by

$$\begin{aligned} H_n &= H_n^+ + E_n^0 = \frac{1}{2}A_n^+A_n^- + E_n^0 \qquad (n=0,1,2,\cdots), \\ H_n &= H_{n-1}^- + E_{n-1}^0 = \frac{1}{2}A_{n-1}^-A_{n-1}^+ + E_{n-1}^0 \quad (n=1,2,\cdots), \end{aligned} \tag{3.1}$$

with the definitions of $A_n^\pm$:

$$A_n^+ = C_n\frac{\mathrm{d}}{\mathrm{d}x}+B_n, \quad A_n^- = -C_n\frac{\mathrm{d}}{\mathrm{d}x}+D_n \qquad (n=0,1,\cdots). \tag{3.2}$$

It is not difficult to find:

1) Since an arbitrary one-dimensional Hamiltonian with second order derivative can be factorized as $H=A^+A^-+E^0$, and can be adopted as H_0, we can always construct a Hamiltonian hierarchy $\{H_n\}$ including a certain Hamiltonian as H_0.

2) The mth eigenvalues E_n^m and the mth eigenfunctions ψ_n^m of the Hamiltonian H_n are linked by the following relations:

$$\begin{aligned} E_n^m &= E_{n-1}^{m+1} = \cdots = E_0^{n+m} \quad (m=0,1,2,\cdots, \quad n=1,2,3,\cdots); \\ \psi_n^m &= \{[E_0^{n+m}-E_0^n]\cdots[E_0^{n+m}-E_0^0]\}^{-\frac{1}{2}}A_{n-1}^-A_{n-2}^-\cdots A_0^-\psi_0^{n+m} \end{aligned} \tag{3.3}$$

3) Two neighbouring Hamiltonians in the hierarchy, H_n and H_{n+1} (or, more exactly H_n^+ and H_n^-) are supersymmetric partner Hamiltonians, i.e.

$$H_n^{\mathrm{SS}} = \begin{pmatrix} H_n^+ & 0 \\ 0 & H_n^- \end{pmatrix}. \tag{3.4}$$

4) If the hierarchy $\{H_n^\pm \mid n=0,1,2,\cdots\}$ can be parametrized by

$$H_n^+ = \frac{1}{2}A_n^+(x,a_n)A_n^-(x,a_n), \quad H_n^- = \frac{1}{2}A_n^-(x,a_n)A_n^+(x,a_n), \tag{3.5}$$

where

$$A^+(x,a_n) = C(x,a_n)\frac{\mathrm{d}}{\mathrm{d}x}+B(x,a_n), \quad A^-(x,a_n) = -C(x,a_n)\frac{\mathrm{d}}{\mathrm{d}x}+D(x,a_n), \tag{3.6}$$

and the shape invariance conditions[6]

$$A^+(x,a_n)A^-(x,a_n) - A^-(x,a_{n-1})A^+(x,a_{n-1}) = -2R(a_n),\ a_n = f(a_{n-1}), \tag{3.7}$$

or

$$\begin{aligned} C(x,a_0) = C(x,a_1) = \cdots = C(x,a_n) \equiv C(x), \quad a_n \quad f(a_{n-1}),\\ D(x,a_0) - B(x,a_0) = D(x,a_1) - B(x,a_1) = \cdots =\\ D(x,a_n) - B(x,a_n) \equiv -2g(x),\\ D(x,a_n)B(x,a_n) - D(x,a_{n-1})B(x,a_{n-1}) +\\ C(x)[B'(x,a_{n-1}) + D'(x,a_n)] = -2R(a_n). \end{aligned} \tag{3.8}$$

are satisfied, then $DB + CD'$ and $DB - CB'$ are called shape invariant potentials. An arbitrary Hamiltonian H with a shape invariant potential can be factorized in the form

$$H = \frac{1}{2}A^+(x,a_0)A^-(x,a_0) + E_0^0 \equiv H_0, \tag{3.9}$$

satisfying the shape invariance conditions Eq. (3.7) or (3.8), and the eigenvalues and the eigenstates can be easily solved:

$$\begin{aligned} &H\psi_n = E^n\psi_n \quad \text{or} \quad H_0\psi_n(x,a_0) = E_0^n\psi_n(x,a_0) \quad (n = 0,1,\cdots),\\ &E_0^n = E_0^n = \sum_{k=1}^{n} R(a_k) + E_0^0,\\ &\psi_n \equiv \psi_n(x,a_0) \propto A^+(x,a_0)A^+(x,a_1)\cdots A^+(x,a_{n-1})\psi_0(x,a_n),\\ &\quad A^-(x,a_n)\psi_0(x,a_n) = 0. \end{aligned} \tag{3.10}$$

4 Examples

1) We take $C_l = 1$, $B_l = \frac{l+2}{r} - \frac{e^2}{l+1}$, $D_l = \frac{l}{r} - \frac{e^2}{l+1}$, $(l = 0,1,\cdots)$, then

$$A_l^+ = \frac{d}{dr} + \frac{l+2}{r} - \frac{e^2}{l+1}, \quad A_l^- = -\frac{d}{dr} + \frac{l}{r} - \frac{e^2}{l+1}, \tag{4.1}$$

where e is to be interpreted as the charge of the electron. This example shows the super-symmetric structure of the radial Schrödinger equation of the hydrogen atom,

$$H_lR_{nl} = \frac{1}{2}\left[-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} - \frac{2e^2}{r}\right]R_{nl} = E_nR_{nl}. \tag{4.2}$$

2) We take

$$C_l = 1, \quad B_l = \frac{l+2}{r} - \omega r, \quad D_l = \frac{l}{r} - \omega r, \tag{4.3}$$

then

$$A_l^+ = \frac{d}{dr} + \frac{l+2}{r} - \omega r, \quad A_l^- = -\frac{d}{dr} + \frac{l}{r} - \omega r. \tag{4.4}$$

This example shows the supersymmetric structure of the radial Schrödinger equation of the three-dimensional isometric harmonic oscillator,

$$H_lR_{nl} = \frac{1}{2}\left[-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} + \omega^2r^2\right]R_{nl} = E_nR_{nl}. \tag{4.5}$$

The detailed study of examples 1) and 2) has been shown in our previous work[5] and the shape invariance was studied in Ref. [6]. Here we will study the following interesting example.

3) The Schrödinger equation in a curved space.

Katayama[7,8] studied the Schrödinger equation in a 3-dimensional space of constant curvature

K. The classical Hamiltonian of the harmonic oscillator is

$$H = \frac{1}{2}\left(1+\frac{1}{4}Kr^2\right)^2\sum_{i=1}^{3}p_i^2 + \left(1-\frac{1}{4}Kr^2\right)^{-2}\beta r^2 \tag{4.6}$$

with the momentum

$$p_i = \left(1+\frac{1}{4}Kr^2\right)^{-2}\frac{\mathrm{d}x^i}{\mathrm{d}t} \quad (i = 1,2,3), \tag{4.7}$$

and then the quantum mechanical one is

$$H = -\frac{1}{2}\left(1+\frac{1}{4}Kr^2\right)^2\sum_{i=1}^{3}\partial_i^2 + \frac{1}{4}K\left(1+\frac{1}{4}Kr^2\right)\sum_{i=1}^{3}\chi^i\partial_i + \left(1-\frac{1}{4}Kr^2\right)^{-2}\beta r^2, \tag{4.8}$$

where β is a constant and $r = \left[\sum_{i=1}^{3}(x^i)^2\right]^{\frac{1}{2}}$. After performing the coordinate transformations

$$\chi^1 = g(y)\cos\theta, \quad \chi^2 = g(y)\sin\theta\cos\varphi, \quad \chi^3 = g(y)\sin\theta\sin\varphi, \tag{4.9}$$

where

$$g(y) = \begin{cases} \frac{2}{\sqrt{K}}\tan\left(\frac{1}{2}\sqrt{K}y\right) & K > 0,\ 0 \leqslant y \leqslant \frac{\pi}{\sqrt{K}} \\ y & K = 0, y \geqslant 0 \\ \frac{2}{\sqrt{-K}}\tanh\left(\frac{1}{2}\sqrt{-K}y\right) & K < 0, y \geqslant 0 \end{cases} \tag{4.10}$$

the Schrödinger equation, $H\psi_n = E_n\psi_n$, can be written as

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2G^2(y)}L^2 - \frac{G'(y)}{G(y)}\frac{\partial}{\partial y} + V(y)\right]\psi_{nl} = E_n\psi_{nl}, \tag{4.11}$$

where

$$L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}, \tag{4.12}$$

$$V(y) = \begin{cases} \frac{\beta}{K}\tan^2(\sqrt{K}y) & K > 0 \\ \beta y^2 & K = 0, \\ -\frac{\beta}{K}\tanh^2(\sqrt{-K}y) & K < 0 \end{cases} \tag{4.13}$$

$$G(y) = \begin{cases} \frac{1}{\sqrt{K}}\sin(\sqrt{K}y) & K > 0 \\ y & K = 0 \\ \frac{1}{\sqrt{-K}}\sinh(\sqrt{-K}y) & K < 0 \end{cases} \tag{4.14}$$

After separating variables

$$\psi_{nl}(y,\theta,\phi) = R_{nl}(y)Y_l(\theta,\phi) \tag{4.15}$$

we get the radial equation

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{G'(y)}{G(y)}\frac{\mathrm{d}}{\mathrm{d}y} + \frac{l(l+1)}{2G^2(y)} + V(y)\right]R_{nl}(y) = E_nR_{nl}(y). \tag{4.16}$$

For the $K<0$ case, let $\chi = \sqrt{-K}y$, then we can rewrite (4.16) as

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} - 2\coth\chi\frac{\mathrm{d}}{\mathrm{d}\chi} + \frac{l(l+1)}{\sinh^2\chi} - \frac{2V}{K}\right]R_{nl} = -\frac{2E_n}{K}R_{nl}. \tag{4.17}$$

We have the weight function $\rho = \sinh^2\chi$. For example, we choose $C_l = 1$, $B_l = (l+2)\coth\chi + \frac{\sqrt{2\beta_l}}{K}\tanh\chi$ and $D_l = l\coth\chi + \frac{\sqrt{2\beta_l}}{K}\tanh\chi$, or by parametrization in section 3, we rewrite

$$\begin{aligned} C(\chi, a_s, b_s) &= 1, \\ B(\chi, a_s, b_s) &= (l + a_s + 2)\coth\chi - b_s\tanh\chi, \\ D(\chi, a_s, b_s) &= (l + a_s)\coth\chi - b_s\tanh\chi, \end{aligned} \tag{4.18}$$

where $s = 0, 1, 2, \cdots$. It is easy to check that Eq. (4.18) satisfies the shape invariance conditions Eq. (3.7) or (3.8), in which

$$\begin{aligned} &C(\chi) = 1, \quad g(\chi) = \coth\chi, \quad R(a_s, b_s) = 2b_s - 2(l + a_s), \\ &= f(a_s) \quad a_s + 1, \quad a_0 = 0, \quad a_s = s, \\ &= h(b_s) \quad b_s - 1, \quad b_0 = -\frac{\sqrt{2\beta_l}}{K}, \quad b_s = -\frac{\sqrt{2\beta_l}}{K} - s. \end{aligned} \tag{4.19}$$

Thus,

$$\begin{aligned} &= \frac{1}{2}A^+(\chi, a_0, b_0)A^-(\chi, a_0, b_0) + E_0^0 = \\ &\frac{1}{2}\left[-\frac{d^2}{d\chi^2} - 2\coth\chi\frac{d}{d\chi} + \frac{l(l+1)}{\sinh^2\chi} + \left(\frac{2\beta_l}{K^2} - \frac{\sqrt{2\beta_l}}{K}\right)\tanh^2\chi + \right. \\ &\left.(2l+3)\frac{\sqrt{2\beta_l}}{K} + l(l+2)\right] - \frac{E_0^0}{K}, \end{aligned} \tag{4.20}$$

and is exactly solvable. Therefore

$$H_0\psi^s(\chi, a_0, b_0) = -\frac{E_0^s}{K}\psi^s(\chi, a_0, b_0) \quad (s = 0, 1, 2, \cdots), \tag{4.21}$$

$$-\frac{E_0^s}{K} = -\frac{E_s^0}{K} = \sum_{k=1}^{s} R(a_k, b_k) - \frac{E_0^0}{K} = 2sb_0 - 2sl - 2s(s+1) - \frac{E_0^0}{K}, \tag{4.22}$$

$$\psi^s(\chi, a_0, b_0) \propto A^+(\chi, a_0, b_0)A^+(\chi, a_1, b_1)\cdots A^+(\chi, a_{s-1}, b_{s-1})\psi^0(\chi, a_s, b_s), \tag{4.23}$$

where

$$\psi^0(\chi, a_s, b_s) \propto \frac{\sinh^{l+a_s}\chi}{\cosh^{b_s}\chi} \tag{4.24}$$

has been derived from $A^-(\chi, a_s, b_s)\psi^0(\chi, a_s, b_s) = 0$. This is the solution of the harmonic oscillator in the curved space,

$$\begin{aligned} &H\psi^s(\chi, a_0, b_0) = -\frac{E^s}{K}\psi^s(\chi, a_0, b_0), \\ &H = \frac{1}{2}\left[-\frac{d^2}{d\chi^2} - 2\coth\chi\frac{d}{d\chi} + \frac{l(l+1)}{\sinh^2\chi} + \frac{2\beta}{K^2}\tanh^2\chi\right] \end{aligned} \tag{4.25}$$

with

$$\frac{2\beta}{K^2} = \frac{2\beta_l}{K^2} - \frac{\sqrt{2\beta_l}}{K} = b_0^2 + b_0, \tag{4.26}$$

where

$$b_0 = -\frac{1}{2} - \frac{1}{K}\sqrt{\frac{K^2}{4} + 2\beta} \tag{4.27}$$

We can derive

$$-\frac{E^s}{K} = -\frac{E_0^s}{K} + \frac{(2l+3)}{2}b_0 - \frac{l(l+2)}{2} + \frac{E_0^0}{K} =$$

$$\left(l + 2s + \frac{3}{2}\right) b_0 - 2sl - \frac{l(l+2)}{2} - 2s(s+1), \tag{4.28}$$

$$E^s = \left(l + 2s + \frac{3}{2}\right)\sqrt{\frac{K^2}{4} + 2\beta} + \frac{K}{2}\left[(l+2s)^2 + 3(l+2s) + \frac{3}{2}\right], \tag{4.29}$$

and $\psi^s(\chi, a_0, b_0)$ is the same as in Eq.(4.23).

Let $n = l + 2s$ ($s = 0,1,2,\cdots$; $n = l,\ l+2,\ l+4,\cdots$), then this solution will be changed into another notation,

$$H\psi_n(\chi, a_0, b_0) = -\frac{E_n}{K}\psi_n(\chi, a_0, b_0), \tag{4.30}$$

where

$$\psi_n(\chi, a_0, b_0) = \psi_{l+2s}(\chi, a_0, b_0) = \psi^s(\chi, a_0, b_0), \tag{4.31}$$

$$E_n = E_{l+2s} = E^s = \left(n + \frac{3}{2}\right)\sqrt{\frac{K^2}{4} + 2\beta} + \frac{K}{2}\left(n^2 + 3n + \frac{3}{2}\right). \tag{4.32}$$

If we choose

$$\begin{aligned} C(\chi, a_s) &= 1, \\ B(\chi, a_s) &= (l + a_s + 2)\coth\chi - \frac{\beta}{\sqrt{-K}(l + a_s + 1)}, \\ D(\chi, a_s) &= (l + a_s)\coth\chi - \frac{\beta}{\sqrt{-K}(l + a_s + 1)}, \end{aligned} \tag{4.33}$$

then Eq.(4.33) satisfies the shape invariance conditions Eq. (3.7) or (3.8) as well, in which

$$\begin{aligned} R(a_s) &= \frac{1}{K}\left[\frac{\beta^2(2l + 2a_s + 1)}{2(l + a_s)^2(l + a_s + 1)^2} + (2l + 2a_s + 1)\frac{K}{2}\right], \\ a_{s+1} &= f(a_s) = a_s + 1, \qquad a_0 = 0, \qquad a_s = s \end{aligned} \tag{4.34}$$

The hierarchy of Hamiltonian $\{H_s^+ \mid s = 0,1,2,\cdots\}$,

$$H_s^+ = \frac{1}{2}A^+(\chi, a_s)A^-(\chi, a_s), \tag{4.35}$$

is related to the Hamiltonian of hydrogen atom in the curved space,

$$\begin{aligned} H &= \frac{1}{2}\left[-\frac{d^2}{d\chi^2} - 2\coth\chi\frac{d}{d\chi} + \frac{l(l+1)}{\sinh^2\chi} - \frac{\beta\coth\chi}{\sqrt{-K}}\right] = \\ &H_0^+ + \frac{1}{2}\left[\frac{\beta^2}{K(l+1)^2} - l(l+2)\right] \quad (l = 0,1,2,\cdots). \end{aligned} \tag{4.36}$$

The Schrödinger equation

$$H\psi^s(\chi, a_0) = -\frac{E^s}{K}\psi^s(\chi, a_0) \tag{4.37}$$

is easy to solve,

$$-\frac{E^s}{K} = \sum_{k=1}^{s} R(a_k) + \frac{1}{2}\left[\frac{\beta^2}{K(l+1)^2} - l(l+2)\right], \tag{4.38}$$

$$E^s = -\frac{\beta^2}{2(l+s+1)^2} + \frac{K}{2}(l+s)(l+s+2) \quad (s = 0,1,2,\cdots), \tag{4.39}$$

$$\psi^s(\chi, a_0) \propto A^+(\chi, a_0)A^+(\chi, a_1)\cdots A^+(\chi, a_{s-1})\psi^s(\chi, a_s), \tag{4.40}$$

where

$$\psi^0(\chi, a_s) \propto \sinh^{l+s}\chi \exp\left[\frac{-\beta\chi}{\sqrt{-K}(l+s+1)}\right]$$

is derived from $A^-(\chi, a_s)\psi^0(\chi, a_s)=0$. This solution could be written in another notation,

$$\begin{aligned} H\psi_n(\chi, a_0) &= E_n\psi_n(\chi, a_0), \\ \psi_n(\chi, a_0) &= \psi_{l+s+1}(\chi, a_0) = \psi^s(\chi, a_0), \\ E_n &= E_{l+s+1} = E^s = -\frac{\beta^2}{2n^2} + \frac{K}{2}(n^2-1), \end{aligned} \tag{4.41}$$

where $n=l+s+1$, $s=0,1,2,\cdots$; $n=l+1$, $l+2,\cdots$. These examples show that the realization (2.10) can be applied to several quantum mechanical systems with a spherically symmetric potential in a 3-dimensional Euclidean space as well as in a constant curved space.

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N=2超对称量子力学的新实现和形状不变性*

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摘要 构造新的超荷和定义权重函数，并研究了 $N=2$ 一维超对称量子力学. 在新的实现中讨论了若干实例.

关键词 超对称量子力学 Hamiltonian 谱系 Schrodinger 方程 形状不变性

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