Exact solutions of the Duffin-Kemmer-Petiau equation with a hyperbolical potential in (1+3) dimensions

H. Hassanabadi[1;1)] Z. Molaee[2] A. Boumali[3]

[1] Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran
[2] Department of Basic sciences. Garmsar Branch, Islamic Azad university, Garmsar, Iran
[3] Laboratoire de Physique Appliquée et Théorique, LPAT, Université de Tébessa, 12000, Algeria

Abstract: We study S-wave solutions of the Duffin-Kemmer-Petiau (DKP) equation in the presence of a hyperbolical potential in (1+3)-dimensional space-time for spin-one particles. The exact analytical Nikiforov-Uvarov (NU) method is used in the calculations to obtain the eigenfunctions and the corresponding eigenvalues. Some figures and numerical values are included to give a better insight to the solutions.

Key words: Duffin-Kemmer-Petiau equation, hyperbolical potential, Nikiforov-Uvarov technique

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1 Introduction

The first-order relativistic Duffin-Kemmer-Petiau (DKP) equation, in separate formulations, describes spin-one and spin-zero particles [1]. It has well accounted for many phenomena including deuteron-nucleus scattering [2], relativistic α-nucleus elastic scattering, meson-nuclear interaction [3, 4], and K^+ nucleus scattering in the presence of the Aharonov-Bohm potential [5]. In quantum mechanics language (relativistic or non-relativistic), determination of the exact solutions of the wave equation are the most essential step as the wave functions and the energy eigenvalues provide us with the requisite information about the quantum system. Here, we will obtain the solutions of the spin-one DKP equation under a hyperbolical interaction via the powerful Nikiforov-Uvarov (NU) technique which solves a large class of linear second-order differential equations frequently present in the study of various interactions under DKP, Klein-Gordon, Dirac and Schrödinger wave equations. Although there are some papers, which discuss the spin-zero version of the DKP equation under common interactions of quantum mechanics such as the Coulomb and harmonic terms, there are only a few papers on the spin-one DKP equation. The main reason for the lack of sufficient papers for the spin-one case is the complicated structure of the version in comparison with the spin-zero one, which posses the same mathematical structure as the Klein-Gordon equation when a scalar term is present [6–18]. Also we can cite some papers about the DKP equation under different interactions [18–22]. Our purpose is to solve the spin-one DKP equation under a hyperbolical potential in (1+3)-dimensional space-time. The results seem more useful when we remember on the one hand the very complicated nature of the Proca equation and the other hand the high number of relativistic spin-one bosons present in various studies of nuclear and particle physics.

2 Overview the Duffin-Kemmer-Petiau equation

The DKP equation for free particles with spin-one and spin-zero is (in natural units $\hbar=c=1$) [21]

$$(\mathrm{i}\beta^{\mu}\partial_{\mu}-m)\Psi=0, \tag{1}$$

β^{μ} are the DKP matrices and satisfy the algebra

$$\beta^{\mu}\beta^{\nu}\beta^{\lambda}+\beta^{\lambda}\beta^{\nu}\beta^{\mu}=g^{\mu\nu}\beta^{\lambda}+g^{\lambda\nu}\beta^{\mu}, \tag{2}$$

with

$$g^{\mu\nu}=\mathrm{diag}(1,-1,-1,-1),\ (g^{\mu\nu})^2=1. \tag{3}$$

The engaged β^{μ} matrices for spin- one particles are,

$$\beta^0=\begin{pmatrix} 0 & \bar{0} & \bar{0} & \bar{0} \\ \bar{0}^{\mathrm{T}} & 0 & \mathrm{I} & 0 \\ \bar{0}^{\mathrm{T}} & \mathrm{I} & 0 & 0 \\ \bar{0}^{\mathrm{T}} & 0 & 0 & 0 \end{pmatrix},\quad \beta^i=\begin{pmatrix} 0 & \bar{0} & \mathrm{e_i} & \bar{0} \\ \bar{0}^{\mathrm{T}} & 0 & 0 & -\mathrm{iS_i} \\ -\mathrm{e_i^T} & 0 & 0 & 0 \\ \bar{0}^{\mathrm{T}} & -\mathrm{iS_i} & 0 & 0 \end{pmatrix}, \tag{4}$$

with

$$\bar{0}=(0,0,0),\ e_1=(1,0,0),\ e_2=(0,1,0),\ e_3=(0,0,1), \tag{5}$$

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1) E-mail: h.hasanabadi@shahroodut.ac.ir; Corresponding author, Tel.:+98 912 5325104; fax: +98 273 3335270

$$\bar{0}^{\mathrm{T}}=\begin{pmatrix}0\\0\\0\end{pmatrix},\ -\mathrm{i}S_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\ -\mathrm{i}S_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\ -\mathrm{i}S_3=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}. \tag{6}$$

More explicitly, we can write the matrices as

$$\beta^0=\begin{pmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix},\quad \beta^1=\begin{pmatrix}0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&0&1&0\\-1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&-1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\end{pmatrix},$$

$$\beta^2=\begin{pmatrix}0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&-1&0&0\\0&0&0&0&0&0&0&0&0&0\\-1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0&0&0\end{pmatrix},\quad \beta^3=\begin{pmatrix}0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&-1&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\-1&0&0&0&0&0&0&0&0&0\\0&0&-1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix}. \tag{7}$$

3 The DKP equation in (1+3) dimensional space-time

The DKP equation in the presence of an interaction term in (1+3)-dimensions is written as

$$(\mathrm{i}\beta^0\partial_0+\mathrm{i}\beta^1\partial_1+\mathrm{i}\beta^2\partial_2+\mathrm{i}\beta^3\partial_3-m-U)\Psi(x,y,z,t)=0, \tag{8}$$

$$\Psi(x,y,z,t)=\exp(-\mathrm{i}E_{n,l}t)\psi_{n,l}(x,y,z), \tag{9}$$

where
$\psi^{\mathrm{T}}_{n,l}(\vec{r})=(\varphi^{(1)}_{n,l}(\vec{r}),\ \varphi^{(2)}_{n,l}(\vec{r}),\ \varphi^{(3)}_{n,l}(\vec{r}),\ \varphi^{(4)}_{n,l}(\vec{r}),\ \varphi^{(5)}_{n,l}(\vec{r}),\ \varphi^{(6)}_{n,l}(\vec{r}),\ \varphi^{(7)}_{n,l}(\vec{r}),\ \varphi^{(8)}_{n,l}(\vec{r}),\ \varphi^{(9)}_{n,l}(\vec{r}),\ \varphi^{(10)}_{n,l}(\vec{r}))^{\mathrm{T}}$,
and choose,
$\varphi^{(1)}_{n,l}=\mathrm{i}\varphi,\ \vec{F}(\vec{r})=(\varphi^{(2)}_{n,l}(\vec{r}),\ \varphi^{(3)}_{n,l}(\vec{r}),\ \varphi^{(4)}_{n,l}(\vec{r})),\ \vec{G}(\vec{r})=(\varphi^{(5)}_{n,l}(\vec{r}),\ \varphi^{(6)}_{n,l}(\vec{r}),\ \varphi^{(7)}_{n,l}(\vec{r}))$,

$$\vec{H}(\vec{r})=(\varphi^{(8)}_{n,l}(\vec{r}),\varphi^{(9)}_{n,l}(\vec{r}),\varphi^{(10)}_{n,l}(\vec{r})). \tag{10}$$

For elastic scattering, the interaction is [23],

$$U=S(r)+PS_{\mu}(r)+\beta^0V(r)+\beta^0PV_{\mathrm{P}}(r), \tag{11}$$

where each term has a specific Lorentz character. Two Lorentz vectors may be written as β^{μ} and $P\beta^{\mu}$ by assuming rotational invariance and parity conservation [1–3]. The term $P=(\beta^{\mu}\beta_{\mu}-2)=$ diag (1, 1, 1, 1, 0, 0, 0, 0, 0, 0) is a projection operator for the spin-one sector of the DKP equation.

From Eq. (8), we have,

$$\mathrm{i}\vec{\nabla}\times\vec{F}=m\vec{H}, \tag{12}$$

$$\vec{\nabla}.\vec{G}=m\varphi, \tag{13}$$

$$E_{n,l}\vec{G}+\mathrm{i}\vec{\nabla}\times\vec{H}=m\vec{F} \tag{14}$$

$$(E_{n,l}-V(r))\vec{F}+\vec{\nabla}\varphi=m\vec{G}. \tag{15}$$

Now we can present some useful formulas by introducing the spinor of the DKP equation as [2], $\psi=(\varphi\ A^1\ A^2\ A^3\ E_1\ E_2\ E_3\ -B_1\ -B_2\ -B_3)^{\mathrm{T}}$. The spin-one KDP equation in a form that resembles Maxwell's equations, in which case one obtains

$$-\mathrm{i}\nabla\times A=mB, \tag{16a}$$

$$\mathrm{i}\frac{\partial A}{\partial t}-\mathrm{i}\nabla\varphi=mE, \tag{16b}$$

$$\mathrm{i}\frac{\partial E}{\partial t}-\mathrm{i}\nabla\times B=mA, \tag{16c}$$

$$\mathrm{i}\nabla.E=m\varphi. \tag{16d}$$

Where the "Maxwell-like" form of the DKP equation is used.

Therefore, by combining the above equations, we have

$$(E_{n,l}(E_{n,l}-V(r))-m_0^2)\vec{F}(\vec{r})+\nabla^2\vec{F}(\vec{r})=0, \tag{17}$$

brings us at the equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}+E_{n,l}^2-E_{n,l}V(r)-m^2-\frac{l(l+1)}{r^2}\right)\vec{F}(\vec{r})=0. \tag{18a}$$

Introducing $\vec{F}=r^{-1}\vec{U}$,

$$\left(\begin{pmatrix}\varphi_{n,l}^2(r)\\ \varphi_{n,l}^3(r)\\ \varphi_{n,l}^4(r)\end{pmatrix}=r^{-1}\begin{pmatrix}U_{n,l}^2(r)\\ U_{n,l}^3(r)\\ U_{n,l}^4(r)\end{pmatrix}\right)$$

to remove the first-order derivative in above equation, then we suppose that $l=0$,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+E_{n,0}^2-E_{n,0}V(r)-m^2\right)\vec{U}(\vec{r})=0. \tag{18b}$$

4 The Nikiforov-Uvarov technique

We now give a brief introduction to the powerful NU technique. In its parametric form, this exact analytical tool solves any differential equation of the form [24]

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{\mathrm{d}}{\mathrm{d}s}+\frac{1}{[s(1-\alpha_3 s)]^2}[-\xi_1 s^2+\xi_2 s-\xi_3]\right\}\times\psi_n(s)=0. \tag{19}$$

According to the NU method, the eigenfunctions are [25]

$$\psi_n(s) = N_n s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}\times(1-2\alpha_3 s). \tag{20}$$

Furthermore in some cases we can use

$$\psi_n(s) = s^{\alpha_{12}^*}(1-\alpha_3 s)^{-\alpha_{12}^*-\frac{\alpha_{13}^*}{\alpha_3}}P_n^{\left(\alpha_{10}^*-1,\frac{\alpha_{11}^*}{\alpha_3}-\alpha_{10}-1\right)}\times(1-2\alpha_3 s), \tag{21}$$

where P_n is the orthogonal Jacobi-polynomial and the engaged parameters are

$$\begin{aligned}
&\alpha_4=\frac{1}{2}(1-\alpha_1),\quad \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),\\
&\alpha_6=\alpha_5^2+\xi_1,\quad \alpha_7=2\alpha_4\alpha_5-\xi_2,\\
&\alpha_8=\alpha_4^2+\xi_3,\quad \alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6,\\
&\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8},\\
&\alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}),\\
&\alpha_{12}=\alpha_4+\sqrt{\alpha_8},\quad \alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}).
\end{aligned} \tag{22}$$

$\alpha_{10}^*=\alpha_1+2\alpha_4-2\sqrt{\alpha_8}$, $\alpha_{11}^*=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}-\alpha_3\sqrt{\alpha_8})$, $\alpha_{12}^*=\alpha_4-\sqrt{\alpha_8}$, $\alpha_{13}^*=\alpha_5-(\sqrt{\alpha_9}-\alpha_3\sqrt{\alpha_8})$.

The energy eigenvalues can be derived from

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0. \tag{23}$$

5 *S*-wave Solutions of the DKP equation

By introducing the hyperbolical potential as,

$$V(r)=V_0(1-\cot gh(\alpha r))=V_0\left(\frac{1+\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right),$$

applying the change of variable $\mathrm{e}^{-2\alpha r}=s$ in Eq. (18b), we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}^2U_{n,0}^{(2)}(r)}{\mathrm{d}s^2}+\frac{1-s}{s(1-s)}\frac{\mathrm{d}U_{n,0}^{(2)}(r)}{\mathrm{d}s}\\
&+\frac{1}{s^2(1-s)^2}\left[\frac{E_{n,0}^2-m^2-E_{n,0}V_0}{4\alpha^2}+\left(\frac{-E_{n,0}^2+m^2}{2\alpha^2}\right)s\right.\\
&\left.+\left(\frac{E_{n,0}^2-m^2+E_{n,0}V_0}{4\alpha^2}\right)s^2\right]U_{n,0}^{(2)}(r)=0.
\end{aligned} \tag{24}$$

Now, a simple comparison of Eq. (24) with Eq. (19) indicates the correspondence

$$\begin{aligned}
\xi_1 &= \frac{-E_{n,0}^2+m^2-E_{n,0}V_0}{4\alpha^2},\ \xi_2=\frac{-E_{n,0}^2+m^2}{2\alpha^2},\\
\xi_3 &= \frac{-E_{n,0}^2+m^2+E_{n,0}V_0}{4\alpha^2},
\end{aligned} \tag{25}$$

and

$$\alpha_1=1,\ \alpha_2=1,\ \alpha_3=1,\ \alpha_4=0,\ \alpha_5=-\frac{1}{2}, \tag{26}$$

$$\begin{aligned}
\alpha_6 &= \frac{1}{4}+\left(\frac{-E_{n,0}^2+m^2-E_{n,0}V_0}{4\alpha^2}\right),\\
\alpha_7 &= \left(\frac{E_{n,0}^2-m^2}{2\alpha^2}\right),\\
\alpha_8 &= \frac{-E_{n,0}^2+m^2+E_{n,0}V_0}{4\alpha^2},\\
\alpha_9 &= \frac{1}{4},\quad \alpha_{10}=1+\frac{\sqrt{m^2-E_{n,0}^2+E_{n,0}V_0}}{\alpha},\\
\alpha_{11} &= 2+2\left(\sqrt{\frac{1}{4}}+\frac{\sqrt{m^2-E_{n,0}^2+E_{n,0}V_0}}{2\alpha}\right),\\
\alpha_{12} &= \frac{\sqrt{m^2-E_{n,0}^2+E_{n,0}V_0}}{2\alpha},\\
\alpha_{13} &= \frac{-1}{2}-\left(\sqrt{\frac{1}{4}}+\frac{\sqrt{m^2-E_{n,0}^2+E_{n,0}V_0}}{2\alpha}\right).
\end{aligned} \tag{27}$$

So, we have for the wave function

$$U_{n,0}^{(2)}(r) = N_n\mathrm{e}^{-\sqrt{-E_{n,0}^2+m^2+E_{n,0}V_0}r}(1-\mathrm{e}^{-2\alpha r})\times P_n^{\left(\frac{\sqrt{-E_{n,0}^2+m^2+E_{n,0}V_0}}{\alpha},1\right)}(1-2\mathrm{e}^{-2\alpha r}). \tag{28}$$

Where the Jacobi polynomials are [26, 27],

$$P_n^{(c,d)}(z) = 2^{-n}\sum_{p=0}^{n}\begin{pmatrix} n+c \\ P \end{pmatrix}\begin{pmatrix} n+d \\ n-p \end{pmatrix}(1-z)^{n-p}(1+z)^{p},$$

$$P_n^{(c,d)}(z) = \frac{\Gamma(n+c+1)}{n!\Gamma(n+c+d+1)}\sum_{r=0}^{n}\begin{pmatrix} n \\ r \end{pmatrix} \times\frac{\Gamma(n+c+d+r+1)}{\Gamma(r+c+1)}\left(\frac{z-1}{2}\right)^{r}, \quad (29)$$

with

$$\begin{pmatrix} n \\ r \end{pmatrix}=\frac{n!}{r!(n-r)!}=\frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$$

the energy spectrum is found to be

$$(n+1)^2+\frac{(\sqrt{m^2-E_{n,0}^2+E_{n,0}V_0})(n+1)}{\alpha}+\frac{E_{n,0}V_0}{2\alpha^2}=0. \quad (30)$$

Table 1. Energy eigenvalues for $\alpha=-0.02$ and $m=1$.

n	$E_{n,0}$ for $V_0=0.1$	$E_{n,0}$ for $V_0=0.5$	$E_{n,0}$ for $V_0=1$
0	0.371316	0.079729	0.039960
1	0.624195	0.157864	0.079681
2	0.766837	0.232953	0.118931
3	0.845280	0.303799	0.157484
4	0.889944	0.369529	0.195133
5	0.916407	0.429604	0.231687
6	0.932467	0.483791	0.266974
7	0.942183	0.532109	0.300849

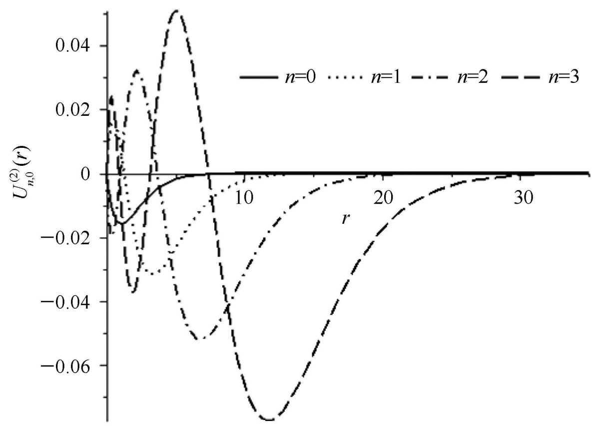

Fig. 1. Wavefunctions vs. r for n=0, 1, 2, 3.

We have plotted the wavefunctions for $n=0$, 1, 2, 3 in Fig. 1. Energy for different values of V_0 is reported in Table 1.

6 Conclusion

We derived the S-wave solutions of the Duffin-Kemmer-Petiau equation for spin-one particles in the presence of a hyperbolical potential in (1+3)-dimensional. The energy eigenvalues and the associated normalized wavefunctions were obtained using the NU method. Our results are useful in the study of relativistic spin-one particles particularly in particle and nuclear physics after a proper fit is performed.

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