Chiral magnetic effect for chiral fermion system*

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Abstract: The chiral magnetic effect is concisely derived by employing the Wigner function approach in the chiral fermion system. Subsequently, the chiral magnetic effect is derived by solving the Landau levels of chiral fermions in detail. The second quantization and ensemble average leads to the equation of the chiral magnetic effect for righthand and lefthand fermion systems. The chiral magnetic effect arises uniquely from the contribution of the lowest Landau level. We carefully analyze the lowest Landau level and find that all righthand (chirality is +1) fermions move along the direction of the magnetic field, whereas all lefthand (chirality is −1) fermions move in the opposite direction of the magnetic field. Hence, the chiral magnetic effect can be explained clearly using a microscopic approach.

Keywords: CME, Landau levels, chiral fermions

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1 Introduction

Quark gluon plasma (QGP) is created in high energy heavy ion collisions, constituting extremely hot and dense matter. An enormous magnetic field can be generated by high energy peripheral collisions [1-3]. One of the predictions in QGP is that positively and negatively charged particles seperate along the direction of the magnetic field, which is related to chiral magnetic effect (CME) [4-6]. Numerous efforts have been made to determine the CME in experiments [7-9]. However, due to background noise, no definite CME has been revealed to date. Numerous theoretical methods likewise investigated the CME, such as AdS/CFT [10, 11], hydrodynamics [12-14], finite temperature field theory [15-18], quantum kinetic theory [19], lattice method [20], *et al*.

In this article, we study the CME in detail by determination of Landau levels. For the massive Dirac fermion system, several studies on CME addressed Landau levels. In Ref. [15], Fukushima *et al*. proposed four methods to derive the CME. One of these methods made use of Landau energy levels for the massive Dirac equation with chemical potential μ and chiral chemical potential μ_5 in a homogeneous magnetic background $\boldsymbol{B} = B\boldsymbol{e}_z$ to construct the thermodynamic potential Ω. The macroscopic electric current j^z along the z-axis can be obtained from the thermodynamic potential Ω. Another study on the CME addressing Landau levels is related to the second quantization of the Dirac field. In Ref. [21], the authors determined the Landau levels and corresponding Landau wavefunctions for the massive Dirac equation in a uniform magnetic field, likewise with chemical potential μ and chiral chemical potential μ_5. Then, they second-quantized the Dirac field and expanded it by these solved Landau wavefunctions and creation/construction operators. The density operator $\hat{\rho}$ can then be determined from Hamiltonian $\hat{H}$ and particle number operator $\hat{N}$ of the system. Finally, they derived the macroscopic electric current j^z along the z-axis through the trace of density operator $\hat{\rho}$ and electric current operator $\hat{j}^z$, which is simply the CME equation.

From the study on CME for massive Dirac fermions through Landau levels, we conclude that the contribution to CME arises uniquely from the lowest Landau level, while the contributions from higher Landau levels cancel each other. However, because of the mass m of the Dirac fermion, the physical picture of the CME for the massive Dirac fermion system is not as clear as in the massless fermion case, as the physical meaning of the chiral chem-

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ical potential μ_5 for the massive fermion case is not entirely understood. To address this issue, we list the lowest Landau level as follows (we set the homogeneous magnetic background $\boldsymbol{B}=B\boldsymbol{e}_z$ along the z-axis and assume $eB>0$, which is also appropriate for following sections),

$$\psi_{0\lambda}(k_y,k_z;\boldsymbol{x})=c_{0\lambda}\begin{pmatrix}\varphi_0\\0\\F_{0\lambda}\varphi_0\\0\end{pmatrix}\frac{1}{L}\mathrm{e}^{\mathrm{i}(yk_y+zk_z)},\ (\lambda=\pm1),\qquad(1)$$

with energy $E=\lambda\sqrt{m^2+k_z^2}$, where $F_{0\lambda}=(\lambda\sqrt{m^2+k_z^2}+k_z)/m$, and φ_0 is the zeroth harmonic oscillator wavefunction along the x-axis. To simplify the following discussions, we set $k_y=0$. The z-component of the spin operator for the single particle is $S^z=\frac{1}{2}\mathrm{diag}(\sigma_3,\sigma_3)$, implying $S^z\psi_{0\lambda}=(+\frac{1}{2})\psi_{0\lambda}$. When $\lambda=+1$, $E=\sqrt{m^2+k_z^2}>0$, then ψ_{0+} in Eq. (1) describes a particle with momentum k_z and spin projection $S^z=+\frac{1}{2}$. When $\lambda=-1$, $E=-\sqrt{m^2+k_z^2}<0$, then ψ_{0-} in Eq. (1) describes an antiparticle with momentum $-k_z$ and spin projection $S^z=-\frac{1}{2}$. Thus, in the homogeneous magnetic background $\boldsymbol{B}=B\boldsymbol{e}_z$, we obtain a picture for the lowest Landau level (with $k_y=0$): All particles spin along the $(+z)$-axis, while all antiparticles spin along the $(-z)$-axis; however, the z-component momentum of particles and anti-particles can be along both the $(+z)$-axis or the$(-z)$-axis. A net electric current is difficult to obtain along the magnetic field direction from the point of view of the lowest Landau level for the massive fermion case.

In this article, we focus on a massless fermion (also referred to as the "chiral fermion") system, where we show that it is easy to obtain a net electric current along the magnetic field direction, seen from the picture of the lowest Landau level. The chiral fermion field can be divided into two independent parts, namely the righthand and lefthand parts. First, we set up the notation. The electric charge of a fermion/antifermion is $\pm e$. The chemical potential for righthand/lefthand fermions is $\mu_{R/L}$, which can be employed to express the chiral and ordinary chemical potentials as $\mu_5=(\mu_R-\mu_L)/2$ and $\mu=(\mu_R+\mu_L)/2$, respectively. The chemical potential μ describes the imbalance of fermions and anti-fermions, while the chiral chemical potential μ_5 describes the imbalance of righthand and lefthand chirality. Notably, the introduction of a chemical potential generally corresponds to a conserved quantity. The conserved quantity corresponding to the ordinary chemical potential μ is total electric charge of the system. However, due to chiral anomaly [22, 23], there is no conserved quantity corresponding to the chiral chemical potential μ_5, which is crucial for the existence of CME [24].

To study the CME in the chiral fermion system, first we show a succinct derivation of CME employing the Wigner function approach, which we can use to obtain the CME as a quantum effect of the first order in the $\hbar$ expansion. Subsequently, we turn to determine the Landau levels for the chiral fermion system. Because chiral fermions are massless, the equations of righthand and lefthand parts of the chiral fermion field decouple with each other, which allows us to deal with righthand and lefthand fermion fields independently. Taking the righthand fermion field as an example, we first solve the energy eigenvalue equation of the righthand fermion field in an external uniform magnetic field and obtain a series of Landau levels. Then, we perform the second quantization for righthand fermion field, which can be expanded by complete wavefunctions of Landau levels. Finally, the CME can be derived through the ensemble average, explicitly indicating that the CME uniquely arises from the lowest Landau level. By analyzing the physical picture for the lowest Landau level, we conclude that all righthand (chirality is +1) fermions move along the positive z-direction, and all lefthand (chirality is -1) fermions move along the negative z-direction. This is the main result of this study. This result can qualitatively explain why a macroscopic electric current occurs along the direction of the magnetic field in a chiral fermion system, called the CME. We emphasize that the CME equation is derived by determining Landau levels, without the approximation of a weak magnetic field.

The rest of this article is organized as follows. In Sec. 2, we present a succinct derivation for the CME using Wigner function approach. In Sec. 3, we determine the Landau levels for the righthand fermion field. In Secs. 4 and 5, we perform the second quantization of the righthand fermion system and obtain CME through the ensemble average. In Sec. 6, we discuss the physical picture of the lowest Landau level. Finally, we summarize this study in Sec. 7. Some derivation details are presented in the appendixes.

Throughout this article, we adopt natural units, where $\hbar=c=k_B=1$. The convention for the metric tensor is $g^{\mu\nu}=\mathrm{diag}(+1,-1,-1,-1)$. The totally antisymmetric Levi-Civita tensor is $\epsilon^{\mu\nu\rho\sigma}$ with $\epsilon^{0123}=+1$, which is in agreement with Peskin [25], but not with Bjorken and Drell [26]. The Greek indices, μ,ν,ρ,σ, run over $0,1,2,3$, or t,x,y,z, whereas Roman indices, i,j,k, run over $1,2,3$ or x,y,z. We use the Heaviside-Lorentz convention for electromagnetism.

2 A succinct derivation of CME using Wigner function approach

We concisely derive the CME using the Wigner function approach for a chiral fermion system. Our starting point is the following covariant and gauge invariant Wigner function,

$$\mathcal{W}_{\alpha\beta}(x,p)=\left\langle :\frac{1}{(2\pi)^4}\int \mathrm{d}^4 y \mathrm{e}^{-\mathrm{i}p\cdot y}\overline{\Psi}_\beta\left(x+\frac{y}{2}\right)U\left(x+\frac{y}{2},x-\frac{y}{2}\right)\times\Psi_\alpha\left(x-\frac{y}{2}\right):\right\rangle, \quad (2)$$

where $\langle:\cdots:\rangle$ represents the ensemble average, $\Psi(x)$ is the Dirac field operator for chiral fermions, α, β are Dirac spinor indices, and $U(x+y/2,x-y/2)$ is the gauge link of a straight line from $(x-y/2)$ to $(x+y/2)$. This specific choice for the path in the gauge link in the definition of the Wigner function was first proposed in Ref. [27], where the authors argued that this type of gauge link can create the variable p in the Wigner function $\mathcal{W}(x,p)$ to represent the kinetic momentum, although in principle the path in the gauge link is arbitrary. The specific choice of the two points $(x\pm y/2)$ in the integrand in Eq. (2) is based on the consideration of symmetry. We can also replace $(x\pm y/2)$ by $(x+sy)$ and $(x-(1-s)y)$, where s is a real parameter [28].

Suppose that the electromagnetic field $F^{\mu\nu}$ is homogeneous in space and time, then from the dynamical equation satisfied by $\Psi(x)$, one can obtain the dynamical equation for $\mathcal{W}(x,p)$ as follows,

$$\gamma\cdot K\mathcal{W}(x,p)=0, \quad (3)$$

where $K_\mu=\frac{\mathrm{i}}{2}\nabla_\mu+p_\mu$ and $\nabla_\mu=\partial^x_\mu-eF_{\mu\nu}\partial^\nu_p$. Because $\mathcal{W}(x,p)$ is a 4×4 matrix, we can decompose it into 16 independent Γ-matrices,

$$\mathcal{W}=\frac{1}{4}\left(\mathcal{F}+\mathrm{i}\gamma^5\mathcal{P}+\gamma^\mu\mathcal{V}_\mu+\gamma^5\gamma^\mu\mathcal{A}_\mu+\frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right). \quad (4)$$

The 16 coefficient functions $\mathcal{F},\mathcal{P},\mathcal{V}_\mu,\mathcal{A}_\mu,\mathcal{S}_{\mu\nu}$ are scalar, pseudoscalar, vector, pseudovector, and tensor, respectively, and they are all real functions because $\mathcal{W}^\dagger=\gamma^0\mathcal{W}\gamma^0$. Vector current and axial vector current can be expressed as the four-momentum integration of $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$,

$$J^\mu_V(x)=\int \mathrm{d}^4 p\mathcal{V}^\mu, \quad (5)$$

$$J^\mu_A(x)=\int \mathrm{d}^4 p\mathcal{A}^\mu. \quad (6)$$

By multiplying Eq. (3) by $\gamma\cdot K$ from the lefthand side, we obtain the quadratic form of Eq. (3) as follows,

$$\left(K^2-\frac{\mathrm{i}}{2}\sigma^{\mu\nu}[K_\mu,K_\nu]\right)\mathcal{W}=0. \quad (7)$$

From Eq. (7), we can obtain two off mass-shell equations for $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ (see Appendix A for details),

$$\left(p^2-\frac{1}{4}\hbar^2\nabla^2\right)\mathcal{V}_\mu=-e\hbar\tilde{F}_{\mu\nu}\mathcal{A}^\nu, \quad (8)$$

$$\left(p^2-\frac{1}{4}\hbar^2\nabla^2\right)\mathcal{A}_\mu=-e\hbar\tilde{F}_{\mu\nu}\mathcal{V}^\nu, \quad (9)$$

where $\tilde{F}_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. We explicitly showed the $\hbar$ factor in Eqs. (8), (9). If we expand $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$ order-by-order in $\hbar$ as

$$\mathcal{V}^\mu=\mathcal{V}^\mu_{(0)}+\hbar\mathcal{V}^\mu_{(1)}+\hbar^2\mathcal{V}^\mu_{(2)}+\cdots, \quad (10)$$

$$\mathcal{A}^\mu=\mathcal{A}^\mu_{(0)}+\hbar\mathcal{A}^\mu_{(1)}+\hbar^2\mathcal{A}^\mu_{(2)}+\cdots, \quad (11)$$

then, at order $o(1)$ and $o(\hbar)$, Eqs. (8), (9) become

$$p^2\mathcal{V}^\mu_{(0)}=0, \quad (12)$$

$$p^2\mathcal{A}^\mu_{(0)}=0, \quad (13)$$

$$p^2\mathcal{V}_{(1)\mu}=-e\hbar\tilde{F}_{\mu\nu}\mathcal{A}^\nu_{(0)}, \quad (14)$$

$$p^2\mathcal{A}_{(1)\mu}=-e\hbar\tilde{F}_{\mu\nu}\mathcal{V}^\nu_{(0)}. \quad (15)$$

The zeroth order solutions $\mathcal{V}^\mu_{(0)}$ and $\mathcal{A}^\mu_{(0)}$ can be derived by directly calculating the Wigner function without the gauge link through the ensemble average in Eq. (2), which was already obtained by one of the authors and his collaborators [29]. The results for $\mathcal{V}^\mu_{(0)}$ and $\mathcal{A}^\mu_{(0)}$ are

$$\mathcal{V}^\mu_{(0)}=\frac{2}{(2\pi)^3}p^\mu\delta(p^2)\sum_s\left[\theta(p^0)\frac{1}{\mathrm{e}^{\beta(p^0-\mu_s)}+1}+\theta(-p^0)\frac{1}{\mathrm{e}^{\beta(-p^0+\mu_s)}+1}\right], \quad (16)$$

$$\mathcal{A}^\mu_{(0)}=\frac{2}{(2\pi)^3}p^\mu\delta(p^2)\sum_s s\left[\theta(p^0)\frac{1}{\mathrm{e}^{\beta(p^0-\mu_s)}+1}+\theta(-p^0)\frac{1}{\mathrm{e}^{\beta(-p^0+\mu_s)}+1}\right], \quad (17)$$

where $\beta=1/T$ is the inverse temperature of the system, $\mu_{R/L}$ is the chemical potential for righthand/lefthand fermions as mentioned in the introduction, and $s=\pm1$ corresponds to the chirality of righthand/lefthand fermions. The zeroth order solutions $\mathcal{V}^\mu_{(0)}$ and $\mathcal{A}^\mu_{(0)}$ satisfy Eqs. (12), (13), which indicates that they are both on shell. From Eqs. (14), (15) we directly obtain the first order solutions,

$$\mathcal{V}^\mu_{(1)}=\frac{2}{(2\pi)^3}e\hbar\tilde{F}^{\mu\nu}p_\nu\delta'(p^2)\times\sum_s s\left[\theta(p^0)\frac{1}{\mathrm{e}^{\beta(p^0-\mu_s)}+1}+\theta(-p^0)\frac{1}{\mathrm{e}^{\beta(-p^0+\mu_s)}+1}\right], \quad (18)$$

$$\mathcal{A}^\mu_{(1)}=\frac{2}{(2\pi)^3}e\hbar\tilde{F}^{\mu\nu}p_\nu\delta'(p^2)\times\sum_s\left[\theta(p^0)\frac{1}{\mathrm{e}^{\beta(p^0-\mu_s)}+1}+\theta(-p^0)\frac{1}{\mathrm{e}^{\beta(-p^0+\mu_s)}+1}\right], \quad (19)$$

where we employ $\delta'(p^2)=-\delta(p^2)/p^2$. Eqs. (18), (19) are the same as the second term in Eq. (3) of Ref. [30].

Now, we can calculate $\boldsymbol{J}_{A/V}$ based on Eqs. (5) and (6). Because $\mathcal{V}^i_{(0)},\mathcal{A}^i_{(0)}$ are odd functions of three-momentum $\boldsymbol{p}$, the nonzero contribution to $J^i_{V/A}$ arises uniquely from $\mathcal{V}^i_{(1)}$ and $\mathcal{A}^i_{(1)}$. We assume that only a uniform magnetic field exists $\boldsymbol{B}=B\boldsymbol{e}_z$, i.e. $F^{12}=-F^{21}=-B$ and $\tilde{F}^{03}=-\tilde{F}^{30}=-B$ (other components of $F^{\mu\nu}$, $\tilde{F}^{\mu\nu}$ are zero), which implies $J^x_{V/A}=J^y_{V/A}=0$. After integration over the z-components of Eqs. (18, 19) we have

$$J^z_V=\int d^4p\mathcal{V}^z_{(1)}=\frac{e\hbar\mu_5}{2\pi^2}B, \tag{20}$$

$$J^z_A=\int d^4p\mathcal{A}^z_{(1)}=\frac{e\hbar\mu}{2\pi^2}B, \tag{21}$$

where $\mu_5=(\mu_R-\mu_L)/2$ and $\mu=(\mu_R+\mu_L)/2$. Eq. (20) indicates that if $\mu_5\neq 0$, a current flows along the magnetic direction. Because$\hbar$ appears in the coefficient of the magnetic field B, an enormous magnetic field is required to produce a macroscopic current, which may be realized in high energy heavy ion collisions. Thus far, we derived the CME in the chiral fermion system using the Wigner function approach, and we observe that the CME is a first order quantum effect in $\hbar$. In fact, the Wigner function approach is a quantum kinetic theory, which implies the presence of quantum effects of a multi-particle system, such as the CME.

3 Landau levels for righthand fermions

In this and the following sections, we derive the CME for a chiral fermion system by determining the Landau levels. The Lagrangian for a chiral fermion field is

$$\mathcal{L}=\overline{\Psi}(x)\mathrm{i}\gamma\cdot D\Psi(x), \tag{22}$$

with the covariant derivative $D^\mu=\partial^\mu+\mathrm{i}eA^\mu$, and the electric charge $\pm e$ for particles/antiparticles. For a uniform magnetic field $\boldsymbol{B}=B\mathbf{e}_z$ along the z-axis, we choose the gauge potential as $A^\mu=(0,0,Bx,0)$. The equation of motion for the field $\Psi(x)$ is

$$\mathrm{i}\gamma\cdot D\Psi(x)=0, \tag{23}$$

which can be written in the form of a Schrödinger equation,

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(t,\boldsymbol{x})=\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{D}\Psi(t,\boldsymbol{x}), \tag{24}$$

with $\boldsymbol{D}=-\nabla+\mathrm{i}e\boldsymbol{A}$, $\boldsymbol{A}=(0,Bx,0)$. In the chiral representation of Dirac γ-matrices, where $\gamma^5=\mathrm{diag}(-1,1)$, $\boldsymbol{\alpha}=\mathrm{diag}(-\boldsymbol{\sigma},\boldsymbol{\sigma})$, we express Ψ in the form $\Psi=(\Psi_L^T,\Psi_R^T)^T$. Then, Eq. (24) becomes

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\Psi_L(t,\boldsymbol{x})\\ \Psi_R(t,\boldsymbol{x})\end{pmatrix}=\begin{pmatrix}-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}\Psi_L(t,\boldsymbol{x})\\ \mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}\Psi_R(t,\boldsymbol{x})\end{pmatrix}, \tag{25}$$

which indicates that the two fields $\Psi_{L/R}$, which correspond to eigenvalues ∓ 1 of the matrix γ^5, decouple with each other. The two fields $\Psi_{L/R}$ are often referred to as lefthand/righthand fermion fields, respectively. Lefthand and righthand fermions are also referred to as chiral fermions.

In the following, we focus on solving the eigenvalue equation for the righthand fermion field Ψ_R (similar results are obtained for the lefthand fermion field Ψ_L).

To determine the Landau levels, we must solve the eigenvalue equation for the righthand fermion field as follows,

$$\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}\psi_R=E\psi_R, \tag{26}$$

with $\boldsymbol{D}=(-\partial_x,-\partial_y+\mathrm{i}eBx,-\partial_z)$. The details for solving Eq. (26) are provided in Appendix B. We list the eigenfunctions and eigenvalues in the following : For $n=0$ Landau level, the wavefunction with energy $E=k_z$ is

$$\psi_{R0}(k_y,k_z;\boldsymbol{x})=\begin{pmatrix}\varphi_0(\xi)\\ 0\end{pmatrix}\frac{1}{L}\mathrm{e}^{\mathrm{i}(yk_y+zk_z)}. \tag{27}$$

For $n>0$ Landau level, the wavefunction with energy $E=\lambda E_n(k_z)$ is

$$\psi_{Rn\lambda}(k_y,k_z;\boldsymbol{x})=c_{n\lambda}\begin{pmatrix}\varphi_n(\xi)\\ iF_{n\lambda}\varphi_{n-1}(\xi)\end{pmatrix}\frac{1}{L}\mathrm{e}^{\mathrm{i}(yk_y+zk_z)}, \tag{28}$$

where $\lambda=\pm 1, E_n(k_z)=\sqrt{2neB+k_z^2}, F_{n\lambda}(k_z)=[k_z-\lambda E_n(k_z)]/\sqrt{2neB}$, normalized coefficient $|c_{n\lambda}|^2=1/(1+F^2_{n\lambda})$, and $\varphi_n(\xi)=\varphi_n(\sqrt{eB}x-k_y/\sqrt{eB})$ is the n-th order wavefunction of a harmonic oscillator.

For $n>0$ Landau levels, the wavefunctions with energies $E=\pm E_n(k_z)$ correspond to fermions and antifermions, respectively. For the lowest Landau level, the wavefunction with energy $E=k_z>0$ corresponds to fermions, whereas that with energy $E=k_z<0$ corresponds to antifermions. The wavefunctions of all Landau levels are orthonormal and complete. For the lefthand fermion field, the eigenfunctions of Landau levels are the same as the righthand case, but with the sign of the eigenvalues changed.

4 Second quantization for righthand fermion field

In this section, we second-quantize the righthand fermion field $\Psi_R(\boldsymbol{x})$, such that it becomes an operator and satisfies following anticommutative relations,

$$\begin{aligned}\{\Psi_R(\boldsymbol{x}),\Psi_R^\dagger(\boldsymbol{x}')\}&=\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'),\\ \{\Psi_R(\boldsymbol{x}),\Psi_R(\boldsymbol{x}')\}&=0.\end{aligned} \tag{29}$$

Because all eigenfunctions for the Hamiltonian of the righthand fermion field are orthonormal and complete, we decompose the righthand fermion field operator $\Psi_R(\boldsymbol{x})$ by these eigenfunctions as

$$\begin{aligned}\Psi_R(\boldsymbol{x})=&\sum_{k_y,k_z}[\theta(k_z)a_0(k_y,k_z)\psi_{R0}(k_y,k_z;\boldsymbol{x})\\ &+\theta(-k_z)b_0^\dagger(k_y,k_z)\psi_{R0}(k_y,k_z;\boldsymbol{x})]\\ &+\sum_{n,k_y,k_z}[a_n(k_y,k_z)\psi_{Rn+}(k_y,k_z;\boldsymbol{x})\\ &+b_n^\dagger(k_y,k_z)\psi_{Rn-}(k_y,k_z;\boldsymbol{x})].\end{aligned} \tag{30}$$

In contrast to the general Fourier decomposition for the second quantization, we place two theta functions $\theta(\pm k_z)$ in front of $a_0(k_y,k_z)$ and $b_0^\dagger(k_y,k_z)$ in the decomposition, which is very important for the subsequent second quantization procedure. From Eq. (29), we obtain following anticommutative relations,

$$\begin{aligned}\{\theta(k_z)a_0(k_y,k_z),\theta(k_z')a_0^\dagger(k_y',k_z')\}&=\theta(k_z)\delta_{k_yk_y'}\delta_{k_zk_z'}\\ \{\theta(-k_z)b_0(k_y,k_z),\theta(-k_z')b_0^\dagger(k_y',k_z')\}&=\theta(-k_z)\delta_{k_yk_y'}\delta_{k_zk_z'}\\ \{a_n(k_y,k_z),a_{n'}^\dagger(k_y',k_z')\}&=\delta_{nn'}\delta_{k_yk_y'}\delta_{k_zk_z'}\\ \{b_n(k_y,k_z),b_{n'}^\dagger(k_y',k_z')\}&=\delta_{nn'}\delta_{k_yk_y'}\delta_{k_zk_z'}.\end{aligned}\tag{31}$$

The two theta functions $\theta(\pm k_z)$ are always attached to the lowest Landau level operators, such as $a_0,a_0^\dagger,b_0,b_0^\dagger$. The Hamiltonian and total particle number of the righthand fermion system are

$$\begin{aligned}H=&\int d^3x\Psi_R^\dagger(\boldsymbol{x})\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}\Psi_R(\boldsymbol{x})=\sum_{k_y,k_z}[k_z\theta(k_z)a_0^\dagger(k_y,k_z)a_0(k_y,k_z)\\ &+(-k_z)\theta(-k_z)b_0^\dagger(k_y,k_z)b_0(k_y,k_z)]+\sum_{n,k_y,k_z}E_n(k_z)[a_n^\dagger(k_y,k_z)\\ &\times a_n(k_y,k_z)+b_n^\dagger(k_y,k_z)b_n(k_y,k_z)],\end{aligned}\tag{32}$$

$$\begin{aligned}N=&\int d^3x\Psi_R^\dagger(\boldsymbol{x})\Psi_R(\boldsymbol{x})\\ =&\sum_{k_y,k_z}[\theta(k_z)a_0^\dagger(k_y,k_z)a_0(k_y,k_z)-\theta(-k_z)b_0^\dagger(k_y,k_z)b_0(k_y,k_z)]\\ &\times\sum_{n,k_y,k_z}[a_n^\dagger(k_y,k_z)a_n(k_y,k_z)-b_n^\dagger(k_y,k_z)b_n(k_y,k_z)],\end{aligned}\tag{33}$$

where we omitted the infinite vacuum term. This can be renormalized in the physics calculation and does not affect our result on the CME coefficient. Evidently, $\theta(k_z)a_0^\dagger(k_y,k_z)a_0(k_y,k_z)$ and $a_n^\dagger(k_y,k_z)a_n(k_y,k_z)$ are the occupied number operators of particles for different Landau levels, and $\theta(-k_z)b_0^\dagger(k_y,k_z)b_0(k_y,k_z)$ and $b_n^\dagger(k_y,k_z)b_n(k_y,k_z)$ are occupied number operators of antiparticles for different Landau levels. Notably, without introduction of the two theta functions $\theta(\pm k_z)$ in front of $a_0(k_y,k_z)$ and $b_0^\dagger(k_y,k_z)$ in the decomposition of $\Psi_R(\boldsymbol{x})$, the second quantization procedure could not be performed successfully. This is different from the massive case [21], where the authors determined the Landau levels and corresponding wavefunctions for the massive Dirac equation in a uniform magnetic field with chemical potential μ and chiral chemical potential μ_5. The wave functions for the massive case are in a four-component Dirac form, and the θ function is not needed for the second quantization.

5 Chiral magnetic effect

Supposing that the system of the righthand ferimons within an external uniform magnetic field $\boldsymbol{B}=B\mathbf{e}_z$ is in equilibrium with a reservior with temperature T and chemical potential μ_R, then the density operator $\hat{\rho}$ for this righthand fermion system is

$$\hat{\rho}=\frac{1}{Z}\mathrm{e}^{-\beta(H-\mu_RN)},\tag{34}$$

where $\beta=1/T$ is the inverse temperature, and Z is the grand canonical partition function,

$$Z=\mathrm{Tr}\,\mathrm{e}^{-\beta(H-\mu_RN)}.\tag{35}$$

The expectation value of an operator $\hat{F}$ in the equilibrium state can be calculated as

$$\langle:\hat{F}:\rangle=\mathrm{Tr}(\hat{\rho}\hat{F}).\tag{36}$$

In the Appendix C, we calculated the expectation values of occupied number operators as

$$\begin{aligned}\langle:\theta(k_z)a_0^\dagger(k_y,k_z)a_0(k_y,k_z):\rangle&=\frac{\theta(k_z)}{\mathrm{e}^{\beta(k_z-\mu_R)}+1}\\ \langle:\theta(-k_z)b_0^\dagger(k_y,k_z)b_0(k_y,k_z):\rangle&=\frac{\theta(-k_z)}{\mathrm{e}^{\beta(-k_z+\mu_R)}+1}\\ \langle:a_n^\dagger(k_y,k_z)a_n(k_y,k_z):\rangle&=\frac{1}{\mathrm{e}^{\beta[E_n(k_z)-\mu_R]}+1}\\ \langle:b_n^\dagger(k_y,k_z)b_n(k_y,k_z):\rangle&=\frac{1}{\mathrm{e}^{\beta[E_n(k_z)+\mu_R]}+1}.\end{aligned}\tag{37}$$

The macroscopic electric current for the righthand fermion system is

$$\boldsymbol{J}_R=\Big\langle:\Psi_R^\dagger(\boldsymbol{x})\boldsymbol{\sigma}\Psi_R(\boldsymbol{x}):\Big\rangle.\tag{38}$$

According to the rotational invariance of this system along the z-axis, $J_R^x=J_R^y=0$. In the following, we calculate J_R^z. Using Eq. (30), we see that

$$\begin{aligned}J_R^z=&\langle:\Psi_R^\dagger(\boldsymbol{x})\sigma^3\Psi_R(\boldsymbol{x}):\rangle\\ =&\sum_{k_y,k_z}\Big(\langle:\theta(k_z)a_0^\dagger(k_y,k_z)a_0(k_y,k_z):\rangle\\ &+\langle:\theta(-k_z)b_0(k_y,k_z)b_0^\dagger(k_y,k_z):\rangle\Big)\\ &\times\psi_{R0}^\dagger(k_y,k_z;\boldsymbol{x})\sigma^3\psi_{R0}(k_y,k_z;\boldsymbol{x})\\ &+\sum_{n,k_y,k_z}\langle:a_n^\dagger(k_y,k_z)a_n(k_y,k_z):\rangle\psi_{Rn+}^\dagger(k_y,k_z;\boldsymbol{x})\sigma^3\psi_{Rn+}(k_y,k_z;\boldsymbol{x})\\ &+\sum_{n,k_y,k_z}\langle:b_n(k_y,k_z)b_n^\dagger(k_y,k_z):\rangle\psi_{Rn-}^\dagger(k_y,k_z;\boldsymbol{x})\sigma^3\psi_{Rn-}(k_y,k_z;\boldsymbol{x})\\ =&\sum_{k_y,k_z}\Big(\frac{\theta(k_z)}{\mathrm{e}^{\beta(k_z-\mu_R)}+1}-\frac{\theta(-k_z)}{\mathrm{e}^{\beta(-k_z+\mu_R)}+1}\Big)\psi_{R0}^\dagger(k_y,k_z;\boldsymbol{x})\sigma^3\psi_{R0}(k_y,k_z;\boldsymbol{x})\\ &+\sum_{n,k_y,k_z}\frac{1}{\mathrm{e}^{\beta[E_n(k_z)-\mu_R]}+1}\psi_{Rn+}^\dagger(k_y,k_z;\boldsymbol{x})\sigma^3\psi_{Rn+}(k_y,k_z;\boldsymbol{x})\\ &-\sum_{n,k_y,k_z}\frac{1}{\mathrm{e}^{\beta[E_n(k_z)+\mu_R]}+1}\psi_{Rn-}^\dagger(k_y,k_z;\boldsymbol{x})\sigma^3\psi_{Rn-}(k_y,k_z;\boldsymbol{x}).\end{aligned}\tag{39}$$

First, we sum over k_y for $\psi_{R0}^\dagger(k_y,k_z;\boldsymbol{x})\sigma^3\psi_{R0}(k_y,k_z;\boldsymbol{x})$ and

$\psi^{\dagger}_{Rn\lambda}(k_y,k_z;\boldsymbol{x})\sigma^3\psi_{Rn\lambda}(k_y,k_z;\boldsymbol{x})$ in Eq. (39), the results are

$$\begin{aligned}&\sum_{k_y}\psi^{\dagger}_{R0}(k_y,k_z;\boldsymbol{x})\sigma^3\psi_{R0}(k_y,k_z;\boldsymbol{x})\\&=\frac{1}{L^2}\sum_{k_y}\left(\ \varphi_0(\xi),\quad 0\ \right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\varphi_0(\xi)\\0\end{pmatrix}\\&=\frac{1}{2\pi L}\int_{-\infty}^{\infty}\mathrm{d}k_y[\varphi_0(\sqrt{eB}x-k_y/\sqrt{eB})]^2\\&=\frac{eB}{2\pi L},\end{aligned}\tag{40}$$

and

$$\begin{aligned}&\sum_{k_y}\psi^{\dagger}_{Rn\lambda}(k_y,k_z;\boldsymbol{x})\sigma^3\psi_{Rn\lambda}(k_y,k_z;\boldsymbol{x})\\&=\frac{1}{2\pi L}\int_{-\infty}^{\infty}\mathrm{d}k_y c^2_{n\lambda}(k_z)\Big([\varphi_n(\xi)]^2-\frac{2neB[\varphi_{n-1}(\xi)]^2}{[k_z+\lambda E_n(k_z)]^2}\Big)\\&=\frac{eB}{2\pi L}c^2_{n\lambda}(k_z)\Big(1-\frac{2neB}{[k_z+\lambda E_n(k_z)]^2}\Big)\\&=\frac{eB}{2\pi L}c^2_{n\lambda}(k_z)[2-c^{-2}_{n\lambda}(k_z)]=\frac{eB}{2\pi L}\cdot\frac{\lambda k_z}{E_n(k_z)}.\end{aligned}\tag{41}$$

Second, we sum over k_z in the third equal sign of Eq. (39),

$$\begin{aligned}J^z_R=&\sum_{k_z}\Big(\frac{\theta(k_z)}{\mathrm{e}^{\beta(k_z-\mu_R)}+1}-\frac{\theta(-k_z)}{\mathrm{e}^{\beta(-k_z+\mu_R)}+1}\Big)\frac{eB}{2\pi L}\\&+\sum_{n,k_z}\Big(\frac{1}{\mathrm{e}^{\beta[E_n(k_z)-\mu_R]}+1}+\frac{1}{\mathrm{e}^{\beta[E_n(k_z)+\mu_R]}+1}\Big)\frac{eB}{2\pi L}\cdot\frac{k_z}{E_n(k_z)}\\=&\frac{eB}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_z\Big(\frac{\theta(k_z)}{\mathrm{e}^{\beta(k_z-\mu_R)}+1}-\frac{\theta(-k_z)}{\mathrm{e}^{\beta(-k_z+\mu_R)}+1}\Big)+0\\=&\frac{eB}{4\pi^2}\mu_R.\end{aligned}\tag{42}$$

Combining Eq. (42) and $J^x_R=J^y_R=0$ yields

$$\boldsymbol{J}_R=\frac{e\mu_R}{4\pi^2}\boldsymbol{B}.\tag{43}$$

From the calculation above, we see that only the lowest Landau level contributes to Eq. (43). A similar calculation for the lefthand fermion system shows that

$$\boldsymbol{J}_L=-\frac{e\mu_L}{4\pi^2}\boldsymbol{B}.\tag{44}$$

We can also obtain Eq. (44) from Eq. (43) under space inversion: $\boldsymbol{J}_R\to-\boldsymbol{J}_L$, $\mu_R\to\mu_L$, $\boldsymbol{B}\to\boldsymbol{B}$. If the system is composed of righthand and lefthand fermions, then the vector current $\boldsymbol{J}_V$ and axial current $\boldsymbol{J}_A$ are

$$\boldsymbol{J}_V=\boldsymbol{J}_R+\boldsymbol{J}_L=\frac{e\mu_5}{2\pi^2}\boldsymbol{B},\tag{45}$$

$$\boldsymbol{J}_A=\boldsymbol{J}_R-\boldsymbol{J}_L=\frac{e\mu}{2\pi^2}\boldsymbol{B},\tag{46}$$

where $\mu_5=(\mu_R-\mu_L)/2$ is the chiral chemical potential and $\mu=(\mu_R+\mu_L)/2$. Thus far, we derived the CME in the chiral fermion system by determining Landau levels. We emphasize that Eqs. (45), (46) are valid for any strength of magnetic field, in contrast to the weak magnetic field approximation through Wigner function approach in Sec. 2.

6 Physical picture of lowest Landau level

We discuss the physical picture of the lowest Landau level. The wavefunction and energy of the lowest Landau level ($n=0$) for the righthand fermion field is

$$\psi_{R0}(k_y,k_z;\boldsymbol{x})=\begin{pmatrix}\varphi_0\\0\end{pmatrix}\frac{1}{L}\mathrm{e}^{\mathrm{i}(yk_y+zk_z)},\quad E=k_z.\tag{47}$$

Setting $k_y=0$ in Eq. (47), we calculate the Hamiltonian, particle number, z-component of momentum, and z-component of the spin angular momentum of the righthand fermion system for the lowest Landau level as follows,

$$\begin{aligned}H=&\sum_{k_z}[k_z\theta(k_z)a^{\dagger}_0(0,k_z)a_0(0,k_z)\\&+(-k_z)\theta(-k_z)b^{\dagger}_0(0,k_z)b_0(0,k_z)],\\N=&\sum_{k_z}[\theta(k_z)a^{\dagger}_0(0,k_z)a_0(0,k_z)\\&+(-1)\theta(-k_z)b^{\dagger}_0(0,k_z)b_0(0,k_z)],\\P_z=&\sum_{k_z}[k_z\theta(k_z)a^{\dagger}_0(0,k_z)a_0(0,k_z)\\&+(-k_z)\theta(-k_z)b^{\dagger}_0(0,k_z)b_0(0,k_z)],\\S_z=&\sum_{k_z}\Big[\frac{1}{2}\theta(k_z)a^{\dagger}_0(0,k_z)a_0(0,k_z)\\&+\Big(-\frac{1}{2}\Big)\theta(-k_z)b^{\dagger}_0(0,k_z)b_0(0,k_z)\Big],\end{aligned}\tag{48}$$

where the definitions of P_z and S_z are

$$\begin{aligned}P_z&=-\mathrm{i}\int\mathrm{d}^3x\Psi^{\dagger}_R(\boldsymbol{x})\frac{\partial}{\partial z}\Psi_R(\boldsymbol{x}),\\S_z&=\frac{1}{2}\int\mathrm{d}^3x\Psi^{\dagger}_R(\boldsymbol{x})\sigma^3\Psi_R(\boldsymbol{x}).\end{aligned}\tag{49}$$

Thus, we have a picture for the lowest Landau level: The operator $\theta(k_z)a^{\dagger}_0(0,k_z)$ produces a particle with charge e, energy $k_z>0$, z-component of momentum $k_z>0$, and z-component of spin angular momentum $+\frac{1}{2}$ (helicity $h=+1$); The operator $\theta(-k_z)b^{\dagger}_0(0,k_z)$ produces a particle with charge $-e$, energy $-k_z>0$, z-component of momentum $-k_z>0$, and z-component of spin angular momentum $-\frac{1}{2}$ (helicity $h=-1$). This picture indicates that all righthand fermions/antifermions move along the (+z)-axis, with righthand fermions spinning along the (+z)-axis and righthand antifermions spinning along the $-z$-axis. If $\mu_R>0$, which indicates that there are more righthand fermions than righthand anti-fermions, a net electric current will move along the (+z)-axis, which is referred to as the CME for the righthand fermion system.

The analogous analysis can be applied to lefthand fermions. The picture of the lowest Landau level for a lefthand fermion is: All lefthand fermions/antifermions move along the $(-z)$-axis, with left fermions spinning along the $(+z)$-axis and lefthand antifermions spinning along the $(-z)$-axis. If $\mu_L > 0$, which indicates that there is more lefthand fermions than lefthand anti-fermions, a net electric current will move along the $(-z)$-axis, which is referred to as the CME for the lefthand fermion system.

Because the total electric current $\boldsymbol{J}_V$ of the chiral fermion system is the summation of the electric current $\boldsymbol{J}_R$ of the righthand fermion system and the electric current $\boldsymbol{J}_L$ of the lefthand fermion system, whether $\boldsymbol{J}_V$ moves along the $(+z)$-axis will only depend on the sign of $(\mu_R-\mu_L)$. Thus, the CME for the chiral fermion system is described microscopically.

7 Summary

CME arises from the lowest Landau level both for the massive Dirac fermion system and the chiral fermion system. For the massive case, the physical picture of how the lowest Landau level contributes to CME is not extensively clear. When the Landau levels are determined for the chiral fermion system in a uniform magnetic field, by performing the second quantization for the chiral fermion field, expanding the field operator by an eigenfunction of Landau levels, and calculating the ensemble average of the vector current operator, we natrually obtain the equation for the CME. Notably, no approximations were made for the strength of magnetic field in the calculation. Further, we introduced two theta functions $\theta(\pm k_z)$ in front of $a_0(k_y,k_z)$ and $b_0^\dagger(k_y,k_z)$ in the decomposition of $\Psi_R(\boldsymbol{x})$, which is crucial for the successful performance of the subsequent procedure of second quantization. When we carefully analyze the lowest Landau level, we find that all righthand (chirality is +1) fermions move along the positive z-direction, and all lefthand (chirality is -1) fermions move along the negative z-direction. Thus, the CME is described microscopically within this picture of the lowest Landau level.

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Appendix A: Vlasov equation and off mass-shell equation

The quadratic form for the equation of motion of the Wigner function $\mathcal{W}(x,p)$ is

$$\left(K^2-\frac{\mathrm{i}}{2}\sigma^{\mu\nu}[K_\mu,K_\nu]\right)\mathcal{W}=0. \tag{A1}$$

Using $K^2=p^2-\frac{1}{4}\nabla^2+\mathrm{i}p\cdot\nabla$ and $[K_\mu,K_\nu]=-\mathrm{i}eF_{\mu\nu}$, Eq. (A1) becomes

$$\left(p^2-\frac{1}{4}\nabla^2+\mathrm{i}p\cdot\nabla-\frac{1}{2}eF_{\mu\nu}\sigma^{\mu\nu}\right)\mathcal{W}=0. \tag{A2}$$

$\mathcal{W}$ and $\sigma^{\mu\nu}$ satisfy $\mathcal{W}=\gamma^0\mathcal{W}^\dagger\gamma^0$ and $\sigma^{\mu\nu}=\gamma^0\sigma^{\mu\nu\dagger}\gamma^0$. Employing the Hermi conjugation and subsequently multiplying γ^0 to both sides of Eq. (A2) yields

$$\left(p^2-\frac{1}{4}\nabla^2-\mathrm{i}p\cdot\nabla\right)\mathcal{W}-\frac{1}{2}eF_{\mu\nu}\mathcal{W}\sigma^{\mu\nu}=0. \tag{A3}$$

Eq. (A2) minus Eq. (A3) yields the Vlasov equation for $\mathcal{W}$,

$$\mathrm{i}p\cdot\nabla\mathcal{W}-\frac{1}{4}eF_{\mu\nu}[\sigma^{\mu\nu},\mathcal{W}]=0. \tag{A4}$$

Eq. (A2) added to Eq. (A3) yields the off mass-shell equation for $\mathcal{W}$,

$$\left(p^2-\frac{1}{4}\nabla^2\right)\mathcal{W}-\frac{1}{4}eF_{\mu\nu}\{\sigma^{\mu\nu},\mathcal{W}\}=0. \tag{A5}$$

To calculate $[\sigma^{\mu\nu},W]$ and $\{\sigma^{\mu\nu},W\}$ in Eqs. (A4) (A5), we list the following useful identities,

$$\begin{aligned}
&[\sigma^{\mu\nu},1]=0,\\
&[\sigma^{\mu\nu},\mathrm{i}\gamma^5]=0,\\
&[\sigma^{\mu\nu},\gamma^\rho]=-2\mathrm{i}g^{\rho[\mu}\gamma^{\nu]},\\
&[\sigma^{\mu\nu},\gamma^5\gamma^\rho]=-2\mathrm{i}g^{\rho[\mu}\gamma^5\gamma^{\nu]},\\
&[\sigma^{\mu\nu},\sigma^{\rho\sigma}]=2\mathrm{i}g^{\mu[\rho}\sigma^{\sigma]\nu}-2\mathrm{i}g^{\nu[\rho}\sigma^{\sigma]\mu},
\end{aligned} \tag{A6}$$

$$\begin{aligned}
&\{\sigma^{\mu\nu},1\}=2\sigma^{\mu\nu},\\
&\{\sigma^{\mu\nu},\mathrm{i}\gamma^5\}=-\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma},\\
&\{\sigma^{\mu\nu},\gamma^\rho\}=2\epsilon^{\mu\nu\rho\sigma}\gamma^5\gamma_\sigma,\\
&\{\sigma^{\mu\nu},\gamma^5\gamma^\rho\}=2\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma,\\
&\{\sigma^{\mu\nu},\sigma^{\rho\sigma}\}=2g^{\mu[\rho}g^{\sigma]\nu}+2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\gamma^5.
\end{aligned} \tag{A7}$$

Then, all matrices appearing in Eqs. (A4) (A5) are the 16 independent Γ-matrices, whose coefficients must be zero. These coefficient equations are the Vlasov equations and the off mass-shell equations for $\mathcal{F},\mathcal{P},\mathcal{V}_\mu,\mathcal{A}_\mu,\mathcal{S}_{\mu\nu}$. The Vlasov equations are

$$\begin{aligned}
&p\cdot\nabla\mathcal{F}=0,\\
&p\cdot\nabla\mathcal{P}=0,\\
&p\cdot\nabla\mathcal{V}_\mu=eF_{\mu\nu}\mathcal{V}^\nu,\\
&p\cdot\nabla\mathcal{A}_\mu=eF_{\mu\nu}\mathcal{A}^\nu,\\
&p\cdot\nabla\mathcal{Q}_{\mu\nu}=eF^\rho_{\ [\mu}\mathcal{Q}_{\nu]\rho},
\end{aligned} \tag{A8}$$

and the off mass-shell equations are

$$\begin{aligned}
&\left(p^2-\frac{1}{4}\nabla^2\right)\mathcal{F}=\frac{1}{2}eF_{\mu\nu}\mathcal{Q}^{\mu\nu},\\
&\left(p^2-\frac{1}{4}\nabla^2\right)\mathcal{P}=\frac{1}{2}e\tilde{F}_{\mu\nu}\mathcal{Q}^{\mu\nu},\\
&\left(p^2-\frac{1}{4}\nabla^2\right)\mathcal{V}_\mu=-e\tilde{F}_{\mu\nu}\mathcal{A}^\nu,\\
&\left(p^2-\frac{1}{4}\nabla^2\right)\mathcal{A}_\mu=-e\tilde{F}_{\mu\nu}\mathcal{V}^\nu,\\
&\left(p^2-\frac{1}{4}\nabla^2\right)\mathcal{Q}_{\mu\nu}=e(F_{\mu\nu}\mathcal{F}-\tilde{F}_{\mu\nu}\mathcal{P}),
\end{aligned} \tag{A9}$$

where $\tilde{F}_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.

Appendix B: Landau levels for righthand fermion field

We solve following eigenvalue equation in detail,

$$\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}\psi_R(\boldsymbol{x}) = E\psi_R(\boldsymbol{x}), \tag{B1}$$

with $\boldsymbol{D} = (-\partial_x, -\partial_y + \mathrm{i}eBx, -\partial_z)$. Because the operator $\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}$ is commutative with $\hat{p}_y = -\mathrm{i}\partial_y, \hat{p}_z = -i\partial_z$, we can choose ψ_R as the commom eigenstate of $\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}$, $\hat{p}_y$ and $\hat{p}_z$ as follows

$$\psi_R(x,y,z) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \frac{1}{L}\mathrm{e}^{\mathrm{i}(yk_y+zk_z)}, \tag{B2}$$

where L is the length of the system in y- and z- directions. The explicit form of $\boldsymbol{\sigma}\cdot\boldsymbol{D}$ is

$$\boldsymbol{\sigma}\cdot\boldsymbol{D} = \begin{pmatrix} -\partial_z & -\partial_x + \mathrm{i}\partial_y + eBx \\ -\partial_x - \mathrm{i}\partial_y - eBx & \partial_z \end{pmatrix}. \tag{B3}$$

Inserting Eq. (B2) (B3) into Eq. (B1), we obtain the group of differential equations for $\phi_1(x)$ and $\phi_2(x)$ as

$$\mathrm{i}(k_z - E)\phi_1 + (\partial_x + k_y - eBx)\phi_2 = 0, \tag{B4}$$

$$(\partial_x - k_y + eBx)\phi_1 - \mathrm{i}(k_z + E)\phi_2 = 0. \tag{B5}$$

From Eq. (B5), we can express ϕ_2 by ϕ_1, then Eq. (B4) becomes

$$\partial_x^2\phi_1 + \Big(E^2 + eB - k_z^2 - e^2B^2\Big(x - \frac{k_y}{eB}\Big)^2\Big)\phi_1 = 0, \tag{B6}$$

which is a typical harmonic oscillator equation. Defining a dimensionless variable $\xi = \sqrt{eB}(x - k_y/eB)$, and $\phi_1(x) = \varphi(\xi)$, then (B6) becomes

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2} + \Big(\frac{E^2 - k_z^2}{eB} + 1 - \xi^2\Big)\varphi = 0. \tag{B7}$$

With the boundary condition $\varphi \to 0$ as $\xi \to \pm\infty$, we must set

$$\frac{E^2 - k_z^2}{eB} + 1 = 2n + 1, \tag{B8}$$

with $n = 0,1,2,\cdots$. Thus, energy E can only assume the following discrete values,

$$E = \pm E_n(k_z) \equiv \pm\sqrt{2neB + k_z^2}, \tag{B9}$$

where we define $E_n(k_z) = \sqrt{2neB + k_z^2}$. The corresponding normalized solution for equation (B6) is

$$\phi_1(x) = \varphi_n(\xi) = N_n\mathrm{e}^{-\xi^2/2}H_n(\xi), \tag{B10}$$

where $N_n = (eB)^{\frac{1}{4}}\pi^{-\frac{1}{4}}(2^n n!)^{-\frac{1}{2}}$, and $H_n(\xi) = (-1)^n\mathrm{e}^{\xi^2}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}\mathrm{e}^{-\xi^2}$. For energy $E = \lambda E_n(k_z)$ $(\lambda = \pm 1)$, we can obtain ϕ_2 as

$$\phi_2(x) = \frac{\sqrt{eB}(\partial_\xi + \xi)\varphi_n(\xi)}{\mathrm{i}(k_z + E)} = \frac{\mathrm{i}[k_z - \lambda E_n(k_z)]}{\sqrt{2neB}}\varphi_{n-1}(\xi), \tag{B11}$$

where we used $(\partial_\xi + \xi)\varphi_n(\xi) = \sqrt{2n}\varphi_{n-1}(\xi)$. Defining $F_{n\lambda}(k_z) = [k_z - \lambda E_n(k_z)]/\sqrt{2neB}$, the eigenfunction with eigenvalue $E = \lambda E_n(k_z)$ becomes

$$\psi_{Rn\lambda}(k_y,k_z;\boldsymbol{x}) = \begin{pmatrix} \varphi_n(\xi) \\ \mathrm{i}F_{n\lambda}(k_z)\varphi_{n-1}(\xi) \end{pmatrix} \frac{1}{L}\mathrm{e}^{\mathrm{i}(yk_y+zk_z)}. \tag{B12}$$

This is very subtle when $n = 0$ in Eq. (B11). When $n = 0, E = k_z$, the first equal sign of Eq. (B11) indicates $\phi_2 = 0$ due to $(\partial_\xi + \xi)\varphi_0(\xi) = 0$. Then, the corresponding eigenfunction becomes

$$\psi_{R0}(k_y,k_z;\boldsymbol{x}) = \begin{pmatrix} \varphi_0(\xi) \\ 0 \end{pmatrix} \frac{1}{L}\mathrm{e}^{\mathrm{i}(yk_y+zk_z)}. \tag{B13}$$

When $n = 0, E = -k_z$, the denominator of the first equal sign of Eq. (B11) becomes zero, in which case we must directly deal with Eqs. (B4) (B5). In this case, Eqs. (B4) (B5) become

$$2\mathrm{i}k_z\phi_1 + (\partial_x + k_y - eBx)\phi_2 = 0, \tag{B14}$$

$$(\partial_x - k_y + eBx)\phi_1 = 0. \tag{B15}$$

Eq. (B15) gives $\phi_1(x) \sim \exp[-\frac{1}{2}eBx^2 + xk_y]$, then Eq. (B14) becomes

$$2\mathrm{i}k_z\exp\Big(-\frac{1}{2}eBx^2 + xk_y\Big) + (\partial_x + k_y - eBx)\phi_2 = 0. \tag{B16}$$

When $x \to \pm\infty$, Eq. (B16) tends to

$$(\partial_x - eBx)\phi_2 = 0, \tag{B17}$$

whose solution is $\phi_2 \sim \exp(\frac{1}{2}eBx^2)$, which is divergent as $x \to \pm\infty$. Thus, there is no physical solution when $n = 0, E = -k_z$.

Thus far, we obtain the eigenfunctions and eigenvalues of the Hamiltonian of the righthand fermion field as follows:

For $n = 0$ Landau level, the wavefunction with energy $E = k_z$ is

$$\psi_{R0}(k_y,k_z;\boldsymbol{x}) = \begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix} \frac{1}{L}\mathrm{e}^{\mathrm{i}(yk_y+zk_z)}. \tag{B18}$$

For $n > 0$ Landau level, the wavefunction with energy $E = \lambda E_n(k_z)$ is

$$\psi_{Rn\lambda}(k_y,k_z;\boldsymbol{x}) = c_{n\lambda}\begin{pmatrix} \varphi_n \\ \mathrm{i}F_{n\lambda}\varphi_{n-1} \end{pmatrix} \frac{1}{L}\mathrm{e}^{\mathrm{i}(yk_y+zk_z)}, \tag{B19}$$

where $\lambda = \pm 1$, $E_n(k_z) = \sqrt{2neB + k_z^2}$, $F_{n\lambda}(k_z) = [k_z - \lambda E_n(k_z)]/\sqrt{2neB}$, $|c_{n\lambda}|^2 = 1/(1 + F_{n\lambda}^2)$.

Appendix C: Expectation value of occupied number operators

We calculate the expectation values of particle number operators. From the expression of the Hamiltonian and the total particle number operator in Eqs. (32, 33), we easily obtain following commutative relations,

$$\begin{aligned} [N, \theta(k_z)a_0^\dagger(k_y,k_z)] &= \theta(k_z)a_0^\dagger(k_y,k_z) \\ [N, \theta(-k_z)b_0^\dagger(k_y,k_z)] &= -\theta(-k_z)b_0^\dagger(k_y,k_z) \\ [N, a_n^\dagger(k_y,k_z)] &= a_n^\dagger(k_y,k_z) \\ [N, b_n^\dagger(k_y,k_z)] &= -b_n^\dagger(k_y,k_z), \end{aligned} \tag{C1}$$

$$\begin{aligned} [H, \theta(k_z)a_0^\dagger(k_y,k_z)] &= k_z\theta(k_z)a_0^\dagger(k_y,k_z) \\ [H, \theta(-k_z)b_0^\dagger(k_y,k_z)] &= (-k_z)\theta(-k_z)b_0^\dagger(k_y,k_z) \\ [H, a_n^\dagger(k_y,k_z)] &= E_n(k_z)a_n^\dagger(k_y,k_z) \\ [H, b_n^\dagger(k_y,k_z)] &= E_n(k_z)b_n^\dagger(k_y,k_z), \end{aligned} \tag{C2}$$

where we employ $[AB,C] = A\{B,C\} - \{A,C\}B$. Defining

$$\theta(k_z)a_0^\dagger(k_y,k_z;\beta) = \mathrm{e}^{-\beta(H-\mu_R N)}\theta(k_z)a_0^\dagger(k_y,k_z)\mathrm{e}^{\beta(H-\mu_R N)}$$

$$\theta(-k_z)b_0^\dagger(k_y,k_z;\beta) = \mathrm{e}^{-\beta(H-\mu_R N)}\theta(-k_z)b_0^\dagger(k_y,k_z)\mathrm{e}^{\beta(H-\mu_R N)}$$

$$a_n^\dagger(k_y,k_z;\beta) = \mathrm{e}^{-\beta(H-\mu_R N)} a_n^\dagger(k_y,k_z)\mathrm{e}^{\beta(H-\mu_R N)}$$
$$b_n^\dagger(k_y,k_z;\beta) = \mathrm{e}^{-\beta(H-\mu_R N)} b_n^\dagger(k_y,k_z)\mathrm{e}^{\beta(H-\mu_R N)}. \tag{C3}$$

For $\theta(k_z)a_0^\dagger(k_y,k_z;\beta)$, we obtain

$$\begin{aligned}\frac{\partial}{\partial\beta}[\theta(k_z)a_0^\dagger(k_y,k_z;\beta)] &= -[H-\mu_R N,\theta(k_z)a_0^\dagger(k_y,k_z;\beta)]\\ &= -\mathrm{e}^{-\beta(H-\mu_R N)}[H-\mu_R N,\theta(k_z)a_0^\dagger(k_y,k_z)]\mathrm{e}^{\beta(H-\mu_R N)}\\ &= -\mathrm{e}^{-\beta(H-\mu_R N)}[(k_z-\mu_R)\theta(k_z)a_0^\dagger(k_y,k_z)]\mathrm{e}^{\beta(H-\mu_R N)}\\ &= -(k_z-\mu_R)[\theta(k_z)a_0^\dagger(k_y,k_z;\beta)],\end{aligned} \tag{C4}$$

with the boundary condition $\theta(k_z)a_0^\dagger(k_y,k_z;0) = \theta(k_z)a_0^\dagger(k_y,k_z)$, which implies

$$\theta(k_z)a_0^\dagger(k_y,k_z;\beta) = \theta(k_z)a_0^\dagger(k_y,k_z)\mathrm{e}^{-\beta(k_z-\mu_R)}. \tag{C5}$$

Similarly, we obtain

$$\theta(-k_z)b_0^\dagger(k_y,k_z;\beta) = \theta(-k_z)b_0^\dagger(k_y,k_z)\mathrm{e}^{-\beta(-k_z+\mu_R)}$$
$$a_n^\dagger(k_y,k_z;\beta) = a_n^\dagger(k_y,k_z)\mathrm{e}^{-\beta[E_n(k_z)-\mu_R]}$$
$$b_n^\dagger(k_y,k_z;\beta) = b_n^\dagger(k_y,k_z)\mathrm{e}^{-\beta[E_n(k_z)+\mu_R]}. \tag{C6}$$

We calculate the expectation value of $\langle:\theta(k_z)a_0^\dagger(k_y,k_z)a_0(k_y,k_z):\rangle$. We see that

$$\begin{aligned}&\langle:\theta(k_z)a_0^\dagger(k_y,k_z)a_0(k_y,k_z):\rangle\\ &=\mathrm{Tr}[\rho\theta(k_z)a_0^\dagger(k_y,k_z)a_0(k_y,k_z)]\\ &=\frac{1}{Z}\mathrm{Tr}\left(\theta(k_z)a_0^\dagger(k_y,k_z;\beta)\mathrm{e}^{-\beta(H-\mu_R N)}a_0(k_y,k_z)\right)\\ &=\frac{1}{Z}\mathrm{Tr}\left(\theta(k_z)a_0(k_y,k_z)a_0^\dagger(k_y,k_z;\beta)\mathrm{e}^{-\beta(H-\mu_R N)}\right)\\ &=\langle:\theta(k_z)a_0(k_y,k_z)a_0^\dagger(k_y,k_z;\beta):\rangle\\ &=\langle:\theta(k_z)a_0(k_y,k_z)a_0^\dagger(k_y,k_z):\rangle\mathrm{e}^{-\beta(k_z-\mu_R)}\\ &=\theta(k_z)\mathrm{e}^{-\beta(k_z-\mu_R)}-\langle:\theta(k_z)a_0^\dagger(k_y,k_z)a_0(k_y,k_z):\rangle\mathrm{e}^{-\beta(k_z-\mu_R)},\end{aligned} \tag{C7}$$

thus, we obtain

$$\langle\theta(k_z)a_0^\dagger(k_y,k_z)a_0(k_y,k_z)\rangle = \frac{\theta(k_z)}{\mathrm{e}^{\beta(k_z-\mu_R)}+1}. \tag{C8}$$

Similar calculations obtain

$$\langle\theta(-k_z)b_0^\dagger(k_y,k_z)b_0(k_y,k_z)\rangle = \frac{\theta(-k_z)}{\mathrm{e}^{\beta(-k_z+\mu_R)}+1}$$
$$\langle a_n^\dagger(k_y,k_z)a_n(k_y,k_z)\rangle = \frac{1}{\mathrm{e}^{\beta[E_n(k_z)-\mu_R]}+1}$$
$$\langle b_n^\dagger(k_y,k_z)b_n(k_y,k_z)\rangle = \frac{1}{\mathrm{e}^{\beta[E_n(k_z)+\mu_R]}+1}. \tag{C9}$$

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