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Quantum electrodynamics with arbitrary charge on a noncommutative space*

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Abstract Using the Seiberg-Witten map, we obtain a quantum electrodynamics on a noncommutative space, which has arbitrary charge and keep the gauge invariance to at the leading order in theta. The one-loop divergence and Compton scattering are reinvestigated. The noncommutative effects are larger than those in ordinary noncommutative quantum electrodynamics.

Key words noncommutative, quantum electrodynamics, charge

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1 Introduction

The idea that coordinates can be noncommutative (NC) has a long history[1]. It has been thought as a way to eliminate the infinities in quantum field theory by providing a natural cut off, but the success of renormalization obscures the idea.

In the past few years, noncommutativity has attracted much attention[2]. It is widely accepted that the open string's end is attached on the D-brane[2, 3], where the background Neveu-Schwarz B field exists. It makes the string's coordinates noncommutative[2, 4—6], and gives an extra phase factor to the scattering amplitude[2]. The open string theory indicates that we must replace the ordinary product in the effective actions by the star product

$$f(x)^{*}g(x)=\exp\left(\frac{\mathrm{i}}{2}\theta^{ij}\frac{\partial}{\partial\xi^{i}}\frac{\partial}{\partial\eta^{j}}\right)f(x+\xi)g(x+\eta)\bigg|_{\varsigma,\eta=0}, \tag{1}$$

where θ^{ij} is a function of the B field. Using the star product, one can obtain the commutator for the coordinates

$$[x^{\mu},{}^{*}x^{\nu}]=x^{\mu *}x^{\nu}-x^{\nu *}x^{\mu}=\mathrm{i}\theta^{\mu\nu}. \tag{2}$$

Replacing the ordinary product in the gauge field actions by the star product, we get a noncommutative gauge theory. The simplest noncommutative gauge theory is the NC$U(1)$ theory[7—13]. It has two important properties[12, 13]. One is that the theory isn't Lorentz invariance, and the other is that new vertices such as three and four photon self-interactions are introduced. It is similar to the non-abelian theory and renormalizable the single loop level.

Unfortunately, there are some problems in the NC gauge theory. First the ordinary $SU(N)$ theory, which is the foundation of the standard model, can't be extended to the NC $SU(N)$ theory because $SU(N)$ group's NC counterpart destroys the closure condition[14, 15]. The only group which admits a simple noncommutative extension is $U(N)$. In order to get the NC $SU(N)$ theory, the authors of Ref. [16] use the Seiberg-Witten map to the get a low energy effective theory. Second, the time components of $\theta^{\mu\nu}$ leads to a the unitarity problem. Another problem is the no-go theorem[14]: the matter fields can transform nontrivially under at most two NC group factors. In other words, the matter fields cannot carry more than two NC gauge group charges. Additionally only the NC $U(1)$ charges 1, 0, −1 are allowed, because the gauge transform is charge dependent

$$\delta\hat{a}_{\mu}=\partial_{\mu}\hat{\lambda}+\mathrm{i}g[\hat{\lambda},{}^{*}\hat{a}_{\mu}]. \tag{3}$$

All this conflicts with observation. In the standard model the matter fields can couple with three gauge

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fields, and the number of $U(1)$ charges is higher than three. We must introduce additional fields for the new charge in the theory, and in addition there are too many degrees of freedom. In the commutative limit $\theta^{\mu\nu} \to 0$, this theory can't coincide with ordinary QED. The latter problem has been partly dealt with in Ref. [17]. Calmet required that the $U(1)$ field depends explicitly on the charge Q, and found that all of the NC $U(1)$ fields could be expressed as a function of the ordinary $U(1)$ field via the Seiberg-Witten map

$$\hat{a}_\xi = a_\xi + \frac{g}{4}\theta^{\mu\nu}\{a_\nu, \partial_\mu a_\xi\} + \frac{g}{4}\theta^{\mu\nu}\{f_{\mu\xi}, a_\nu\}, \quad (4)$$

$$\hat{\lambda} = \lambda + \frac{g\theta^{\mu\nu}}{4}\{\partial_\mu \lambda, a_\nu\}. \quad (5)$$

They got an effective theory of the Standard Model. In the effective theory all fields are ordinary fields. But there is a flaw. The original NC $U(1)$ action isn't gauge invariant, the reason is the same as the Eq. (3). The new covariant derivative does not transform covariantly.

This paper is aimed to constuct a NC $U(1)$ theory with arbitrary charge in a noncommutative space. To avoid a problem with unitarity we assume that only the space-space components of $\theta^{\mu\nu}$ are nonzero, namely $\theta^{0\nu} = 0$. It is organized as follows. In Section 2, we construct the generalized form of the NC $U(1)$ theory. In Section 3, we discuss the one-loop divergence of the theory, which has minimal deviations from ordinary quantum electrodynamics, the β function. In Section 4, we calculate noncommutative corrections of the Compton scattering amplitude. Section 5 is the conclusion.

2 Noncommutative QED

In order to make different NC $U(1)$ fields degenerate to the same ordinary $U(1)$ field in the commutative limite, we should require that all the NC $U(1)$ fields are functions of the classical gauge field a_ξ, as Calmet did in Ref. [17].

$$\begin{aligned}\hat{a}_\xi &= a_\xi + \frac{g}{4}\theta^{\mu\nu}\{a_\nu, \partial_\mu a_\xi\} + \frac{g}{4}\theta^{\mu\nu}\{f_{\mu\xi}, a_\nu\} + O(\theta), \\ \hat{a}'_\xi &= a_\xi + \frac{g'}{4}\theta^{\mu\nu}\{a_\nu, \partial_\mu a_\xi\} + \frac{g'}{4}\theta^{\mu\nu}\{f_{\mu\xi}, a_\nu\} + O(\theta).\end{aligned} \quad (6)$$

It is easy to see that the different NC $U(1)$ fields can be expressed as local functions of each other. We are interesting in the special NC $U(1)$ field which provides kinetic terms for all NC gauge fields. Its strength is given by

$$\hat{f}_{\mu\nu} = \partial_\mu \hat{a}_\nu - \partial_\nu \hat{a}_\mu - \mathrm{i}g[\hat{a}_\mu, {}^* \hat{a}_\nu]. \quad (7)$$

The charge g is the NC $U(1)$ field's inherent property. The action of the gauge field is invariant under the gauge transformation.

Considering Eq. (6) , other NC $U(1)$ gauge can be rewritten as

$$\hat{a}'_\xi = \hat{a}_\xi + \frac{(g'-g)}{4}\theta^{\mu\nu}\{\hat{f}_{\mu\xi} + \partial_\mu \hat{a}_\xi, {}^* \hat{a}_\nu\} + O(\theta) \ . \quad (8)$$

The action of the full NC $U(1)$ theory is

$$\begin{aligned}L = &\int \overline{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi + g'\overline{\psi}^* \gamma^\mu a_\mu{}^* \psi + \\ &g'\frac{(g'-g)}{4}\theta^{\mu\nu}\overline{\psi}^*\{\hat{f}_{\mu\xi} + \partial_\mu \hat{a}_\xi, {}^* \hat{a}_\nu\}^* \psi \mathrm{d}^4 x - \\ &\frac{1}{4}\int f_{\mu\nu}{}^* f^{\mu\nu} \mathrm{d}^4 x + O(\theta) + L_{\text{othermatter}} \ . \end{aligned} \quad (9)$$

Here and in the following part of the paper, we ignore the hat on the noncommutative field except in special mention case, ψ is the matter with charge g'. $L_{\text{othermatter}}$ stands for the action of the matter with different $U(1)$ charges.

This NC $U(1)$ theory isn't invariant under the ordianary NC gauge transformation

$$\delta\psi = \mathrm{i}g'\lambda^* \psi \ . \quad (10)$$

Noticing from Eq. (5) that λ is also charge dependent, we must modify (10) as follows

$$\delta\psi = \mathrm{i}g'\lambda'^* \psi \ , \quad (11)$$

where

$$\lambda' = \lambda + \frac{g'-g}{4}\theta^{\eta\nu}\{\partial_\mu \lambda, {}^* a_\nu\} + O(\theta). \quad (12)$$

It is not hard to check that the NC $U(1)$ theory (Eq. (9)) is invariant under the gauge transformation Eqs. (3) and (11) to first order in theta. Following the same steps, we can go to the arbitrary order of θ, and this way obtain an infinite contribution. So this theory is a low energy effective theory and unrenormalizable. We will only consider the tree and single loop processes, and ignore the terms of order $O(\theta)$. The theory reduces to the ordinary $U(1)$ theory in the commutative limite.

By introducing a gauge fixing term and aghost term, the effective action is[11]

$$L_{\text{eff}} = L + \int\left\{-\frac{1}{2}\partial_\mu a^{\mu *}\partial_\nu a^\nu + \partial^\mu \overline{c}^* (\partial_\mu c - \mathrm{i}g[a_\mu, {}^* c])\right\}\mathrm{d}^4 x \ . \quad (13)$$

The corresponding Feynman rules in the Feynman gauge are given in Fig. 1.

The action (13) and Fig. 1 indicate that the theory depends on the inherent charge g chosen. The minial deviation from the ordinary $U(1)$ theory is achieved for $g = 0$; then the poton self-interaction and the ghost field disappear (Fig. 1(f)—(h)). We will discuss this model in the next sections.

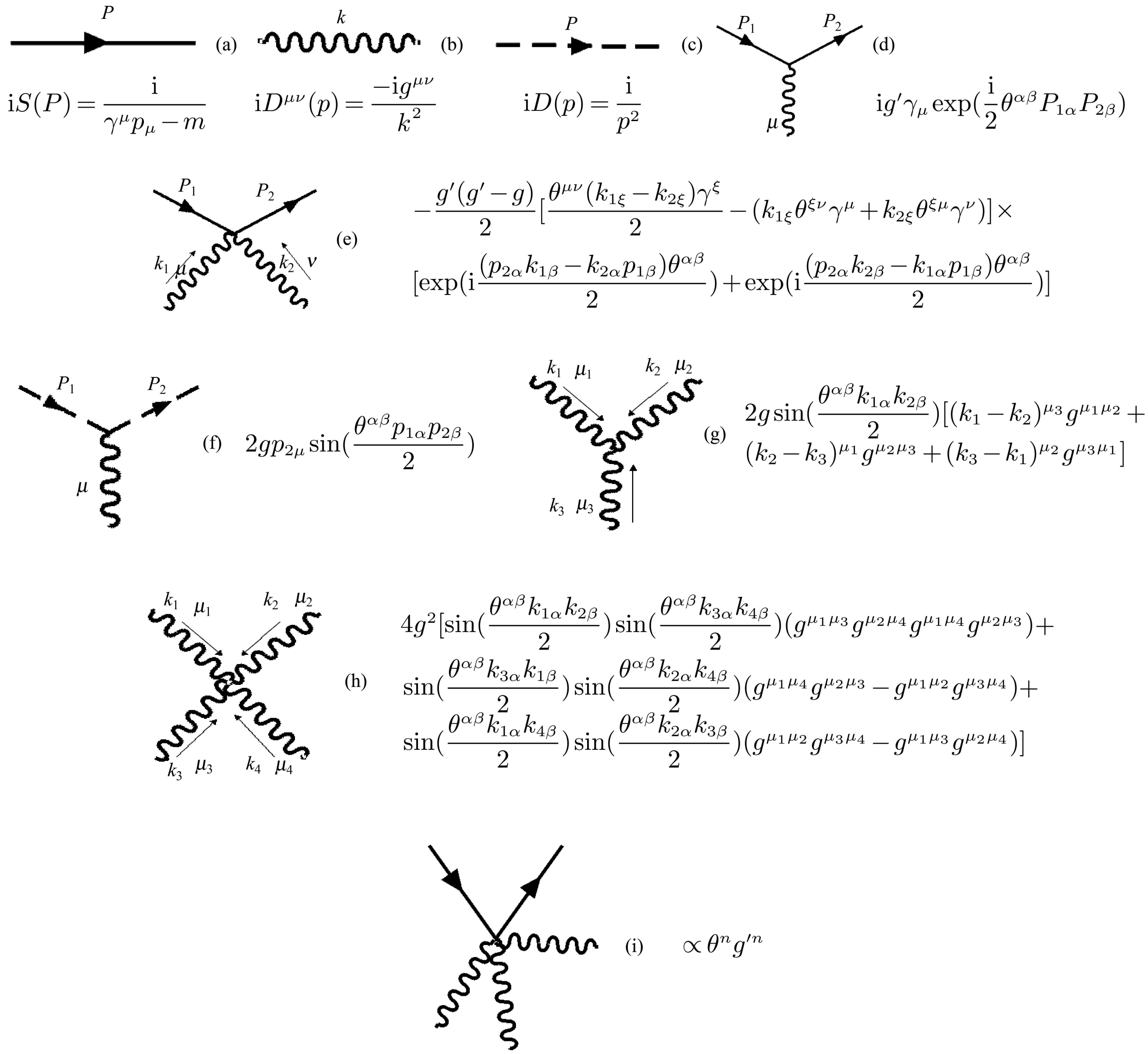


Fig. 1. (a) Fermion propagator; (b) Photon propagator; (c) Ghost propagator; (d) Fermion photon vertex; (e) Two photon Fermion vertex; (f) Ghost photon vertex; (g) Three photon vertex; (h) Four photon vertex; (i) N photon Fermion vertex.

3 The one loop renormalised NC $U(1)$ theory

Using the Feynman rules of Fig. 1,we obtain corrections to the fermion self-energy (Fig. 2), photon self-energy (Fig. 3) and fermion photon vertex Fig. 4.

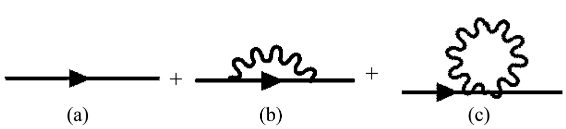


Fig. 2. Fermion self-energy.

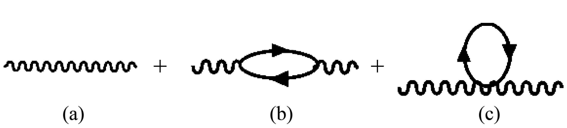


Fig. 3. The photon self-energy.

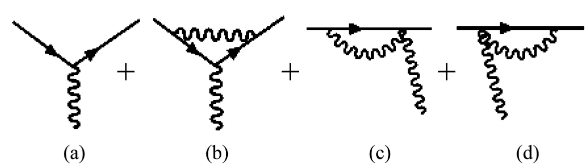


Fig. 4. The corrections to the fermion photon vertex.

The fermion self-energy graph Fig. 2(b) is the same as in ordinary QED[11, 13]. The phase factors cancel with each other. The third Feynman graph is zero. The renormalisation constant is

$$Z_2 = 1-\frac{g'^2}{16\pi^2\varepsilon'} \ . \tag{14}$$

The phase factors also cancel in Fig. 3(b), and the contribution of Fig. 3(c) is equal to 0. The correction to photon self-energy coincides with ordinary QED. The renormalisation constant is[11, 13]

$$Z_3 = 1-\frac{g'^2}{16\pi^2}\frac{4N_f}{3\varepsilon'} \ . \tag{15}$$

Next we consider the corrections to the vertex. Fig. 4(b) is no-planar graph, its phase factor contains internal momentum. The integral is finite[13]. The contributions of Fig. 4(c) and (d) are

$$\Lambda^\mu=-\frac{g'^2}{2}\exp\left(\frac{\mathrm{i}\theta^{\alpha\beta}P_{1\alpha}P_{2\beta}}{2}\right)\int\Big\{\left[\frac{\theta^{\mu\nu}(k_{1\xi}-k_{2\xi})\gamma^\xi}{2}-(k_{1\xi}\theta^{\xi\nu}\gamma^\mu+k_{2\xi}\theta^{\xi\mu}\gamma^\nu)\right]\frac{\mathrm{i}}{\gamma^\alpha(P_{1\alpha}-k_{2\alpha})-m}\mathrm{i}g'\gamma_\nu+ \mathrm{i}g'\gamma_\nu\frac{\mathrm{i}}{\gamma^\alpha(P_{2\alpha}-k_{2\alpha})-m}\left[\frac{\theta^{\mu\nu}(k_{1\xi}+k_{2\xi})\gamma^\xi}{2}-(k_{1\xi}\theta^{\xi\nu}\gamma^\mu-k_{2\xi}\theta^{\xi\mu}\gamma^\nu)\right]\Big\}\frac{-\mathrm{i}}{k_2^2}[1+\exp(\mathrm{i}\theta^{\alpha\beta}k_{1\alpha}k_{2\beta})]\mathrm{d}^4k_2\ . \tag{16}$$

The integral is infinite. We can separate the infinite part

$$\exp\left(\frac{\mathrm{i}\theta^{\alpha\beta}P_{1\alpha}P_{2\beta}}{2}\right)\Lambda^\mu=-\frac{g'\pi^2}{\varepsilon'}\exp\left(\frac{\mathrm{i}\theta^{\alpha\beta}P_{1\alpha}P_{2\beta}}{2}\right)\Big\{\left[\frac{\theta^{\mu\nu}\gamma^\xi k_{1\xi}}{2}-k_{1\xi}\theta^{\xi\nu}\gamma^\mu\right]\left(\frac{\gamma^\xi P_{1\xi}}{2}+m\right)\gamma_\nu+ \left[\frac{\theta^{\mu\nu}\gamma^\alpha}{2}-\theta^{\alpha\mu}\gamma^\nu\right](\gamma^\xi P_{1\xi}+m)\gamma_\nu P_{1\alpha}+\left[\frac{\theta^{\mu\nu}\gamma^\alpha}{2}+\theta^{\alpha\mu}\gamma^\nu\right]\gamma^\beta\gamma_\nu\left[\frac{g_{\alpha\beta}}{2}\left[\frac{P_1^2}{6}-\frac{m^2}{2}\right]+P_{1\alpha}P_{1\beta}\right]+ \gamma_\nu\left(\frac{\gamma^\xi P_{2\xi}}{2}+m\right)\left[\frac{\theta^{\mu\nu}\gamma^\xi k_{1\xi}}{2}-k_{1\xi}\theta^{\xi\nu}\gamma^\mu\right]+\gamma_\nu(\gamma^\xi P_{2\xi}+m)\left[\frac{\theta^{\mu\nu}\gamma^\alpha}{2}+\theta^{\alpha\mu}\gamma^\nu\right]P_{2\alpha}+ \gamma_\nu\gamma^\alpha\left[\frac{\theta^{\mu\nu}\gamma^\beta}{2}+\theta^{\beta\mu}\gamma^\nu\right]\left[\frac{g_{\alpha\beta}}{2}\left[\frac{P_2^2}{6}-\frac{m^2}{2}\right]+P_{2\alpha}P_{2\beta}\right]\Big\}+\text{finite}\ . \tag{17}$$

It is momentum dependent, and we can't extract γ^μ. The renormalisation constant $Z_1\gamma^\mu$ of ordinary QED must be replaced by

$$Z_1\gamma^\mu=\gamma^\mu-\Lambda^\mu\ . \tag{18}$$

At this stage the theory is renormalised at the single loop level.

As in ordinary QED, the bare parameter s are

$$g_0=g(\mu)Z_1Z_2^{-1}Z_3^{-\frac{1}{2}}\ , \tag{19}$$

$$\theta_0^{\mu\nu}=\theta^{\mu\nu}Z_2^{-1}\ . \tag{20}$$

Notice that Λ^μ contains θ as a factor. It is smaller than the other renormalisation constants and it can be discarded when investigating the asymptotic behavior. The running coupling constant and noncommutativity constant satisfy

$$\beta(\mu)=\mu\frac{\partial}{\partial\mu}g(\mu)=\frac{g'^2}{16\pi^2}(2+\frac{4}{3}N_f)\ , \tag{21}$$

$$\frac{\partial\theta^{\alpha\beta}}{\partial t}=\mu\frac{\partial\theta^{\alpha\beta}}{\partial\mu}=\frac{g'^2}{16\pi^2}\theta^{\alpha\beta}\ . \tag{22}$$

They represent the infrared freedom. In the low energy limit the noncommutative effect is small.

4 The correction to Compton scattering

The two photon fermion vertex (Fig. 1(e)) must be taken into account in Comptom scattering. Its contribution may reveal the noncommutativity of space. Comptom scatterings is described by Fig. 5.

In the high-energy limit, the amplitude in the center of mass frame is proportional to

$$|M_{\mathrm{NC}}|^2=|M_{\mathrm{C}}|^2+\delta|M|^2\ . \tag{23}$$

where $|M_{\mathrm{C}}|^2$ is the contribution of the ordinary QED, and the $\delta|M|^2$ term is the effect of noncommutativity.

$$\begin{aligned}\delta|M|^2 = \frac{-g'^4}{2m^2}\big\{&[(\boldsymbol{k}\times\boldsymbol{k}')\cdot\boldsymbol{\theta}]\frac{(\boldsymbol{k}\cdot\boldsymbol{k}')}{E^2}+\\&(E^2-(\boldsymbol{k}\cdot\boldsymbol{k}'))[(\boldsymbol{k}\cdot\boldsymbol{\theta})^2+(\boldsymbol{k}'\cdot\boldsymbol{\theta})^2]+\\&(5E^2-(\boldsymbol{k}\cdot\boldsymbol{k}'))(\boldsymbol{k}\cdot\boldsymbol{\theta})(\boldsymbol{k}'\cdot\boldsymbol{\theta})-\\&(\boldsymbol{\theta}\cdot\boldsymbol{\theta})(E^2-(\boldsymbol{k}\cdot\boldsymbol{k}'))(\boldsymbol{k}\cdot\boldsymbol{k}')\big\}.\end{aligned} \tag{24}$$

where $\theta^i=\varepsilon^{ijk}\theta^{jk}$, E is the energy of the photon. We can rewrite θ,k,k' as

$$\boldsymbol{\theta} = (\theta\sin\alpha,\ 0,\ \theta\cos\alpha), \tag{25}$$

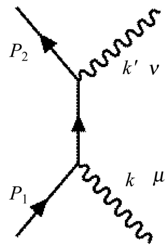
$$\boldsymbol{k} = (0,\ 0,\ E), \tag{26}$$

$$\boldsymbol{k}' = (E\sin\varphi\cos\psi,\ E\sin\varphi\sin\psi,\ E\cos\varphi). \tag{27}$$

The $\delta|M|^2$ term is given by

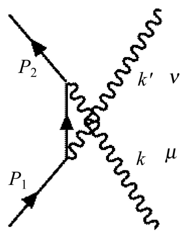
$$\begin{aligned}\delta|M|^2 = \frac{-g'^4E^4\theta^2}{2m^2}\big\{&(\sin\varphi\sin\psi\sin\alpha)^2\cos\varphi+\\&(1-\cos\varphi)[\cos^2\alpha+(\sin\alpha\sin\varphi\cos\psi+\\&\cos\alpha\cos\varphi)^2]+\\&(5-\cos\varphi)\cos\alpha(\sin\varphi\cos\psi\sin\alpha+\\&\cos\alpha\cos\varphi)-(1-\cos\varphi)\cos\varphi\big\}.\end{aligned} \tag{28}$$

The vertex in Fig. 1(i) isn't taken into account in the calculation, because it given no contribution at



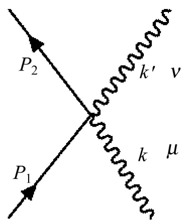
(a)

$$M_s=-\mathrm{i}g'^2\overline{U}(p_2)\gamma_\nu\frac{1}{\gamma^\xi(p_{1\xi}+k_\xi)-m}\gamma_\mu U(p_1)$$
$$\varepsilon^\nu(k')\,\varepsilon^\mu(k)\exp\frac{\mathrm{i}\theta^{\alpha\beta}(k'_\alpha p_{2\beta}+p_{1\alpha}k_\beta)}{2}$$



(b)

$$M_u=-\mathrm{i}g'^2\overline{U}(p_2)\gamma_\mu\frac{1}{\gamma^\xi(p_{1\xi}-k'_\xi)-m}\gamma_\nu U(p_1)$$
$$\varepsilon^\nu(k')\,\varepsilon^\mu(k)\exp\frac{\mathrm{i}\theta^{\alpha\beta}(k'_\alpha p_{1\beta}+p_{2\alpha}k_\beta)}{2}$$



(c)

$$M'=-\frac{g'^2}{2}\overline{U}(p_2)\gamma_\mu\Big[\frac{\gamma^\xi(k_\xi+k'_\xi)}{2}-(k_\xi\theta^{\xi\nu}\gamma^\mu-k'_\xi\theta^{\xi\mu}\gamma^\nu)\Big]$$
$$U(p_1)\;\varepsilon^\nu(k')\varepsilon^\mu(k)\Big[\exp\frac{\mathrm{i}\theta^{\alpha\beta}(k'_\alpha p_{1\beta}+p_{2\alpha}k_\beta)}{2}+$$
$$\exp\frac{\mathrm{i}\theta^{\alpha\beta}(-k_\alpha p_{1\beta}-p_{2\alpha}k'_\beta)}{2}\Big]$$

Fig. 5.

the tree level. So the correction is exact at the order of θ^2. The correction depends on the direction of θ. So the Lorentz invariance is violated. The right hand of Eq. (28) is proportional to E^4 which is bigger than the result of Ref. [12], which is only proportional to E^2. The noncommutativity effect is more significant than in the previous theory.

5 Conclusion

In this paper we construct a noncommutative $U(1)$ theory. Different from the previous theory, various charges can couple with the same gauge field. It degenerates to the ordinary $U(1)$ theory in the commutative limite. But the gauge transformation of the matter fields is nonlinear and charge dependent. We must introduce an infinite vertex. The process in our work is similar to Calmet's[17], but there is a significant difference. First all of the fields are noncommutative variable. If we calculate the total processes at low energies, the phase factor has a nontrivial effect on the loop integral. Second if we rewrite Calmet's action in noncommutative space with the Seiberg-Witten inverse map, we obtain a similar action except the terms of Fig. 1(e), (i).

We also study the minimal modification model which requires that the inherent charge of NC $U(1)$ field be equal to 0. The minimal modification model has no self-interactiom vertex and the running coupling constant and noncommutativity constant lead to infrared freedom. At the single loop level and in leading order of theta the theory is renormalized. The study of Comptom scattering indicates that Lorentz invariance is breakdown violated and the noncommutative effect is more significant than in the previous theory.

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