On two-point boundary correlations of the gl(1|1) supersymmetric vertex model

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Abstract The gl(1|1) supersymmetric vertex model with domain wall boundary conditions (DWBC) on an $N\times N$ square lattice is considered. We obtain the reduction formulae for the two-point boundary correlation functions of the model.

Key words gl(1|1) supersymmetric vertex model, boundary correlation function, domain wall boundary conditions

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1 Introduction

The computation of correlation functions is one of the challenging problems in the theory of quantum integrable lattice models [1, 2]. This is a central question which can both enlarge the range of applications of some models in the realm of condensed matter physics and make us understand their mathematical structures better. Although several important advances have been made in recent years, to our knowledge there does not exist a general method that could give exact and explicit manageable expressions for the correlation functions. In Ref. [3], the representation theory of Drinfeld twists for the quantum affine algebra in finite dimensional modules was used to obtain the expression of the adjacent sites correlation functions for the XXZ model. In Refs. [4, 5], by means of the Drinfeld twists, the determinant representation of the correlation functions for the gl(1|1) and gl(2|1) supersymmetric vertex models was derived.

Using the algebraic Bethe ansatz, Bogoliubov, Pronko and Zvonarev introduced a definition of boundary one-point correlation functions of the six-vertex model with domain wall boundary conditions [6], and these functions were expressed as determinants of $N\times N$ matrices. In Ref. [7], Coloman and Pronko gave two-point boundary correlation functions of the six-vertex model with DWBC, and also computed them in determinant form. In our preceding paper [8], we considered one-point boundary correlation functions of the gl(1|1) supersymmetric vertex model with DWBC, and also derived the representations for these correlation functions as determinants of $N\times N$ matrices by using the algebraic Bethe ansatz. In this paper we will consider the two-point boundary correlations of the gl(1|1) supersymmetric vertex model with DWBC.

The article is organized as follows. In Section 2, we describe the gl(1|1) supersymmetric vertex model with DWBC and the algebraic Bethe ansatz. In Section 3, we recall the one-point boundary correlation functions. In Section 4, we derive the expressions for two-point boundary correlation function $H_N^{(r_1,r_2)}$. Some remarks are given in Section 5.

2 The gl(1|1) supersymmetric vertex model with DWBC and Algebraic Bethe ansatz

Let $\mathbb{C}^2$ be a Z_2-graded 2-dimensional vector space with FB grading, i.e. [1]=1, [2]=0.

We firstly introduce an L-matrix:

$$L_{\alpha k}(\lambda_\alpha,\nu_k)=\begin{pmatrix}\lambda_\alpha-\nu_k-\eta\left(3E^{11}_{(k)}+E^{22}_{(k)}\right) & E^{21}_{(k)}2\eta\\ E^{12}_{(k)}2\eta & \lambda_\alpha-\nu_k-\eta\left(E^{11}_{(k)}-E^{22}_{(k)}\right)\end{pmatrix},\tag{1}$$

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where $E^{ij}_{(k)}$ are the generators of the superalgebra gl(1|1), acting on the k-th space. Here α and k denote α-th copy of $\mathbb{C}^2$ in $\mathcal{H}$ and the k-th copy in $\mathcal{V}$, the total space of the vertical lines is $\mathcal{V}=(\mathbb{C}^2)^{\otimes N}$ and the total space of the horizontal lines is $\mathcal{H}=(\mathbb{C}^2)^{\otimes N}$. With each vertex of the lattice one associates an operator acting in the full space $\mathcal{V}\otimes\mathcal{H}$. The monodromy matrix $T_\alpha(\lambda_\alpha)$ is defined by

$$T_\alpha(\lambda_\alpha) = L_{\alpha N}(\lambda_\alpha,\nu_N)\ldots L_{\alpha 1}(\lambda_\alpha,\nu_1)= \begin{pmatrix} A(\lambda_\alpha) & B(\lambda_\alpha) \\ C(\lambda_\alpha) & D(\lambda_\alpha)\end{pmatrix}. \tag{2}$$

According to the graded Yang-Baxter relation

$$R_{\alpha\beta}L_{\alpha k}L_{\beta k}=L_{\beta k}L_{\alpha k}R_{\alpha\beta}. \tag{3}$$

The R-matrix is, in the FB grading, given by

$$R_{\alpha\beta}(\lambda,\lambda')=\begin{pmatrix} c & 0 & 0 & 0\\ 0 & a & b & 0\\ 0 & b & a & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{4}$$

where the functions $c(\lambda,\lambda')$, $a(\lambda,\lambda')$ and $b(\lambda,\lambda')$ are

$$c(\lambda,\lambda')=\frac{(\lambda-\lambda'-2\eta)}{(\lambda-\lambda'+2\eta)},\quad a(\lambda,\lambda')=\frac{\lambda-\lambda'}{(\lambda-\lambda'+2\eta)},$$
$$b(\lambda,\lambda')=\frac{2\eta}{(\lambda-\lambda'+2\eta)}. \tag{5}$$

In the following, we consider the gl(1|1) vertex model on a 2-d $N\times N$ square lattice. It is a model that the arrows reside on the edges of the lattice, with the same number of incoming and outgoing arrows through each lattice vertex. For this model, there are altogether 6 possible weights corresponding to a vertex configuration. We give the possible configurations and their corresponding Boltzmann weights w_i. See Fig. 1.

Here let

$$w_1=\lambda_\alpha-\nu_k+\eta,\quad w_2=\lambda_\alpha-\nu_k-3\eta,$$
$$w_3=w_4=\lambda_\alpha-\nu_k-\eta,\quad w_5=w_6=2\eta, \tag{6}$$

then the Boltzmann weights correspond to the elements of the gl(1|1) L-matrix.

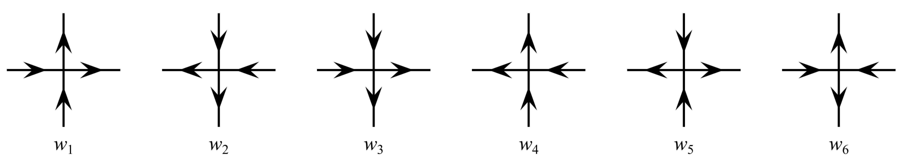

Fig. 1. The vertex configurations and their Boltzmann weights for the gl(1|1) vertex model.

For the domain wall boundary condition, it implies that all arrows on the top and bottom of the lattice are pointing inwards while all arrows on the left and right boundaries are pointing outwards.

Next, we recall the Algebraic Bethe ansatz. Since the L-matrix satisfies the graded Yang-Baxter relation, the monodromy matrix satisfies the following intertwining relation

$$R_{\alpha\beta}(\lambda_\alpha,\lambda_\beta)T_\alpha(\lambda_\alpha)T_\beta(\lambda_\beta)= T_\beta(\lambda_\beta)T_\alpha(\lambda_\alpha)R_{\alpha\beta}(\lambda_\alpha,\lambda_\beta),\quad \alpha\neq\beta. \tag{7}$$

With the help of the above relation, one may find the complete list of the commutation relations. For our purposes we only write four of them:

$$C(\lambda)C(\lambda') = -c(\lambda,\lambda')C(\lambda')C(\lambda), \tag{8}$$

$$D(\lambda)C(\lambda') = f(\lambda,\lambda')C(\lambda')D(\lambda)+ g(\lambda',\lambda)C(\lambda)D(\lambda'), \tag{9}$$

$$A(\lambda)C(\lambda') = f(\lambda,\lambda')C(\lambda')A(\lambda)+ g(\lambda',\lambda)C(\lambda)A(\lambda'), \tag{10}$$

$$B(\lambda)C(\lambda') = -C(\lambda')B(\lambda)+ g(\lambda,\lambda')[D(\lambda)A(\lambda')-D(\lambda')A(\lambda)]. \tag{11}$$

With

$$f(\lambda,\lambda') = \frac{1}{a(\lambda',\lambda)}=\frac{\lambda'-\lambda+2\eta}{(\lambda'-\lambda)},$$
$$g(\lambda,\lambda') = \frac{b(\lambda',\lambda)}{a(\lambda',\lambda)}=\frac{2\eta}{(\lambda'-\lambda)}. \tag{12}$$

For the generating vector, it is convenient to use the state either with all spins up or with all spins down. Here we have introduced the notation

$$|\Uparrow\rangle = \otimes^N_{k=1}|\uparrow\rangle_k=\otimes^N_{k=1}\begin{pmatrix}1\\0\end{pmatrix}_{(k)},$$

$$|\Downarrow\rangle = \otimes_{k=1}^{N}|\downarrow\rangle_k = \otimes_{k=1}^{N}\begin{pmatrix}0\\1\end{pmatrix}_{(k)}. \qquad (13)$$

Applying the elements of the monodromy matrix to these vectors and their dual, we easily obtain

$$C(\lambda)|\Uparrow\rangle = 0, \quad B(\lambda)|\Downarrow\rangle = 0,$$
$$\langle\Uparrow|B(\lambda) = 0, \quad \langle\Downarrow|C(\lambda) = 0, \qquad (14)$$

$$A(\lambda)|\Downarrow\rangle = \prod_{k=1}^{N}(\lambda-\nu_k-\eta)|\Downarrow\rangle,$$

$$D(\lambda)|\Downarrow\rangle = \prod_{k=1}^{N}(\lambda-\nu_k+\eta)|\Downarrow\rangle. \qquad (15)$$

Thus, all vectors are generated by multiple action of operators $C(\lambda_\alpha)$ on the state $|\Downarrow\rangle$

$$\prod_{\alpha=1}^{M}C(\lambda_\alpha)|\Downarrow\rangle \equiv C(\lambda_M)\dots C(\lambda_1)|\Downarrow\rangle, \quad M\leqslant N. \qquad (16)$$

In Ref. [9], the partition function $Z_N = Z_N$ $(\lambda_1,\dots,\lambda_N;\nu_1,\dots,\nu_N)$ on an $N\times N$ square lattice with DWBC can be written in the form

$$Z_N = \langle\Uparrow|C(\lambda_N)\dots C(\lambda_1)|\Downarrow\rangle. \qquad (17)$$

The determinant representation is given by

$$Z_N = \prod_{k<j}a^{-1}(\nu_k,\nu_j)\det\mathcal{Z}(\{\underline{\lambda}\},\{\underline{\nu}\}), \qquad (18)$$

where the entries of the matrix $\mathcal{Z}$ are given by

$$\mathcal{Z}_{\alpha k} = \phi(\lambda_\alpha,\nu_k) = b(\lambda_\alpha,\nu_k)\prod_{\gamma=1}^{\alpha-1}a(\lambda_\gamma,\nu_k),$$
$$\alpha,k=1,\dots,N. \qquad (19)$$

3 One-point boundary correlation functions

In this section, we will review two kinds of one-point boundary correlation functions $G_N^{(M)}$ and $H_N^{(r)}$ (which have been given in Ref. [8]). These can be defined by

$$G_N^{(M)} = Z_N^{-1}\langle\Uparrow|C(\lambda_N)\dots$$
$$C(\lambda_{M+1})E_{(1)}^{11}C(\lambda_M)\dots C(\lambda_1)|\Downarrow\rangle, \qquad (20)$$

$$H_N^{(M)} = Z_N^{-1}\langle\Uparrow|C(\lambda_N)\dots C(\lambda_{M+1})E_{(1)}^{11}C(\lambda_M)\times$$
$$E_{(1)}^{22}C(\lambda_{M-1})\dots C(\lambda_1)|\Downarrow\rangle, \qquad (21)$$

where $E_{(1)}^{11}$ and $E_{(1)}^{22}$ are the projectors on the spin up and the spin down state. Due to $E_{(1)}^{11}+E_{(1)}^{22}=I$, these correlation functions $H_N^{(M)}$ and $G_N^{(M)}$ are related by

$$G_N^{(M)} = H_N^{(M)}+H_N^{(M-1)}+\dots+H_N^{(1)}, \qquad (22)$$

$$H_N^{(M)} = G_N^{(M)}-G_N^{(M-1)}. \qquad (23)$$

In the following, we will directly give the main results. Two reduction formulae can be expressed as (introducing $d(\lambda)=\prod_{k=1}^{N}(\lambda-\nu_k+\eta)$)

$$H_N^{(M)} = Z_N^{-1}2\eta\prod_{\alpha=1}^{M-1}(\lambda_\alpha-\nu_1-\eta)\prod_{\alpha=M+1}^{N}(\lambda_\alpha-\nu_1-3\eta)\times$$
$$\sum_{\beta=1}^{M}d_2(\lambda_\beta)\frac{g(\lambda_\beta,\lambda_M)}{f(\lambda_\beta,\lambda_M)}(-1)^{M-\beta-1}\prod_{\gamma=\beta+1}^{M-1}c(\lambda_\gamma,\lambda_\beta)\prod_{\substack{\gamma=1\\ \gamma\neq\beta}}^{M}f(\lambda_\beta,\lambda_\gamma)Z_{N-1}\left(\{\lambda_\alpha\}_{\alpha=1,\alpha\neq\beta}^{N};\{\nu_k\}_{k=2}^{N}\right). \qquad (24)$$

$$G_N^{(M)} = Z_N^{-1}\prod_{\alpha=1}^{M}(\lambda_\alpha-\nu_1-\eta)\prod_{\alpha=M+1}^{N}(\lambda_\alpha-\nu_1-3\eta)\times$$
$$\sum_{\beta=1}^{M}\frac{d_2(\lambda_\beta)2\eta}{(\lambda_\beta-\nu_1-\eta)}\prod_{\substack{\gamma=1\\ \gamma\neq\beta}}^{M}f(\lambda_\beta,\lambda_\gamma)(-1)^{M-\beta}\prod_{\gamma=\beta+1}^{M}c(\lambda_\gamma,\lambda_\beta)Z_{N-1}\left(\{\lambda_\alpha\}_{\alpha=1,\alpha\neq\beta}^{N};\{\nu_k\}_{k=2}^{N}\right). \qquad (25)$$

Accordingly, the determinant representation for these boundary correlation functions may be represented as

$$H_N^{(M)} = \frac{2\eta\prod_{j=2}^{N}(\nu_1-\nu_j)\prod_{\alpha=1}^{M-1}(\lambda_\alpha-\nu_1-\eta)\prod_{\alpha=M+1}^{N}(\lambda_\alpha-\nu_1-3\eta)}{\prod_{j=2}^{N}(\nu_1-\nu_j+2\eta)}\frac{\det_N\mathcal{H}}{\det_N\mathcal{Z}}, \qquad (26)$$

$$G_N^{(M)} = \frac{\prod_{j=2}^{N}(\nu_1-\nu_j)\prod_{\alpha=1}^{M}(\lambda_\alpha-\nu_1-\eta)\prod_{\alpha=M+1}^{N}(\lambda_\alpha-\nu_1-3\eta)}{\prod_{j=2}^{N}(\nu_1-\nu_j+2\eta)}\frac{\det_N\mathcal{G}}{\det_N\mathcal{Z}}. \quad (27)$$

4 Two-point boundary correlation functions

In this section, we will use the algebraic Bethe ansatz to calculate the two-point boundary correlation function $H_N^{(r_1,r_2)}$, which describes the probability of finding vertices of type $i=5$ both at the r_1-th position (counted from the right) of the first row and at the r_2-th position of the last row, (see Fig. 2).

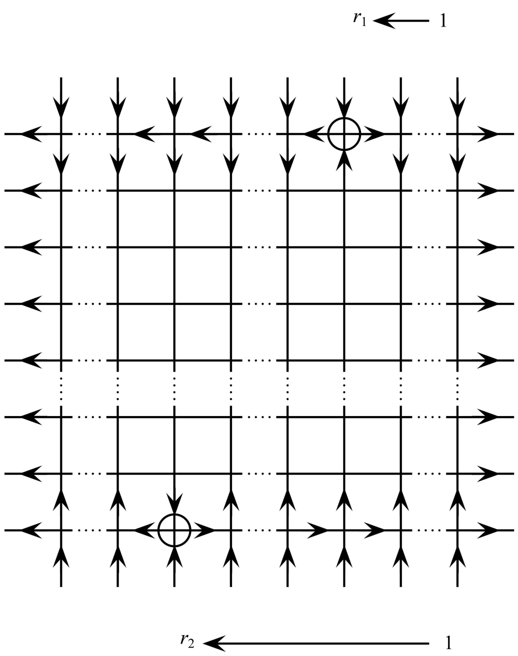

Fig. 2. The two-point correlation function $H_N^{(r_1,r_2)}$.

If $r_1<r_2$, this correlation function can be defined by

$$H_N^{(r_1,r_2)} = Z_N^{-1}\langle\Uparrow|C(\lambda_N)\cdots C(\lambda_{r_2+1})\,E_{(N)}^{11}C(\lambda_{r_2})\times E_{(N)}^{22}\,C(\lambda_{r_2-1})\cdots C(\lambda_{r_1+1})E_{(1)}^{11}\times C(\lambda_{r_1})\,E_{(1)}^{22}\,C(\lambda_{r_1-1})\cdots C(\lambda_1)|\Downarrow\rangle. \quad (28)$$

It is worth pointing out there are has analogous expressions for $r_1=r_2$ and $r_1>r_2$. Similarly, the correlation function $G_N^{(r_1,r_2)}$ describes the probability of finding arrows pointing left on the first and the last rows. Moreover, the functions $H_N^{(r_1,r_2)}$ and $G_N^{(r_1,r_2)}$ are related to each other just as relations (22) in the one-point case, so we shall mainly consider the function $H_N^{(r_1,r_2)}$. The result for $G_N^{(r_1,r_2)}$ can be obtained by its obvious relation with $H_N^{(r_1,r_2)}$.

In the following, we will express the correlation functions of the model on an $N\times N$ square lattice through the sum over partition functions of the models on $(N-2)\times(N-2)$ square sublattices. In order to derive the reduction formulae for the correlation functions $H_N^{(r_1,r_2)}$, they must be witten in a form suitable for applying commutation relations (9), (10) and (11). The monodromy matrix $T_\alpha(\lambda_\alpha)$ can be decomposed into the matrix product of three monodromy matrices in the α-th space [2]:

$$T_\alpha(\lambda_\alpha)=T_{\alpha N}(\lambda_\alpha)\widetilde{T}_\alpha(\lambda_\alpha)T_{\alpha 1}(\lambda_\alpha), \quad (29)$$

with

$$T_{\alpha N}(\lambda_\alpha) = L_{\alpha N}(\lambda_\alpha,\nu_N)=\begin{pmatrix}A_N(\lambda_\alpha) & B_N(\lambda_\alpha)\\ C_N(\lambda_\alpha) & D_N(\lambda_\alpha)\end{pmatrix}_{[\alpha]}, \quad (30)$$

$$\widetilde{T}_\alpha(\lambda_\alpha):=\overleftarrow{\prod_{k=2}^{N-1}}L_{\alpha,k}(\lambda_\alpha,\nu_k)=\begin{pmatrix}\widetilde{A}(\lambda_\alpha) & \widetilde{B}(\lambda_\alpha)\\ \widetilde{C}(\lambda_\alpha) & \widetilde{D}(\lambda_\alpha)\end{pmatrix}_{[\alpha]}, \quad (31)$$

$$T_{\alpha 1}(\lambda_\alpha) = L_{\alpha 1}(\lambda_\alpha,\nu_1)=\begin{pmatrix}A_1(\lambda_\alpha) & B_1(\lambda_\alpha)\\ C_1(\lambda_\alpha) & D_1(\lambda_\alpha)\end{pmatrix}_{[\alpha]}. \quad (32)$$

The entries of $T_{\alpha N}(\lambda)$ and $T_{\alpha 1}(\lambda)$ act nontrivially in the last and in the first "vertical" space, and depend on the "vertical" variable ν_N and the "vertical" variables ν_1 respectively, while $\widetilde{T}_\alpha(\lambda_\alpha)$ act in the rest $N-2$ "vertical" spaces and depend on the "vertical" variable $\nu_2,\dots,\nu_{N-1}$. Because they act nontrivially in different spaces, each set of operators entering the monodromy matrices satisfies the commutation relations (9), (10) and (11). Accordingly, the generating vectors $|\Uparrow\rangle$ ($|\Downarrow\rangle$) can be represented as the direct product of three generating vectors, e.g., $|\Uparrow\rangle=|\Uparrow_N\rangle\otimes|\widetilde{\Uparrow}\rangle\otimes|\Uparrow_1\rangle$, where $|\widetilde{\Uparrow}\rangle=\otimes_{k=2}^{N-1}|\uparrow\rangle_k$ and $|\Uparrow_N\rangle\equiv|\uparrow\rangle_N$ $|\Uparrow_1\rangle\equiv|\uparrow\rangle_1$. The states $|\widetilde{\Uparrow}\rangle$ ($|\widetilde{\Downarrow}\rangle$), $|\Uparrow_N\rangle$($|\Downarrow_N\rangle$) and $|\Uparrow_1\rangle$ ($|\Downarrow_1\rangle$) have the same properties, so the eigenvalues of operator $D_2(\lambda)$ and $A_2(\lambda)$ on the state $|\widetilde{\Downarrow}\rangle$ are

$$\widetilde{A}(\lambda)|\widetilde{\Downarrow}\rangle = \prod_{k=2}^{N-1}(\lambda-\nu_k-\eta)|\widetilde{\Downarrow}\rangle,$$

$$\widetilde{D}(\lambda)|\widetilde{\Downarrow}\rangle = \prod_{k=2}^{N-1}(\lambda-\nu_k+\eta)|\widetilde{\Downarrow}\rangle. \tag{33}$$

Due to (30), (31), (32) and (2), in particular, one gets

$$C(\lambda) = D_N(\lambda)\widetilde{D}(\lambda)C_1(\lambda)+C_N(\lambda)\widetilde{A}(\lambda)A_1(\lambda)+ D_N(\lambda)\widetilde{C}(\lambda)A_1(\lambda)+C_N(\lambda)\widetilde{B}(\lambda)C_1(\lambda). \tag{34}$$

According to L-operator (1), the operators $C_1(\lambda)$, $A_1(\lambda)$, $C_N(\lambda)$ and $D_N(\lambda)$ are expressed as

$$C_1(\lambda) = E^{12}_{(1)}2\eta,$$

$$A_1(\lambda) = \lambda-\nu_1-\eta(3E^{11}_{(1)}+E^{22}_{(1)}), \tag{35}$$

$$C_N(\lambda) = E^{12}_{(N)}2\eta,$$

$$D_N(\lambda) = \lambda-\nu_N-\eta(E^{11}_{(N)}-E^{22}_{(N)}). \tag{36}$$

In the following, we shall express the correlation function $H_N^{(r_1,r_2)}$ in terms of partition functions on $(N-2)\times(N-2)$ lattices. We use Eq. (34), (35) and (36) to reduce the problem of calculating the scalar products in the right hand sides of expressions (28) and get the expression

$$H_N^{(r_1,r_2)} = Z_N^{-1}(2\eta)^2\prod_{\alpha=r_1+1}^{N}(\lambda_\alpha-\nu_1-3\eta)\prod_{\alpha=1}^{r_1-1}(\lambda_\alpha-\nu_1-\eta)\prod_{\alpha=r_2+1}^{N}(\lambda_\alpha-\nu_N-\eta)\prod_{\alpha=1}^{r_2-1}(\lambda_\alpha-\nu_N+\eta)\times$$
$$\langle\widetilde{\Uparrow}|\widetilde{C}(\lambda_N)\cdots\widetilde{C}(\lambda_{r_2+1})\widetilde{A}(\lambda_{r_2})\widetilde{C}(\lambda_{r_2-1})\cdots\widetilde{C}(\lambda_{r_1+1})\widetilde{D}(\lambda_{r_1})\widetilde{C}(\lambda_{r_1-1})\cdots\widetilde{C}(\lambda_1)|\widetilde{\Downarrow}\rangle. \tag{37}$$

Acting first the operator $\widetilde{D}(\lambda_{r_1})$ on the right, using

$$\widetilde{D}(\lambda_r)\prod_{\alpha=1}^{r-1}\widetilde{C}(\lambda_\alpha)|\widetilde{\Downarrow}\rangle = \sum_{\alpha=1}^{r}\prod_{k=2}^{N-1}(\lambda_\alpha-\nu_k+\eta)(-1)^{r-\alpha-1}\prod_{\beta=\alpha+1}^{r-1}c(\lambda_\beta,\lambda_\alpha)\times$$
$$\frac{g(\lambda_\alpha,\lambda_r)}{f(\lambda_\alpha,\lambda_r)}\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{r}f(\lambda_\alpha,\lambda_\beta)\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{r}\widetilde{C}(\lambda_\beta)|\widetilde{\Downarrow}\rangle, \tag{38}$$

and next acting the operator $\widetilde{A}(\lambda_{r_2})$ on the right, using

$$\widetilde{A}(\lambda_r)\prod_{\alpha=1}^{r-1}\widetilde{C}(\lambda_\alpha)|\widetilde{\Downarrow}\rangle = \sum_{\alpha=1}^{r}\prod_{k=2}^{N-1}(\lambda_\alpha-\nu_k-\eta)(-1)^{r-\alpha-1}\prod_{\beta=\alpha+1}^{r-1}c(\lambda_\beta,\lambda_\alpha)\times$$
$$\frac{g(\lambda_\alpha,\lambda_r)}{f(\lambda_\alpha,\lambda_r)}\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{r}f(\lambda_\alpha,\lambda_\beta)\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{r}\widetilde{C}(\lambda_\beta)|\widetilde{\Downarrow}\rangle, \tag{39}$$

one can get the expression

$$H_N^{(r_1,r_2)} = Z_N^{-1}(2\eta)^2\prod_{\alpha=r_1+1}^{N}(\lambda_\alpha-\nu_1-3\eta)\prod_{\alpha=1}^{r_1-1}(\lambda_\alpha-\nu_1-\eta)\prod_{\alpha=r_2+1}^{N}(\lambda_\alpha-\nu_N-\eta)\prod_{\alpha=1}^{r_2-1}(\lambda_\alpha-\nu_N+\eta)\times$$
$$\sum_{\alpha=1}^{r_1}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{r_2}\prod_{k=2}^{N-1}(\lambda_\alpha-\nu_k+\eta)(-1)^{r_1-\alpha-1}\prod_{\gamma=\alpha+1}^{r_1-1}c(\lambda_\gamma,\lambda_\alpha)\frac{g(\lambda_\alpha,\lambda_{r_1})}{f(\lambda_\alpha,\lambda_{r_1})}\prod_{\substack{\gamma=1\\ \gamma\neq\alpha}}^{r_1}f(\lambda_\alpha,\lambda_\gamma)\times$$
$$\prod_{k=2}^{N-1}(\lambda_\beta-\nu_k-\eta)(-1)^{r_2-\beta-1}\prod_{\substack{\gamma=\beta+1\\ \gamma\neq\alpha}}^{r_2-1}c(\lambda_\gamma,\lambda_\beta)\frac{g(\lambda_\beta,\lambda_{r_2})}{f(\lambda_\beta,\lambda_{r_2})}\prod_{\substack{\gamma=1\\ \gamma\neq\alpha,\beta}}^{r_2}f(\lambda_\beta,\lambda_\gamma)\times$$
$$Z_{N-2}\Big(\{\lambda_\delta\}_{\delta=1,\delta\neq\alpha,\beta}^{N};\{\nu_k\}_{k=2}^{N-1}\Big). \tag{40}$$

If $r_1>r_2$, then one should first act the operator $A(\lambda_{r_2})$ on the right and next the operator $D(\lambda_{r_1})$. Thus it can be easily verified that the result has the same expression with (40), which shows that Eq. (40) is valid for $r_1\neq r_2$.

If $r_1=r_2=r$, one can get the following formula instead of (37) in this case:

$$H_N^{(r,r)} = Z_N^{-1}(2\eta)^2\prod_{\alpha=r+1}^{N}(\lambda_\alpha-\nu_1-3\eta)(\lambda_\alpha-\nu_N-\eta)\prod_{\alpha=1}^{r-1}(\lambda_\alpha-\nu_1-\eta)(\lambda_\alpha-\nu_N+\eta)\times \langle\widetilde{\Uparrow}|\widetilde{C}(\lambda_N)\cdots\widetilde{C}(\lambda_{r+1})\widetilde{B}(\lambda_r)\widetilde{C}(\lambda_{r-1})\cdots\widetilde{C}(\lambda_1)|\widetilde{\Downarrow}\rangle. \tag{41}$$

With the help of (11), one can get the expression

$$H_N^{(r,r)} = Z_N^{-1}(2\eta)^2\prod_{\alpha=r+1}^{N}(\lambda_\alpha-\nu_1-3\eta)(\lambda_\alpha-\nu_N-\eta)\prod_{\alpha=1}^{r-1}(\lambda_\alpha-\nu_1-\eta)(\lambda_\alpha-\nu_N+\eta)\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{r}\prod_{k=2}^{N-1}(\lambda_\alpha-\nu_k+\eta)\times \prod_{k=2}^{N-1}(\lambda_\beta-\nu_k-\eta)(-1)^{2r-\alpha-\beta-2}\prod_{\gamma=\alpha+1}^{r-1}c(\lambda_\gamma,\lambda_\alpha)\prod_{\substack{\gamma=\beta+1\\ \gamma\neq\alpha}}^{r-1}c(\lambda_\gamma,\lambda_\beta)\frac{g(\lambda_\alpha,\lambda_r)g(\lambda_\beta,\lambda_r)}{f(\lambda_\alpha,\lambda_r)f(\lambda_\beta,\lambda_r)}\times f(\lambda_\alpha,\lambda_\beta)\prod_{\substack{\gamma=1\\ \gamma\neq\alpha,\beta}}^{r}f(\lambda_\alpha,\lambda_\gamma)f(\lambda_\beta,\lambda_\gamma)Z_{N-2}\Big(\{\lambda_\delta\}_{\delta=1,\delta\neq\alpha,\beta}^{N};\{\nu_k\}_{k=2}^{N-1}\Big). \tag{42}$$

Comparing (42) with (40), one has no trouble in finding that the expression (40) is valid for all values of r_1 and r_2.

Next, by substituting the determinant representation (18) for the partition functions Z_N and Z_{N-2} in the right hand sides of relation (40), respectively, and canceling the resulting common factors, we can get exactly the equation

$$H_N^{(r_1,r_2)} = \prod_{j=2}^{N-1}\frac{(\nu_1-\nu_j)(\nu_j-\nu_N)(\nu_1-\nu_N)}{(\nu_1-\nu_j+2\eta)(\nu_j-\nu_N+2\eta)(\nu_1-\nu_N+2\eta)}\times \frac{(2\eta)^2\prod_{\alpha=1}^{r_1-1}(\lambda_\alpha-\nu_1-\eta)\prod_{\alpha=r_1+1}^{N}(\lambda_\alpha-\nu_1-3\eta)\prod_{\alpha=1}^{r_2-1}(\lambda_\alpha-\nu_N+\eta)\prod_{\alpha=r_2+1}^{N}(\lambda_\alpha-\nu_N-\eta)}{\det_N\mathcal{Z}}\times \sum_{\alpha=1}^{r_1}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{r_2}(-1)^{r_1+r_2-\alpha-\beta-2}\frac{w_{r_1}(\lambda_\alpha)\tilde{w}_{r_2}(\lambda_\beta)}{e(\lambda_\alpha,\lambda_\beta)}\det{}_{N-2}\mathcal{Z}_{(\alpha,\beta;1,N)}. \tag{43}$$

Here, $\mathcal{Z}_{(\alpha,\beta;1,N)}$, denotes the $(N-2)\times(N-2)$ matrix obtained from $\mathcal{Z}$ by eliminating the α-th and β-th rows and the first and the last columns. The functions $w_{r_1}(\lambda_\alpha)$, $\tilde{w}_{r_2}(\lambda_\beta)$ and $e(\lambda_\alpha,\lambda_\beta)$ are defined as

$$w_{r_1}(\lambda_\alpha) = \frac{\prod_{k=2}^{N-1}(\lambda_\alpha-\nu_k+\eta)\prod_{\gamma=\alpha+1}^{r_1-1}(\lambda_\gamma-\lambda_\alpha-2\eta)\prod_{\gamma=1}^{\alpha}(\lambda_\gamma-\lambda_\alpha+2\eta)}{\prod_{\substack{\gamma=1\\ \gamma\neq\alpha}}^{r_1}(\lambda_\gamma-\lambda_\alpha)}, \tag{44}$$

$$\tilde{w}_{r_2}(\lambda_\beta) = \frac{\prod_{k=2}^{N-1}(\lambda_\beta-\nu_k-\eta)\prod_{\substack{\gamma=\beta+1\\ \gamma\neq\alpha}}^{r_2-1}(\lambda_\gamma-\lambda_\beta-2\eta)\prod_{\gamma=1}^{\beta}(\lambda_\gamma-\lambda_\beta+2\eta)}{\prod_{\substack{\gamma=1\\ \gamma\neq\alpha,\beta}}^{r_2}(\lambda_\gamma-\lambda_\beta)}, \tag{45}$$

$$e(\lambda_\alpha,\lambda_\beta) = (\lambda_\alpha-\lambda_\beta+2\eta). \tag{46}$$

Obtaining the Eq. (43) for two-point boundary correlation functions $H_N^{(r_1,r_2)}$ is our main aim for the present paper.

5 Conclusion

In this paper, we have proposed the gl(1|1) supersymmetric vertex models with the so-called domain wall boundary conditions. The reduction formulae and the determinant representation for the one-point boundary correlation functions $G_N^{(M)}$ and $H_N^{(M)}$ are simply reviewed. Moreover, we consider the two-point boundary correlation functions and get the reduction expression for the correlation Eq. (43).

Here we should point out that, contrary to the formulae for the partition function and one-point boundary correlation function, the two-point boundary correlation function cannot be expressed as a determinant in the inhomogeneous model. However, the two-point boundary correlation function can be expressed as a determinant in the homogeneous limit, which we may consider in the future.

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