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Lorentz violation constrained by triplicity of lepton families and neutrino oscillations

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Abstract In this paper we postulate an algebraic model to relate the triplet characteristic of lepton families to Lorentz violation. Inspired by the two-to-one mapping between the group $SL(2,C)$ and the Lorentz group via the Pauli grading (the elements of $SL(2,C)$ expressed by direct sum of unit matrix and generators of $SU(2)$ group), we grade the $SL(3,C)$ group with the generators of $SU(3)$, i. e. the Gell-Mann matrices, then express the $SU(3)$ group in terms of three $SU(2)$ subgroups, each of which stands for a lepton species and is mapped into the proper Lorentz group as in the case of the group $SL(2,C)$. If the mapping from group $SL(3,C)$ to the Lorentz group is constructed by choosing one $SU(2)$ subgroup as basis, then the other two subgroups display their impact only by one more additional generator to that of the original Lorentz group. Applying the mapping result to the Dirac equation, it is found that only when the kinetic vertex $\gamma_\mu\partial^\mu$ is extended to encompass $\gamma_5\gamma_\mu\partial^\mu$ can the Dirac-equation-form be conserved. The generalized vertex is useful in producing neutrino oscillations and mass differences.

Key words Lorentz violation, neutrino oscillation, lepton-family symmetry

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1 Introduction

The possible breaking of Lorentz symmetry (BLS) has been put forward from different aspects[1—4] as a low energy limit of a more fundamental theory at sufficiently high energy scale. Practically and intriguingly, the BLS has been applied to the neutrino sector to study the oscillation and the origin of masses[1, 5—7]. There the Dirac equation is written with a general form to include all possible gamma matrices[1],

$$(\mathrm{i}\Gamma^\nu_{AB}\partial_\nu - M_{AB})\nu_B = 0\,, \qquad (1)$$

where the indices A, B are responsible for the neutrino species, and Γ^ν_{AB}, M_{AB} are 4×4 matrices in spinor space including all the possible bases of γ matrices. The most recent MiniBooNE experiment[8] convincingly denied the possible existence of sterile neutrinos observed previously by liquid scintillator neutrino detector (LSND)[9]. Therefore the above mentioned mechanism to account for the oscillations of sterile neutrinos via generalizing Γ^ν_{AB} becomes unnecessary.

The lepton families of the standard model are also of much interest[10—16]. In this paper we will not be very much involved in the origin[10—15] of the three families or the possibility of the existence of more lepton families[16]. We straightforwardly employ the fact that there are just three families of leptons and associate it with the Lorentz violation. It is found that under the constraints from the triplicity of lepton families and the denial of sterile neutrinos, speculations on Lorentz violation (extension) as in Eq. (1) are still possible.

The remainder of the paper is arranged as follows: in the next section, we briefly review the mapping between the group $SL(2,C)$ and the Lorentz group. In the third section, we construct a similar mapping between $SL(3,C)$ and the Lorentz group, and also consider the question of what would be the consequence if one chooses one $SU(2)$ subgroup as a mapping basis (the concept “mapping basis” will be elucidated later). The fourth section is dedicated to apply the result of the previous section to the Dirac Equation. The concluding remarks are presented in the last section.

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2 Review of the relationship between the group $SL(2,C)$ and the Lorentz group

Each element of the group $SL(2,C)$ has the form

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2}$$

where a, b, c, d are complex numbers and $\det g = ad-bc=1$, which means $\mathrm{Re}(\det g)$=1 and $\mathrm{Im}(\det g)$=0 and thus leaves from the original 8 parameters of g 6 which can be chosen free. Equivalently Eq. (2) can be expressed in the form

$$g=g_\mu\sigma^\mu, \tag{3}$$

where g_μ ($\mu=0, 1, 2, 3$) are complex numbers, and

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

In the above expressions a contravariant vector component $W^\mu = (W_0,\vec{W})$ and its covariant components $W_\mu = (W_0,\vec{W})$ are conventionally linked by $W^\mu=g^{\mu\nu}W_\nu$, where

$$g^{\mu\nu}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Let's introduce a matrix form Q related to the coordinates of space-time $x_\mu=(x_0,-\vec{x})$ as follows

$$Q=x_\mu\sigma^\mu, \tag{5}$$

from which we obviously obtain

$$\det Q=x_0^2-x_1^2-x_2^2-x_3^2. \tag{6}$$

We further define a transformation of Q by

$$Q'=gQg^+, \tag{7}$$

where $Q'=x'_\mu\sigma^\mu$, and x_μ transforms under a Lorentz transformation according to

$$x'_\mu=\Lambda^\alpha_\mu x_\alpha. \tag{8}$$

From Eqs. (6), (8) we conclude that $\det Q$ ($=\det Q'$) is an invariant quantity. From Eqs. (5), (7), the transformation of x_μ yields

$$x'^\alpha = \delta^\alpha_\beta x'^\beta=\frac{1}{2}\mathrm{Tr}(\sigma^\alpha\sigma^\beta)x'_\beta=\frac{1}{2}\mathrm{Tr}(\sigma^\alpha Q')= \frac{1}{2}\mathrm{Tr}(\sigma^\alpha gQg^+)=\frac{1}{2}\mathrm{Tr}(\sigma^\alpha g\sigma^\beta g^+)x_\beta. \tag{9}$$

Then, comparing Eq. (9) with the transformation Eq. (8), we can obtain a mapping between the Lorentz group and the elements g of $SL(2,C)$,

$$\Lambda^{\alpha\beta}=\frac{1}{2}\mathrm{Tr}(\sigma^\alpha g\sigma^\beta g^+), \tag{10}$$

for example, $\Lambda^0_0=|g_0|^2+\sum_{k=1}^{3}|g_k|^2$. From Eq. (10) we can also obtain g expressed through the elements of the Lorentz transformation[17]:

$$g = g_0\sigma^0+\sum_{k=1}^{3}g_k\sigma_k = D^{-1}\left[\mathrm{Tr}\sigma^0+\sum_{k=1}^{3}\Lambda^k_0+\Lambda^0_k-\mathrm{i}\varepsilon^{0k\rho}_{\lambda}\Lambda^\lambda_\rho\right]\sigma^k, \tag{11}$$

where $D^2=4-\mathrm{Tr}\Lambda^2+(\mathrm{Tr}\Lambda)^2-\mathrm{i}\varepsilon^{\mu\lambda}_{\rho\tau}\Lambda^\tau_\lambda\Lambda^\rho_\mu$.

According to Eqs. (10), (11), the following Lorentz transformations and elements of $SL(2,C)$ can be derived which are equivalent. The typical two to one mapping shows up in the arguments of κ and $\kappa/2$.

elements of $SL(2,C)$	Lorentz transformations Λ^μ_ν
$\pm\begin{pmatrix} \cosh\frac{\kappa}{2} & \sinh\frac{\kappa}{2} \\ \sinh\frac{\kappa}{2} & \cosh\frac{\kappa}{2} \end{pmatrix}$	$\begin{pmatrix} \cosh\kappa & \sinh\kappa & 0 & 0 \\ \sinh\kappa & \cosh\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\pm\begin{pmatrix} \cosh\frac{\kappa}{2} & \mathrm{i}\sinh\frac{\kappa}{2} \\ -\mathrm{i}\sinh\frac{\kappa}{2} & \cosh\frac{\kappa}{2} \end{pmatrix}$	$\begin{pmatrix} \cosh\kappa & 0 & \sinh\kappa & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\kappa & 0 & \cosh\kappa & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\pm\begin{pmatrix} \exp\left[\frac{\kappa}{2}\right] & 0 \\ 0 & \exp\left[-\frac{\kappa}{2}\right] \end{pmatrix}$	$\begin{pmatrix} \cosh\kappa & 0 & 0 & \sinh\kappa \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\kappa & 0 & 0 & \cosh\kappa \end{pmatrix}$
$\pm\begin{pmatrix} \cos\frac{\kappa}{2} & \mathrm{i}\sin\frac{\kappa}{2} \\ \mathrm{i}\sin\frac{\kappa}{2} & \cos\frac{\kappa}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\kappa & -\sin\kappa \\ 0 & 0 & \sin\kappa & \cos\kappa \end{pmatrix}$

$$\pm\begin{pmatrix}\cos\frac{\kappa}{2} & -\sin\frac{\kappa}{2}\\ \sin\frac{\kappa}{2} & \cos\frac{\kappa}{2}\end{pmatrix} \qquad \begin{pmatrix}1&0&0&0\\0&\cos\kappa&0&\sin\kappa\\0&0&1&0\\0&-\sin\kappa&0&\cos\kappa\end{pmatrix}$$

$$\pm\begin{pmatrix}\exp\left[\mathrm{i}\frac{\kappa}{2}\right] & 0\\ 0 & \exp\left[-\mathrm{i}\frac{\kappa}{2}\right]\end{pmatrix} \begin{pmatrix}1&0&0&0\\0&\cos\kappa&-\sin\kappa&0\\0&\sin\kappa&\cos\kappa&0\\0&0&0&1\end{pmatrix} \tag{12}$$

where the first three matrices represent Lorentz boosts, and last three matrices represent Lorentz rotations, and κ stands for the rapidity. Notice that all the elements in the first column are independent and can be written in the form of Eq. (3). Therefore the two columns can be seen as the generating elements of the Lorentz group and $SL(2,C)$. We obtain the generators for the two groups by taking the derivatives of these matrices with respect to κ at $\kappa=0$.

3 Associating the group $SL(3,C)$ with the Lorentz group

It is known that the group $SL(3,C)$ has Pauli gradings[18—20], and it is easy to prove that the Pauli grading matrices and the generators of the group $SU(3)$ can be mutually lineally expressed. We use here the generators of the group $SU(3)$ in the Gell-Mann representation

$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},$$

$$\lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad \lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},$$

$$\lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},\quad \lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$\lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \lambda_8=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}. \tag{13}$$

We can divide the $SU(3)$ group into three relevant parts by grouping the generators as follows

$$\Gamma_1=\{\lambda_1,\lambda_2,\lambda_3\},\quad \Gamma_2=\{\lambda_4,\lambda_5,\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\},$$

$$\Gamma_3=\{\lambda_6,\lambda_7,\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}\}, \tag{14}$$

where $\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$ and $\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$ can be derived by combining λ_3 and λ_8. It is obvious that Γ_1, Γ_2, Γ_3 are bases for three $SU(2)$ groups satisfying the commutations of Pauli matrices. Inspired by the Eq. (3), (4), adding a unit matrix $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ separately to Γ_1, Γ_2, Γ_3 we have three bases

$$\Gamma_1=\{I_{3\times3},\lambda_1,\lambda_2,\lambda_3\},$$

$$\Gamma_2=\{I_{3\times3},\lambda_4,\lambda_5,\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\},$$

$$\Gamma_3=\{I_{3\times3},\lambda_6,\lambda_7,\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}\}, \tag{15}$$

which can be mapped to the Lorentz group as shown in the last section, for example, the Lorentz group mapped from Γ_1 has the same form as in Eq. (10)

$$\Lambda^{\alpha\beta}=\frac{1}{2}\mathrm{Tr}(\lambda^\alpha g\lambda^\beta g^+)\,. \tag{16}$$

We note that now the element g of $SL(2,C)$ comes from the linear combination of $I_{3\times3}$, λ_1, λ_2, λ_3. Hereafter we name the λ^α, λ^β in Eq. (16) the basis of the mapping. From Eq. (16), it can be proved that except for $\Lambda^0_0=\frac{3}{2}|g_0|^2+\sum_{k=1}^{3}|g_k|^2$, which has a different coefficient in front of $|g_0|^2$ and therefore differs from Λ^0_0 in Eq. (10) [This remnant coefficient $\frac{3}{2}$, which can be cancel by an additional boost, will not affect the generating of Lorentz group], the other components such as Λ^0_i, Λ^i_0 and Λ^j_k are all the same as those originating from Eq. (10). Similar mappings from Γ_2 and Γ_3 to the Lorentz group can be constructed and lead to the same result. We use these three mappings to characterized the three lepton families.

According to Eq. (16), we can construct the parallelism between the set $\{I_{3\times3}, \lambda_1, \lambda_2, \lambda_3\}$ and Lorentz group by calculating the following matrices which resemble those of Eq. (2):

$$\begin{array}{cc}
\text{elements of} SL(2,C) & \text{Lorentz transformations } \Lambda^{\alpha}_{\beta} \\
\pm\begin{pmatrix} \cosh\frac{\kappa}{2} & \sinh\frac{\kappa}{2} & 0 \\ \sinh\frac{\kappa}{2} & \cosh\frac{\kappa}{2} & 0 \\ 0 & 0 & \cosh\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} \frac{5}{4}\cosh\kappa+\frac{1}{4} & \sinh\kappa & 0 & 0 \\ \sinh\kappa & \cosh\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\pm\begin{pmatrix} \cosh\frac{\kappa}{2} & \mathrm{i}\sinh\frac{\kappa}{2} & 0 \\ -\mathrm{i}\sinh\frac{\kappa}{2} & \cosh\frac{\kappa}{2} & 0 \\ 0 & 0 & \cosh\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} \frac{5}{4}\cosh\kappa+\frac{1}{4} & 0 & -\sinh\kappa & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\kappa & 0 & \cosh\kappa & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\pm\begin{pmatrix} \exp\left[\frac{\kappa}{2}\right] & 0 & 0 \\ 0 & \exp\left[-\frac{\kappa}{2}\right] & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \cosh\kappa & 0 & 0 & \sinh\kappa \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\kappa & 0 & 0 & \cosh\kappa \end{pmatrix} \\
\pm\begin{pmatrix} \cos\frac{\kappa}{2} & \mathrm{i}\sin\frac{\kappa}{2} & 0 \\ \mathrm{i}\sin\frac{\kappa}{2} & \cos\frac{\kappa}{2} & 0 \\ 0 & 0 & \cos\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} 1+\frac{1}{2}\cos^2\frac{\kappa}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\kappa & \sin\kappa \\ 0 & 0 & -\sin\kappa & \cos\kappa \end{pmatrix} \\
\pm\begin{pmatrix} \cos\frac{\kappa}{2} & -\sin\frac{\kappa}{2} & 0 \\ \sin\frac{\kappa}{2} & \cos\frac{\kappa}{2} & 0 \\ 0 & 0 & \cos\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} 1+\frac{1}{2}\cos^2\frac{\kappa}{2} & 0 & 0 & 0 \\ 0 & \cos\kappa & 0 & \sin\kappa \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\kappa & 0 & \cos\kappa \end{pmatrix} \\
\pm\begin{pmatrix} \exp\left[\mathrm{i}\frac{\kappa}{2}\right] & 0 & 0 \\ 0 & \exp\left[-\mathrm{i}\frac{\kappa}{2}\right] & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\kappa & \sin\kappa & 0 \\ 0 & -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{array} \tag{17}$$

Here we have used as generating elements of $SL(2,C)$ the 3×3 matrices originating from the set $\{I_{3\times3}, \lambda_1, \lambda_2, \lambda_3\}$. We note that in the second column, for the Lorentz group, some matrix elements have undergone subtle changes as compared with those of Eq. (12). All changes occur in the (1, 1) elements. In the first matrix for example, the original (1, 1) element $\cosh\kappa$ is replaced by $\frac{5}{4}\cosh\kappa+\frac{1}{4}$. But by taking the derivative, one finds this sort of changes doesn't alter the generators of the Lorentz group, and thus these changes are trivial.

The displayed matrices in Eq. (17) show the mapping from Γ_1 to the Lorentz group. In nature, only one Lorentz group should occur. So how the mappings from Γ_2 and Γ_3 to the Lorentz group manifest their existence is worth studying. Merely in form, we denote the three equivalent Lorentz groups, which are separately produced from Γ_1, Γ_2 and Γ_3, by L_1, L_2, L_3. Now we design a mapping that projects Γ_2 and

Γ_3 also into L_1. Based on Eq. (16), we choose the basis λ^α, λ^β, still from the set $I_{3\times3}$, λ_1, λ_2,λ_3, but the matrix g from the combination of the elements of set Γ_2 or Γ_3. For example, if the matrix g comes from the linear combination of elements in Γ_2, we obtain in analogy to the matrices in Eq. (17), the following result:

$$
\begin{array}{ll}
\text{elements of } SL(2,C) & \text{Quasi-Lorentz tansformations} \\
 & \left[\chi=\cosh^2\frac{\kappa}{2}, \varsigma=\sinh^2\frac{\kappa}{2}, \bar{\chi}=\cos^2\frac{\kappa}{2} \text{ and } \bar{\varsigma}=\sin^2\frac{\kappa}{2}\right] \\
\begin{pmatrix} \cosh\frac{\kappa}{2} & 0 & \sinh\frac{\kappa}{2} \\ 0 & \cosh\frac{\kappa}{2} & 0 \\ \sinh\frac{\kappa}{2} & 0 & \cosh\frac{\kappa}{2} \end{pmatrix} &
\begin{pmatrix} \frac{3}{2}\chi+\varsigma & 0 & 0 & \frac{1}{2}\varsigma \\ 0 & \chi & 0 & 0 \\ 0 & 0 & \chi & 0 \\ \frac{1}{2}\varsigma & 0 & 0 & \chi \end{pmatrix} \\
\begin{pmatrix} \cosh\frac{\kappa}{2} & 0 & \mathrm{i}\sinh\frac{\kappa}{2} \\ 0 & \cosh\frac{\kappa}{2} & 0 \\ -\mathrm{i}\sinh\frac{\kappa}{2} & 0 & \cosh\frac{\kappa}{2} \end{pmatrix} &
\begin{pmatrix} \frac{3}{2}\chi+\varsigma & 0 & 0 & \frac{1}{2}\varsigma \\ 0 & \chi & 0 & 0 \\ 0 & 0 & \chi & 0 \\ \frac{1}{2}\varsigma & 0 & 0 & \chi \end{pmatrix} \\
\begin{pmatrix} \exp\left[\frac{\kappa}{2}\right] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \exp\left[-\frac{\kappa}{2}\right] \end{pmatrix} &
\begin{pmatrix} \cosh\kappa & 0 & 0 & \frac{1}{2}\exp[\kappa] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}\exp[\kappa] & 0 & 0 & \frac{1}{2}\exp[\kappa] \end{pmatrix} \\
\begin{pmatrix} \cos\frac{\kappa}{2} & 0 & \mathrm{i}\sin\frac{\kappa}{2} \\ 0 & \cos\frac{\kappa}{2} & 0 \\ \mathrm{i}\sin\frac{\kappa}{2} & 0 & \cos\frac{\kappa}{2} \end{pmatrix} &
\begin{pmatrix} 1+\frac{1}{2}\bar{\chi} & 0 & 0 & \frac{1}{2}\bar{\varsigma} \\ 0 & \bar{\chi} & 0 & 0 \\ 0 & 0 & \bar{\chi} & 0 \\ \frac{1}{2}\bar{\varsigma} & 0 & 0 & \bar{\chi} \end{pmatrix} \\
\begin{pmatrix} \cos\frac{\kappa}{2} & 0 & -\sin\frac{\kappa}{2} \\ 0 & \cos\frac{\kappa}{2} & 0 \\ \sin\frac{\kappa}{2} & 0 & \cos\frac{\kappa}{2} \end{pmatrix} &
\begin{pmatrix} 1+\frac{1}{2}\bar{\chi} & 0 & 0 & \frac{1}{2}\bar{\varsigma} \\ 0 & \bar{\chi} & 0 & 0 \\ 0 & 0 & \bar{\chi} & 0 \\ \frac{1}{2}\bar{\varsigma} & 0 & 0 & \bar{\chi} \end{pmatrix} \\
\begin{pmatrix} \exp\left[\mathrm{i}\frac{\kappa}{2}\right] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \exp\left[-\mathrm{i}\frac{\kappa}{2}\right] \end{pmatrix} &
\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} .
\end{array}
\tag{18}
$$

Almost the same matrices would appear if in Eq. (16) the mapping basis comes as before from Γ_1, but the matrix g results from a linear combination of elements of Γ_3, with some unimportant signs altered. In the column of the Quasi-Lorentz transformations, we find that after performing the derivatives with respect to κ at $\kappa=0$, only the third matrix leads to a nontrivial generator,

$$
\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} . \tag{19}
$$

Remembering that $\begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}$ is also a generator of original Lorentz group, so the effective part of this new generator in Eq. (19) may be written as

$$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \end{pmatrix} . \quad (20)$$

We call this matrix an extension of the Lorentz generators. The Quasi-Lorentz matrices due to the mapping of $\mathit{\Gamma}_3$ give the same results. If in Eq. (16) we choose λ^α, λ^β from the set $\mathit{\Gamma}_2$, and the matrix g in turn from the combination of the elements in set $\mathit{\Gamma}_1$ and $\mathit{\Gamma}_3$, we will obtain the same results, and again the same if we choose λ^α, λ^β from the set $\mathit{\Gamma}_3$, and so forth. In what follows we confine ourselves to use the results from Eqs. (18) and (19) without loss of the generality.

4 The extension of the kinetic vertex of the Dirac equation

Free leptons in each of the families should satisfy the Dirac equation,

$$\gamma_\mu \mathrm{i}\partial^\mu \psi = m\psi , \quad (21)$$

but this equation will not be accurate if the above extension of the Lorentz generators is accepted. We will elucidate this point of view in this section.

We here define the γ_μ as the kinetic vertex of the Dirac equation (the concrete forms of γ_μ and γ_5 used here are those standard ones presented in many text books, see for example the appendix A of Ref. [21]). Performing the Lorentz transformation Eq. (8) on both sides of the Dirac equation and at the same time assuming that ψ transforms according to

$$\psi'(x') = S^{-1}\psi(x) , \quad (22)$$

where S is a nonsingular 4×4 matrix, then one concludes[21] that

$$S^{-1}\gamma_\mu S = \gamma_\nu \mathit{\Lambda}^\nu_\mu . \quad (23)$$

Let us introduce in explicit form the infinitesimal Lorentz transformation $\mathit{\Lambda}^\nu_\mu = \delta^\nu_\mu + \omega^\nu_\mu$, where the ω^ν_μ are the infitesimal parameters of the Lorentz transformation and can be written in terms of an infiniteseimal antisymmetric tensor $\varepsilon_{\lambda\mu}$ as $\omega^\nu_\mu = g^{\nu\lambda}\varepsilon_{\lambda\mu}$[21]. Substituting this into Eq. (23) and making some minor manipulations, we get

$$\gamma_\mu S - S\gamma_\mu = [\gamma_\mu,\, S] = S\,\omega^\nu_\mu \gamma_\nu . \quad (24)$$

If we write now S in terms of the infinitesimal Lorentz transformation

$$S = 1 + \varepsilon_{\mu\nu} S^{\mu\nu} , \quad (25)$$

then to first order, Eq. (24) can be written

$$[\gamma_\mu,\, \varepsilon_{\mu\nu} S^{\mu\nu}] = \omega^\nu_\mu \gamma_\nu , \quad (26)$$

from this we find the solution of Eq. (26) as:

$$S^{\mu\nu} = \frac{1}{2}\gamma^\mu\gamma^\nu . \quad (27)$$

The above formulae of this section are based on the conventional Lorentz generators. If a generator like matrix the Eq. (20) appears, then to preserve the form of the Dirac equation

$$\mathit{\Gamma}_\mu \partial^\mu \psi = m\psi , \quad (28)$$

its transforming relation Eq. (26) must be altered correspondingly. Roughly we can first rewrite the general form as

$$[\mathit{\Gamma}_\mu,\, \bar{\varepsilon}_{\mu\nu} S^{\mu\nu}] = \bar{\omega}^\nu_\mu \gamma_\nu , \quad (29)$$

where $\bar{\omega}^\nu_\mu$ now only $\bar{\omega}^3_3$ is nontrivial, and $\bar{\varepsilon}_{\mu\nu}$ may not be antisymmetric any longer. If we extend the matrix Eq. (20) to the unit matrix, i.e.

$$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} , \quad (30)$$

then it is found that with $\mathit{\Gamma}_\mu = \gamma_\mu(1+\gamma_5)$ and $\bar{\varepsilon}_{\mu\nu}S^{\mu\nu} = \frac{\bar{\omega}^\nu_\mu}{2}\gamma_5$, the Eq. (29) can be satisfied. This means that the kinetic vertex of the Dirac equation has been extended to include $\gamma_\mu(1+\gamma_5)$, and the Lorentz generators in spinor space $\{\frac{1}{2}\gamma^\mu\gamma^\nu\}$ have been extended to include γ_5. We call the set $\{\gamma_5, \{\frac{1}{2}\gamma^\mu\gamma^\nu\}\}$ an extended Lorentz group (in spinor representation): the group Closure condition is kept by recognizing that the product of γ_5 and any $\frac{1}{2}\gamma^\mu\gamma^\nu$is still in the set $\{\frac{1}{2}\gamma^\mu\gamma^\nu\}$, and their commutator $[\gamma_5, \frac{1}{2}\gamma^\mu\gamma^\nu] = 0$; γ_5 and $\{\frac{1}{2}\gamma^\mu\gamma^\nu\}$ actually form a group, in which γ_5 turns out to be an identity element. And the antisymmetric characteristic of elements of Lorentz group is lost due to the feature of γ_5. So now the Dirac equation can be written

$$\gamma_\mu[(1+w)+w\gamma_5]\mathrm{i}\partial^\mu\psi=m\psi\,, \qquad (31)$$

where w is a little parameter determined according to specific situation. If we don't make the extension (30), then $\bar{\omega}_3^3$ only allows γ_3 to appear in the right hand of Eq. (29), and the original general solution shrinks to

$$\left[\gamma_3(1+\gamma_5),\,\frac{\bar{\omega}_3^3}{2}\gamma_5\right]=\bar{\omega}_3^3\gamma_3\,, \qquad (32)$$

and correspondingly the Dirac equation in the form Eq. (31) becomes

$$\gamma_\mu(1+g^{3\,\mu}w\gamma_5)\mathrm{i}\partial^\mu\psi=m\psi\,. \qquad (33)$$

5 Summary and discussion

On the basis of Eq. (16), a mapping from the group $SL(3,C)$ to the Lorentz group is proposed. By this mapping we find that the lepton families and Lorentz violation can be of relevance. The mapping leads to a high degeneracy of the elements in the group $SL(3,C)$ since we finally map three $SU(2)$ subgroups into one Lorentz group: we grade the $SL(3,C)$ group with the generators of $SU(3)$, i. e. with the Gell-Mann matrices, then express the $SU(3)$ group with three $SU(2)$ subgroups, each of which stands for a lepton species and can be mapped into the proper Lorentz group as in the case of the group $SL(2,C)$. If the mapping from $SL(3,C)$ to the Lorentz group is constructed by choosing one $SU(2)$ subgroup as a mapping basis, then the other two subgroups display their impacts only by one more additional generator as compared to the original Lorentz group.

Subsequently we apply the results of this mapping to the generalized form of Dirac equation, and find only if the kinetic vertex $\gamma_\mu\partial^\mu$ is extended to encompass the part $\gamma_5\gamma_\mu\partial^\mu$ can the Dirac-equation-form be conserved. This is the constraint from the families' triplicity. In view of Eq. (31) it still leaves some possibilities allows us to freely adjust the parameter w. As in Ref. [1], by taking out the term $\mathrm{i}\partial_0$ in Eq. (31) by multiplying with a certain matrix A, then the Hamiltonian can be obtained, and subsequently ΔH can be derived. Obviously, in our case, only if the parameter w is not equal to $-\frac{1}{2}$ (when the rank of matrix before $\mathrm{i}\partial_0$ is not larger than 3), we can give the effective ΔH, and thus following the steps of Ref. [1] the oscillations due to sterile neutrinos can be obtained. So, to avoid the existence of sterile neutrinos, the parameter w must be $-\frac{1}{2}$. In this respect it shows that we get an apparently covariant Dirac equation of the form

$$\gamma_\mu(1-\gamma_5)\mathrm{i}\partial^\mu\psi=2m\psi\,, \qquad (34)$$

but we should take care to transform it according to the generalized Lorentz group with one more generator shown in Eq. (20). This is the constraint from the denial of sterile neutrinos. If the form Eq. (33) is not extended, the effective ΔH would exist anyway since the coefficient $g^{3\mu}w$ induces both the Lorentz violation and CPT violation. Despite the discussion of sterile neutrinos, these violations at least provide a mechanism to account for neutrino oscillations and produce mass differences. In this context many possible corollaries are open to be addressed by experiments.

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