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Critical behavior of higher cumulants of order parameter in the 3D-Ising universality class*

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Abstract: QCD deconfinement phase transition is supposed to be the same universality class as the 3D-Ising model. According to the universality of critical behavior, the Binder-like ratios and ratios of higher cumulants of order parameter near the critical temperature in the 3D-Ising model are studied. The Binder-like ratio is shown to be a step function of temperature. The critical point is the intersection of the ratios of different system sizes between two platforms. The normalized cumulant ratios, like the Skewness and Kurtosis, do not diverge with correlation length, contrary to the corresponding cumulants. Possible applications of these characters in locating critical point in relativistic heavy ion collisions are discussed.

Key words: QCD deconfinement phase transition, critical point, 3D-Ising model, Binder-like ratios, ratios of higher cumulants

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1 Introduction

One of the main goals of current relativistic heavy ion experiments is to locate the critical point of QCD deconfinement phase transition. The critical character is that the correlation length ξ becomes infinitely larger in an infinite system. For a finite system, like the one formed in relativistic heavy ion collisions, the correlation length should be a finite maximum at the critical point. Therefore, various correlation length related observables were suggested for relativistic heavy ion collisions [1].

It has been recently shown that near the critical point, the density-density correlator of the baryon number follows the same power law behavior as the correlator of the sigma field, which is associated with the chiral order parameter [2, 3]. Therefore, the baryon number is considered as an equivalent order parameter of the system formed in nuclear collisions [4].

From statistical physics, it also shows that the susceptibilities of order parameter are directly related to the cumulants of conserved charges, e.g.,

$$\langle \delta N^2 \rangle = VT\chi_2. \tag{1}$$

χ_2 is the second susceptibility. $\langle \delta N^2 \rangle = \langle (N-\bar{N})^2 \rangle$ is the fluctuations of the conserved charge N. For three flavor QCD, the conserved charges are baryon-number, strangeness, and electric charge [5].

The third and fourth cumulants of conserved charges are defined respectively as,

$$K_3 = \langle \delta N^3 \rangle, \quad K_4 = \langle \delta N^4 \rangle - 3\langle \delta N^2 \rangle^2. \tag{2}$$

In the vicinity of the critical point, they are argued to be proportional to the higher power of correlation length, i.e., $\xi^{4.5}$ and ξ^7 [6, 7], respectively. So they are more sensitive to the correlation length, and highly recommended for locating the critical point in relativistic heavy ion collisions.

In experiments [8], properly normalized cumu-

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lants, i.e., Skewness and Kurtosis,

$$K_3/K_2^{3/2}=\frac{\langle\delta N^3\rangle}{\langle\delta N^2\rangle^{3/2}},\quad K_4/K_2^2=\frac{\langle\delta N^4\rangle}{\langle\delta N^2\rangle^2}-3, \quad (3)$$

were actually presented. As the second cumulant in denominator is also proportional to a certain power of correlation length [9], whether such normalized skewness and kurtosis still diverge with the correlation length is not clear from the theoretical point of view.

From the theoretical side, the ratios of higher-order cumulants to the second one, e.g.,

$$\begin{aligned} R_{3,2}&=\frac{K_3}{K_2}=\frac{\langle\delta N^3\rangle}{\langle\delta N^2\rangle},\\ R_{4,2}&=\frac{K_4}{K_2}=\frac{\langle\delta N^4\rangle}{\langle\delta N^2\rangle}-3\langle\delta N^2\rangle, \end{aligned} \quad (4)$$

are estimated [10–14]. The lattice QCD with two light quark degrees of freedom shows that these ratios of the baryon number, strangeness, and electric charge have pronounced peaks from low to high temperature in the transition region of chiral symmetry breaking [10]. The effective potential models in the mean-field approximation also shows that there are peak, valley, and oscillating structures near the deconfinement and chiral phase transitions [11, 12, 14]. However, all these are obtained under some approximations due to the difficulties in lattice QCD calculations [15] and model estimations [13].

The universality argument indicates that the static critical exponents of the second-order phase transition are only determined by the dimensionality and symmetry of the system. The critical end point of the QCD deconfinement phase transition, if it exists, belongs to the same universality class as liquid-gas phase transitions and the 3D-Ising model [4, 16–18]. Its universal critical properties are discussed to be valid for various models and relevant to heavy ion collisions [19–23], in particular the event-by-event fluctuation of baryon number [21].

Therefore, if the formed system in heavy ion collisions reaches thermal equilibrium [24], the freeze-out curve is close to the transition line [25], and the critical fluctuations survive in the final state [6, 7]. The critical behavior of corresponding higher cumulant ratios of order parameter in the 3D-Ising model may serve as good guidance in locating the critical point in heavy ion experiments.

It is known in statistical physics that the Binder-like ratio of order parameter is a direct location of critical point [26]. Generally, the Binder-like ratios are normalized raw moments of order parameter. The third and fourth Binder-like ratios can be simply defined as,

$$B_3=\frac{\langle|M|^3\rangle}{\langle M^2\rangle^{3/2}},\quad B_4=\frac{\langle M^4\rangle}{\langle M^2\rangle^2}. \quad (5)$$

Here we take the 3D-Ising model as an example. The order parameter in the model is the magnet M of spin $\vec{s}$ in all lattice sites $N_{\rm L}$, i.e., $M=\sum_{i=1}^{N_{\rm L}}\vec{s_i}/N_{\rm L}$.

Equivalently, in relativistic heavy ion collisions, the order parameter is the baryon number [4]. The incident energy, or the controlling parameter, may be directly mapped to the temperature and baryon chemical potential [27]. The size of the formed system is mainly determined by the overlapped area, i.e., centralities. So if incident energy passes through the critical region, the Binder-like ratios of net-baryon number may serve as a good location of the critical point of QCD deconfinement phase transition.

In this paper, we first present the critical behavior of Binder-like ratios in the 3D-Ising model, and demonstrate why they are helpful, in particular, in locating the critical point in relativistic heavy ion collisions. Then, the critical behavior of skewness and kurtosis, $R_{3,2}$ and $R_{4,2}$, are presented and discussed, respectively. Meanwhile, from finite-size scaling of the susceptibilities, the critical behavior of those ratios are estimated and modeled independently. Finally, the summary is presented in Section 4.

The simulation of the 3D-Ising model is based on the single-cluster algorithm formulated by Wolff [28]. It is a global update algorithm with a much smaller autocorrelation time and dynamical exponent. At a given temperature, 1.0 million independent configurations, each one with 10 intervals, are used in the calculations. The samples of 4 lattice sizes, L =8, 12, 16, 20, are simulated. From the generated samples, the critical temperature can be determined with a very good precision [29].

2 Critical behavior of Binder-like ratios of the order parameter

The critical behavior of Binder-like ratios, B_3 and B_4, in the 3D-Ising for 4 different lattice sizes are presented in Fig. 1(a) and (b), respectively. We can see that both B_3 and B_4 show a step jump in the vicinity of critical temperature. The physical meaning of this jump is clear. When the temperature is much lower than the critical one, the system is almost ordered and the fluctuation of the order parameter is very small, i.e.,

$$\langle|M|^n\rangle\sim\langle|M|\rangle^n \quad (\text{for } n=2,3,4\cdots). \quad (6)$$

So it results in the lower platform, which is 1 for all orders of Binder-like ratios at all system sizes, as shown in Fig. 1.

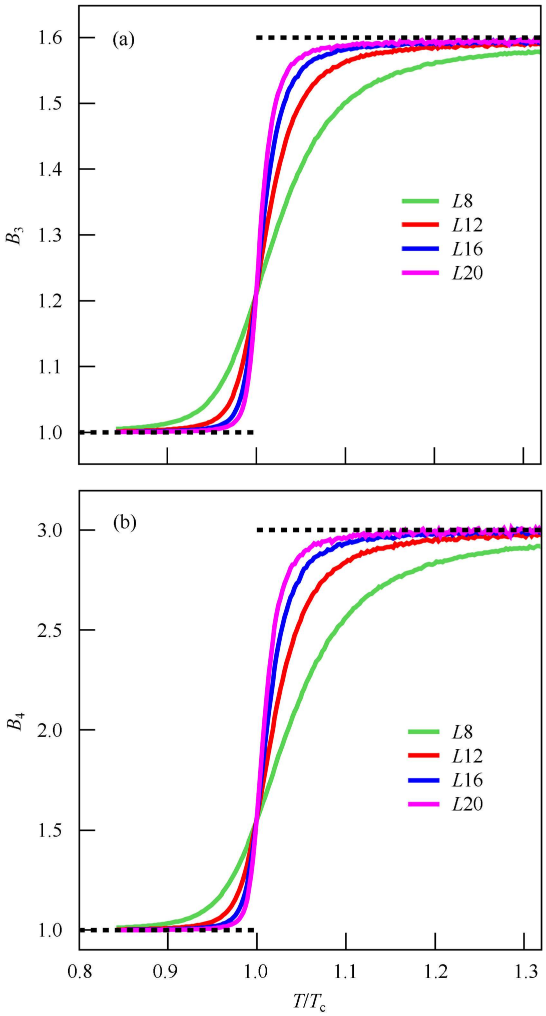

Fig. 1. (color online) The temperature dependence of Binder ratios in Eq. (5) in the vicinity of critical temperature in the 3D-Ising model for 4 different lattice sizes.

When the temperature approaches the critical one, the correlation length starts to increase with temperature and the fluctuations become larger and larger. Their critical behavior is system size dependent and described by finite-size scaling, i.e., $\langle|M|^n\rangle$ can be written in a scaling form [30],

$$\langle|M|^n\rangle = L^{-n\beta/\nu}F_n(tL^{1/\nu}), \tag{7}$$

where $\beta(0.3262)$ and $\nu(0.6297)$ [29] are the critical exponents of order parameter and correlation length, respectively. $t=(T-T_\mathrm{c})/T_\mathrm{c}$ is the reduced temperature and T_c is the critical temperature. Correspondingly, the Binder-like ratios become,

$$B_n = U_n(tL^{1/\nu}),\ n=3,4. \tag{8}$$

All size curves intersect at the fixed point only at the critical temperature, where they are system size independent [31], as shown in Fig. 1.

When the temperature is much higher than the critical one, the system is totally disordered. It again approaches to a constant. This forms the platform at high temperature. It is 1.6 and 3 times larger than the lower platforms for the third and fourth Binder-like ratios, respectively. So the higher the order of Binder-like ratio, the larger the gap of the step function is.

This step function of Binder-like ratios of baryon number may serve as a good locator of the critical point in relativistic heavy ion collisions, where the critical incident energy is unknown a priori. If we scan incident energies, and observe two platforms in low and high energy regions, respectively, then the critical one is most probably between them. We can finely tune the incident energy in the region and precisely determine the critical energy and exponents.

3 Critical behavior of the ratios of higher cumulants of order parameter

If we replace net-baryon, δN, by $L^d(|M|-\langle|M|\rangle)$ in Eq. (3), the skewness and kurtosis of the order parameter in the 3D-Ising model become

$$\begin{aligned}\text{Skewness} &= \frac{\langle(|M|-\langle|M|\rangle)^3\rangle}{\langle(|M|-\langle|M|\rangle)^2\rangle^{3/2}},\\ \text{Kurtosis} &= \frac{\langle(|M|-\langle|M|\rangle)^4\rangle}{\langle(|M|-\langle|M|\rangle)^2\rangle^{2}}-3,\end{aligned} \tag{9}$$

where $L^d=N_\mathrm{L}$, and d is the spacial dimension. The skewness and kurtosis from the 3D-Ising model for 4 different lattice sizes are presented in Fig. 2(a) and (b), respectively. We can see from the figure that they change sharply in the vicinity of the critical temperature. The skewness first drops down and then goes up, and kurtosis oscillates with temperature. Their values are system size dependent. Their signs change, respectively, near the critical point. The former in Fig. 2(a) changes from negative to positive when the temperature is increased through the critical point, while the latter in Fig. 2(b) becomes negative only when the temperature is close to the critical point. The sign changes in skewness and kurtosis are observed in effective models [32, 33].

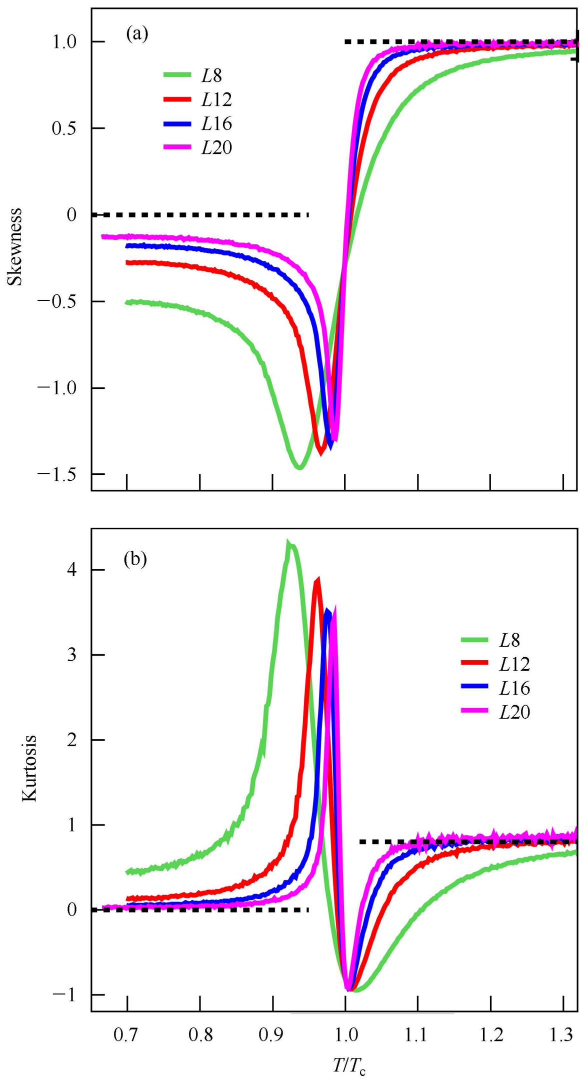

Fig. 2. (color online) The temperature dependence of skewness (a) and kurtosis (b) in Eq. (3) in the vicinity of critical temperature in the 3D Ising model for 4 different lattice sizes.

As we know, the skewness and kurtosis measure the symmetry and sharpness of the distribution, respectively. The distributions of order parameter M at system size $L=8$ near various critical temperatures are shown in Fig. 3. We can clearly see that the long tail of the distributions changes from the left to the right side when the temperature is increased through the critical point, and the peak of the distribution varies from sharp to flat when the temperature approaches the critical one. In the 3D bond percolation model, the same changes of the largest cluster size distribution have been observed in the transition region [34].

Those sign changes can also serve as a signal associated with the appearance of the critical point in relativistic heavy ion collisions. If we observe the sign change of baryon number skewness and kurtosis in a certain incident energy region, it most probably indicates the appearance of the critical point in the nearby incident energy region [32, 33].

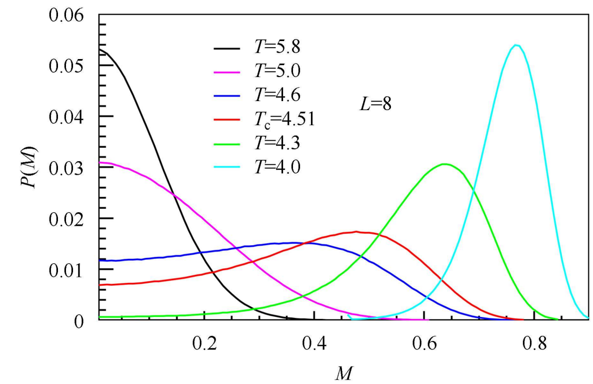

Fig. 3. (color online) The distributions of order parameter near the critical temperatures in the 3D-Ising model at system size $L=8$.

The skewness and kurtosis also converge to two constants when the temperature is away from the critical point, as shown in Fig. 2(a) and (b). But the two constants at low and high temperatures are between zero and 1. The gap between them is small and does not change very much with the order of cumulants, unlike the Binder-like ratio.

From the definition $\delta N \equiv L^d(|M|-\langle|M|\rangle)$ and the scaling form of $\langle|M|^n\rangle$, the ith cumulants of Ising model can be written as

$$K_i = L^{i(d-\beta/\nu)} P_i(tL^{1/\nu})\,. \tag{10}$$

At the critical point $t=0$, $K_3 \sim L^{3(d-\beta/\nu)}$ and $K_4 \sim L^{4(d-\beta/\nu)}$. Correspondingly, the skewness and kurtosis have the scaling forms

$$K_3/K_2^{3/2} = F_{\rm S}(tL^{1/\nu}), \quad K_4/K_2^2 = F_{\rm K}(tL^{1/\nu}). \tag{11}$$

They no longer diverge with correlation length, or system size. At the critical temperature $t=0$, the scaling function, i.e., $F_{\rm S}(0)$ or $F_{\rm K}(0)$, is a system size independent constant. All size curves intersect at the fixed point [31].

The $R_{3,2}$, and $R_{4,2}$ of the order parameter in the 3D-Ising model for 4 different lattice sizes are presented in Fig. 4(a) and (b), respectively. We can see again from Fig. 4(a) that $R_{3,2}$ changes its value sharply from negative to positive when the temperature is increased through the critical point. $R_{4,2}$ in Fig. 4(b) oscillates greatly with temperature near the critical point. These qualitative features, i.e., the sign change in the third moment and oscillating structure in the forth cumulants, are consistent with the estimations of the effective models [32, 33].

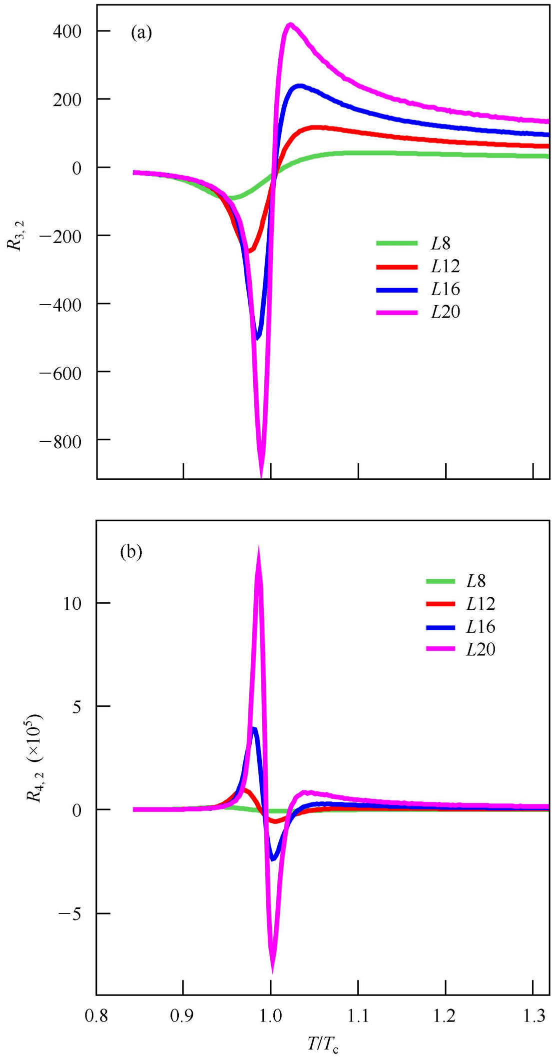

Fig. 4. (color online) The temperature dependence of $R_{3,2}$ (a), and $R_{4,2}$ (b) in the vicinity of critical temperature in the 3D-Ising model for 4 different lattice sizes.

$R_{3,2}$ and $R_{4,2}$ are very sensitive to the system size or correlation length. Their values become very large when the system size increases. From Eq. (10), the finite-size scaling forms of $R_{3,2}$, and $R_{4,2}$ are

$$R_{3,2} = L^{d-\beta/\nu} F_{3,2}(tL^{1/\nu}),$$
$$R_{4,2} = L^{2(d-\beta/\nu)} F_{4,2}(tL^{1/\nu}). \tag{12}$$

They diverge with the system size as $L^{d-\beta/\nu}$ and $L^{2(d-\beta/\nu)}$, respectively.

4 Summary

In this paper, using the 3D-Ising model, the same universality class of QCD de-confinement phase transition, the critical behavior of Binder-like ratios, and higher cumulant ratios of order parameter are presented. It is shown that near the critical temperature, the Binder-like ratios are step functions of temperature. The gap of the step function is 1.6 and 3 times wider for the third and fourth-order Binder-like ratios, respectively. The critical point is the intersection of Binder-like ratios of different size systems between two platforms.

The critical behaviors of Skewness, Kurtosis, $R_{3,2}$ and $R_{4,2}$ at various system sizes are also studied by using the 3D-Ising model, and estimated by finite size scaling. When the temperature is increased through the critical point, the ratios of the third-order cumulants change their values from negative to positive in a valley shape, and the ratios of the fourth-order cumulants oscillate around zero. All size curves of skewness (kurtosis) intersect at the critical point. The normalized ratios, like the skewness and kurtosis, do not diverge with correlation length. While, un-normalized ratios, $R_{3,2}$ and $R_{4,2}$, are divergent with correlation length. They are proportional to $\xi^{3-\beta/\nu}$ and $\xi^{6-2\beta/\nu}$, respectively, and very sensitive to the system size near the critical temperature.

In relativistic heavy ion collisions, the corresponding measurements are higher cumulant ratios of the net-baryon. If the formed system in heavy ion collisions reaches thermal equilibrium, the incident energy passes through the critical region, and the critical net-baryon fluctuations survive in the final state. All those critical characters observed in the 3D-Ising model may show up correspondingly. So it is interesting to observe the Binder-like ratios and higher cumulant ratios of net-baryon at RHIC beam energy, SPS, and FAIR.

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