

Matrix elements in the $SU(1,1) \otimes SO(6)$ limit of the proton-neutron symplectic model

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Abstract: The matrix elements along the reduction chain $Sp(12,R) \supset SU(1,1) \otimes SO(6) \supset U(1) \otimes SU_{pn}(3) \otimes SO(2) \supset SO(3)$ of the proton-neutron symplectic model (PNSM) are considered. Closed analytical expressions are obtained for the matrix elements of the basic building blocks of the PNSM and the $Sp(12,R)$ symplectic generators, allowing the computation of matrix elements of other physical operators as well. The computational technique developed in the present study generally provides us with the required algebraic tool for performing realistic symplectic-based shell-model calculations of nuclear collective excitations. Utilizing two simple examples, we illustrate the application of the theory.

Keywords: matrix elements, proton-neutron symplectic model, $SU(1,1) \otimes SO(6)$ limit, $Sp(12, R)$ dynamical algebra

DOI: 10.1088/1674-1137/ac1c67

I. INTRODUCTION

A fully microscopic proton-neutron symplectic model (PNSM) of nuclear collective motion with $Sp(12,R)$ dynamical algebra was introduced few years ago by considering the symplectic geometry and possible collective flows in the two-component many-particle nuclear system [1]. Its more general motion group $GL(6,R) \subset Sp(12,R)$ introduces the wider class of allowed classical motions, including both the in-phase and out-of-phase proton-neutron collective excitations; by utilizing them, the PNSM naturally generalizes the one-component $Sp(6,R)$ symplectic model [2, 3]. The collective states in the PNSM were initially classified by the basis states of the six-dimensional harmonic oscillator by considering the following dynamical symmetry reduction chain: $Sp(12,R) \supset U(6) \supset SU_p(3) \otimes SU_n(3) \supset SU(3) \supset SO(3)$. Using this chain, the PNSM was applied for the simultaneous description of the microscopic structure of the lowest ground, β , and γ bands in ^{166}Er [4], ^{152}Sm [5], ^{154}Sm [6], and ^{238}U [7]. The results for the microscopic structure of negative-parity states of the lowest $K^\pi = 0_1^-$ and $K^\pi = 1_1^-$ bands in ^{152}Sm , ^{154}Sm and ^{238}U were also reported [5, 8, 9], including the low-energy $B(E1)$ interband transition strengths between the states of the ground band and $K^\pi = 0_1^-$ band [5, 9] for these three nuclei. A significant achievement of the presented approach is the simultaneous

description of low-lying $B(E2)$ and $B(E1)$ transition strengths without introducing an effective charge.

The PNSM $Sp(12,R)$ dynamical algebra has many subalgebra chains, which can be divided into two types: the shell-model and the collective-model chains. The first type of reduction chains focuses on the many-fermion nature of the nuclear systems and relates them to the microscopic shell-model theory of nuclear structure, whereas the second type of subalgebraic chains of $Sp(12,R)$ reveals the dynamical content of the possible collective motions in atomic nuclei. The shell-model reduction chains also define different shell-model coupling schemes in which different collective Hamiltonians can be diagonalized. In the present paper, we consider the PNSM matrix elements with respect to the following reduction chain $Sp(12,R) \supset SU(1,1) \otimes SO(6) \supset U(1) \otimes SU_{pn}(3) \otimes SO(2) \supset SO(3)$, which was recently shown to correspond to a microscopic version [10] of the Bohr-Mottelson [11] collective model. A characteristic property of this dynamical symmetry limit is that the collective dynamics separates into radial and orbital parts. According to this, the Hilbert space of the nucleus in the PNSM can be represented as a direct sum of many-particle shell-model subspaces labeled by the $SO(6)$ seniority quantum number ν , which carry an irreducible representation (irrep) of a direct product group $SU(1,1) \otimes SO(6)$.

The general structure $SU(1,1) \otimes SO(d)$ appears in the

Received 30 June 2021; Accepted 11 August 2021; Published online 6 September 2021

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central-force problems in d -dimensional Euclidean space [12]. Then, the $SO(d)$ -invariant Hamiltonians can be diagonalized with the $SU(1,1)$ Lie algebra used as a spectrum generating algebra. The specific cases of $d = 3, 5$ were widely considered in the literature. For example, the algebraic solutions of the three-dimensional central-force problems were widely studied in terms of the algebra of the direct product group $SU(1,1) \otimes SO(3)$ (see, e.g., [13, 14]). This algebraic structure accounts for the Davidson potential, originally introduced to describe the rotation-vibrational excitations of diatomic molecules in three-dimensional space [15]. In nuclear physics, the five-dimensional Davidson potential that has an $SU(1,1) \otimes SO(5)$ algebraic structure was used by many authors [16–19] in relation with the rotations and vibrations of atomic nuclei. Recently, the Algebraic Collective Model (ACM), also based on the $SU(1,1) \otimes SO(5)$ algebraic structure, was formulated [20, 21] as an algebraic version of the Bohr-Mottelson [11] collective model which has the valuable property of diagonalizing [16] the γ -unstable Wilits-Jean-type Hamiltonians [22].

The $SU(1,1) \otimes SO(5)$ structure was also used in the boson quasi-spin formalism [23], related to the IBM [24], treating boson particles in $l = 2$ orbital level (i.e., considering d bosons). Thus, this algebraic structure considers the pairing between the d bosons in IBM and diagonalizes an $L = 0$ pairing interaction which is an $SO(5)$ invariant. Additionally, using the duality relationships [25] of $U(1) \subset SU(1,1)$ and $SU(d) \supset SO(d)$ structures, two-level pairing boson Hamiltonians expressed in terms of the boson quasi-spin $SU(1,1)$ algebra have been considered in [26].

The five-dimensional Davidson potential within the framework of the (one-component) $Sp(6, R)$ symplectic model was used to compute the detailed microscopic collective shell-model wave functions of the ground band in ^{166}Er in [27]. The six-dimensional Davidson potential, based on the $SU(1,1) \otimes SO(6)$ algebraic structure, was initially introduced in Ref. [28] in the context of the phenomenological Interacting Vector Boson Model [29], which is a trivial closed-shell irrotational-flow collective submodel of the PNSM of Bohr-Mottelson type.

The purpose of the present work is to develop a computational technique for the reduction chain $Sp(12, R) \supset SU(1,1) \otimes SO(6) \supset U(1) \otimes SU_{pn}(3) \otimes SO(2) \supset SO(3)$ of the PNSM, which will allow performing realistic shell-model calculations for determining the microscopic structure of heavy mass nuclei with various collective properties. This reduction chain provides an alternative basis for the shell-model diagonalization of an arbitrary collective Hamiltonian, which can be expressed as a polynomial in the many-particle position and momentum coordinates of the two-component proton-neutron nuclear systems, or is given in the algebraic form. The nuclear wave functions on this basis are presented as a direct product of $SU(1,1)$

radial wave functions with the $SO(6)$ spherical harmonics. Then, if one knows the matrix elements of the basic many-particle Jacobi position $x_{is}(\alpha)$ and momentum $p_{is}(\alpha)$ operators, the $SU(1,1) \otimes SO(6)$ algebraic structure allows us to obtain the matrix elements of any function of them (lying in the enveloping algebra of the $Sp(12, R)$ or $WSp(12, R)$ dynamical groups; cf. Ref. [9]) in the closed analytic form. In particular, the matrix elements of the basic building blocks of the PNSM and the $Sp(12, R)$ symplectic generators are obtained along the considered reduction chain. Using the latter, one can compute other more complicated matrix elements of different physical operators. For example, the matrix elements of the $SU(3)$ mixing interactions that couple various shell-model configurations within and between different oscillator shells are also given. This will extend the applicability of the PNSM in describing the collective properties in various nuclei. This is illustrated by considering a simple application of this theory to the microscopic description of the rotational states of the lowest positive-parity bands in ^{106}Ru and ^{158}Gd .

II. THE PROTON-NEUTRON SYMPLECTIC MODEL

A certain shell-model reduction chain of the PNSM is naturally explored if the following raising and lowering operators of harmonic oscillator quanta, i.e.,

$$\begin{aligned} b_{i\alpha,s}^\dagger &= \sqrt{\frac{m_\alpha \omega}{2\hbar}} \left(x_{is}(\alpha) - \frac{i}{m_\alpha \omega} p_{is}(\alpha) \right), \\ b_{i\alpha,s} &= \sqrt{\frac{m_\alpha \omega}{2\hbar}} \left(x_{is}(\alpha) + \frac{i}{m_\alpha \omega} p_{is}(\alpha) \right) \end{aligned} \quad (1)$$

are introduced. Then, a basis for the $Sp(12, R)$ algebra is given by the all bilinear combinations of these harmonic oscillator operators [30]:

$$F_{ij}(\alpha, \beta) = \sum_{s=1}^m b_{i\alpha,s}^\dagger b_{j\beta,s}^\dagger, \quad (2)$$

$$G_{ij}(\alpha, \beta) = \sum_{s=1}^m b_{i\alpha,s} b_{j\beta,s}, \quad (3)$$

$$A_{ij}(\alpha, \beta) = \frac{1}{2} \sum_{s=1}^m (b_{i\alpha,s}^\dagger b_{j\beta,s} + b_{j\beta,s} b_{i\alpha,s}^\dagger), \quad (4)$$

which are $O(m)$ -scalar operators with $m = A - 1$ and $i, j = 1, 2, 3$; $\alpha, \beta = p, n$. The operators (4) and (2)–(3) are related to the proton-neutron valence-shell and giant resonance degrees of freedom, respectively.

We also introduce the operators [9]:

$$B_i^\dagger(\alpha) = \sum_s b_{i\alpha,s}^\dagger \quad (5)$$

and $B_i(\alpha) = (B_i^\dagger(\alpha))^\dagger$, which together with the identity op-

erator close the six-dimensional Heisenberg-Weyl algebra $hw(6) = \{B_i^\dagger(\alpha), B_i(\alpha), I\}$ and provides the basic building blocks of the PNSM.

We classify the shell-model nuclear states by the following reduction chain [10]:

$$\begin{array}{cccccccc} Sp(12, R) & \supset & SU(1, 1) & \otimes & SO(6) & \supset & U(1) & \otimes & SU_{pn}(3) & \otimes & SO(2) & \supset & SO(3), \\ \langle \sigma \rangle & & \lambda_\nu & & \nu & & p & & (\lambda, \mu) & & \nu & & q \quad L \end{array} \quad (6)$$

where below the different subgroups, the quantum numbers that characterize their irreducible representations are given. We also note that the two groups $SU_{pn}(3)$ and $SO(2)$ are mutually complementary within the space of fully symmetric $SO(6)$ irreps [31]; they form a direct-product subgroup $SU_{pn}(3) \otimes SO(2) \subset SO(6)$. Hence, the $SU_{pn}(3)$ irrep labels (λ, μ) are in one-to-one correspondence with the $SO(6)$ and $SO(2)$ quantum numbers ν and ν , given by the following expression [10]:

$$(v)_6 = \bigoplus_{\nu=\pm\nu, \pm(\nu-2), \dots, 0(\pm 1)} \left(\lambda = \frac{\nu+\nu}{2}, \mu = \frac{\nu-\nu}{2} \right) \otimes (v)_2. \quad (7)$$

The $SU(1,1)$ Lie algebra, related to the radial dynamics, is generated by the shell-model operators [10]:

$$S_+^{(\lambda_\nu)} = \frac{1}{2} \sum_\alpha F^0(\alpha, \alpha), \quad (8)$$

$$S_-^{(\lambda_\nu)} = \frac{1}{2} \sum_\alpha G^0(\alpha, \alpha), \quad (9)$$

$$S_0^{(\lambda_\nu)} = \frac{1}{2} \sum_\alpha A^0(\alpha, \alpha), \quad (10)$$

with the following commutation relations

$$[S_+^{(\lambda_\nu)}, S_-^{(\lambda_\nu)}] = -2S_0^{(\lambda_\nu)}, \quad [S_0^{(\lambda_\nu)}, S_\pm^{(\lambda_\nu)}] = \pm S_\pm^{(\lambda_\nu)}. \quad (11)$$

The operator $S_0^{(\lambda_\nu)}$ itself generates the $U(1)$ subgroup of $SU(1,1)$.

The $SU(1,1)$ algebra has unitary representations with orthonormal basis states $\{|\lambda_\nu, p\rangle; p=0, 1, 2, \dots\}$ that are defined by the equations

$$S_+^{(\lambda_\nu)} |\lambda_\nu, p\rangle = \sqrt{(2\lambda_\nu + p)(p+1)} |\lambda_\nu, p+1\rangle, \quad (12)$$

$$S_-^{(\lambda_\nu)} |\lambda_\nu, p\rangle = \sqrt{(\lambda_\nu - 1 + p)p} |\lambda_\nu, p-1\rangle, \quad (13)$$

$$S_0^{(\lambda_\nu)} |\lambda_\nu, p\rangle = \frac{1}{2} (\lambda_\nu + 2p) |\lambda_\nu, p\rangle, \quad (14)$$

for any value of λ_ν ($\lambda_\nu > 1$), which generally defines the modified oscillator $SU(1,1)$ irreps [12]. From Eqs. (4), (10) and (14), it is evident that the operator $2S_0 \equiv H_0$ is the shell-model harmonic oscillator Hamiltonian whose eigenvalue is given by $(\lambda_\nu + 2p)$ in units of $\hbar\omega$; the $U(1) \subset SU(1,1)$ quantum number p is related to the $U(6) \supset SO(6)$ irrep labels E and ν (cf. Eqs. (37) and (32)) via $p = (E - \nu)/2$.

The group $SO(6)$ can be expressed through the number-preserving generators $A^{LM}(\alpha, \beta)$ (4) of $U(6)$ in the standard way by taking their antisymmetric combination [10]:

$$\Lambda^{LM}(\alpha, \beta) = A^{LM}(\alpha, \beta) - (-1)^L A^{LM}(\beta, \alpha). \quad (15)$$

The generators of different $SO(6)$ subgroups can be obtained by considering different tensor operators with respect to $SO(3)$ that can be constructed from (15). In this way, the generators of the $SU_{pn}(3)$ group are defined by the following set of operators:

$$\tilde{q}^{2M} = \sqrt{3}i[A^{2M}(p, n) - A^{2M}(n, p)], \quad (16)$$

$$Y^{1M} = \sqrt{2}[A^{1M}(p, p) + A^{1M}(n, n)], \quad (17)$$

which are rank-2 and rank-1 tensors, respectively. The single infinitesimal operator of $SO(2)$ is proportional to the $SO(3)$ scalar operator $\Lambda^0(\alpha, \beta)$:

$$M = \Lambda^0(\alpha, \beta) = i[A^0(\alpha, \beta) - A^0(\beta, \alpha)]. \quad (18)$$

The operators given in Eqs. (16)-(18) are 9 in number, which are obtained from $\Lambda^{LM}(\alpha, \beta)$ when $\alpha \neq \beta$ for $L=0, 2$ and $\alpha = \beta$ for $L=1$. The operators Y^{1M} (17) are actually a linear combination of the independent rank-1 tensors $A^{1M}(p, p)$ and $A^{1M}(n, n)$, which are 6 in number. The Lie algebra of $SO(6)$ is 15-dimensional, so the remaining 3 generators are obtained from $\Lambda^{LM}(\alpha, \beta)$ for $\alpha \neq \beta$ and

$L=1$, which represent the components of rank-1 tensor $A^{1M}(p, n) + A^{1M}(n, p)$. Using the relations $L_{1M}^p = \sqrt{2}A^{1M}(p, p)$ and $L_{1M}^n = \sqrt{2}A^{1M}(n, n)$, which define the angular momentum operators of the proton and neutron many-particle subsystems, respectively, the $SO(6)$ group is thus generated by the following set of operators $SO(6) = \{\tilde{q}^{2M}, Y^{1M}, M, L_{1M}^p - L_{1M}^n, A^{1M}(p, n) + A^{1M}(n, p)\}$. The operators $L_{1M}^p - L_{1M}^n$ correspond to the infinitesimal generators of the isovector out-of-phase angular-momentum displacements of the proton subsystem with respect to the neutron one, i.e. the scissors mode excitation operators.

The basis functions along the chain (6) can thus be written in the form [10]

$$\Psi_{\lambda, p; \nu v q L M}(r, \Omega_5) = R_p^{\lambda, \nu}(r) Y_{\nu q L M}^{\nu}(\Omega_5), \quad (19)$$

where $Y_{\nu q L M}^{\nu}(\Omega_5)$ are the $SO(6)$ Dragt's spherical harmonics [32, 33]. However, the explicit form of the latter is not required for our purposes because the calculation of the $SO(6)$ -reduced matrix elements involving the $SO(6)$ spherical harmonics can be carried out in a purely algebraic way. Due to the complementarity relationship [25, 31] of radial and orbital dynamics groups, $SU(1,1)$ and $SO(6)$, the states of seniority ν of the PNSM span a unitary representation of the direct product group $SU(1,1) \otimes SO(6)$ and satisfy the equations

$$S_+^{(\lambda_\nu)} R_p^{\lambda_\nu}(r) Y_{\nu q L M}^{\nu}(\Omega_5) = \sqrt{(\lambda_\nu + p)(p + 1)} R_{p+1}^{\lambda_\nu}(r) Y_{\nu q L M}^{\nu}(\Omega_5), \quad (20)$$

$$S_-^{(\lambda_\nu)} R_p^{\lambda_\nu}(r) Y_{\nu q L M}^{\nu}(\Omega_5) = \sqrt{(\lambda_\nu - 1 + p)p} R_{p-1}^{\lambda_\nu}(r) Y_{\nu q L M}^{\nu}(\Omega_5), \quad (21)$$

$$S_0^{(\lambda_\nu)} R_p^{\lambda_\nu}(r) Y_{\nu q L M}^{\nu}(\Omega_5) = \frac{1}{2}(\lambda_\nu + 2p) R_p^{\lambda_\nu}(r) Y_{\nu q L M}^{\nu}(\Omega_5), \quad (22)$$

where for the harmonic oscillator series $\lambda_\nu = \nu + 6/2$. The full many-particle Hilbert space of the nucleus thus can be represented as a direct sum

$$\mathbb{H} = \bigoplus_{\nu} \mathbb{H}_{\nu}^{S U(1,1)} \otimes \mathbb{H}_{\nu}^{S O(6)} \quad (23)$$

of Hilbert spaces labeled by a seniority quantum number ν , each carrying an irrep of the direct product group $SU(1,1) \otimes SO(6)$.

III. THE BASIC MATRIX ELEMENTS

The Cartesian coordinates $\{x_a \equiv x_i(\alpha) = \sum_s x_{is}(\alpha)\}$ acting in the Euclidean space \mathbb{R}^6 can be expressed in the

form

$$x_a = r Q_a \equiv r Q_i(\alpha), \quad (24)$$

which is their expression in the $SO(6)$ spherical polar coordinates. The operators $B_i^{\dagger}(\alpha)$ (5) and their conjugates $B_i(\alpha) = (B_i^{\dagger}(\alpha))^{\dagger}$ transform according to the $U(6)$ fundamental irreducible representation $[1]_6$ and its conjugate $[1]_6^* = [1, 1, 1, 1, 1, 0] \equiv [-1]_6$, respectively. Under the $SO(6)$ subgroup, both $U(6)$ representations reduce to the irrep $(1)_6$. Then, from $x_i(\alpha) = \frac{1}{\sqrt{2}}[B_i^{\dagger}(\alpha) + B_i(\alpha)]$ and the factorization of the position coordinates into radial and polar parts, one obtains that $Q \equiv Q_i(\alpha)$ is the basic $\nu = 1$ $SO(6)$ tensor. Any operator that can be represented as a polynomial in x_a will then have matrix elements that factor into radial and orbital parts.

A. The radial matrix elements

The radial matrix elements of even powers of r can be obtained utilizing only the properties of the $SU(1,1)$ algebra. Taking into account the fact that r^2 is simply an element of the latter

$$r^2 = S_+^{(\lambda_\nu)} + S_-^{(\lambda_\nu)} + 2S_0^{(\lambda_\nu)}, \quad (25)$$

the matrix elements of r^2 are immediately obtained from Eqs. (20)–(22) to be

$$\begin{aligned} F_{\lambda, p; \lambda, p'}(r^2) &= \langle \lambda_\nu p | r^2 | \lambda_\nu p' \rangle \\ &= \delta_{p, p'} (\lambda_\nu + 2p)/2 + \delta_{p, p-1} \sqrt{(\lambda_\nu - 1 + p)p} \\ &\quad + \delta_{p, p+1} \sqrt{(\lambda_\nu + p)(p + 1)}. \end{aligned} \quad (26)$$

The higher powers of r^2 can then be obtained by summing over intermediate states, e.g.,

$$\begin{aligned} F_{\lambda, p; \lambda, p'}(r^4) &= \sum_{p_c} F_{\lambda, p; \lambda, p_c}(r^2) F_{\lambda, p_c; \lambda, p'}(r^2) \\ &= \delta_{p, p'} \left((\lambda_\nu + 2p)^2 + (\lambda_\nu + p)(p + 1) + (\lambda_\nu + p - 1)p \right) \\ &\quad + \delta_{p, p+1} \left((2\lambda_\nu + 4p + 2) \sqrt{(\lambda_\nu + p)(p + 1)} \right) \\ &\quad + \delta_{p, p-1} \left((2\lambda_\nu + 4p - 2) \sqrt{(\lambda_\nu + p - 1)p} \right) \\ &\quad + \delta_{p, p+2} \sqrt{(\lambda_\nu + p + 1)(\lambda_\nu + p)(p + 2)(p + 1)} \\ &\quad + \delta_{p, p-2} \sqrt{(\lambda_\nu + p - 2)(\lambda_\nu + p - 1)p(p - 1)}. \end{aligned} \quad (27)$$

The matrix elements of $1/r^2$, rd/dr , d^2/dr^2 , r , $1/r$, d/dr are given in Ref. [12]. These radial matrix elements

are sufficient for deriving the matrix elements of any rotationally invariant polynomial in the many-particle collective position and momentum observables $\{x_i(\alpha) = \sum_s x_{is}(\alpha), p_i(\alpha) = \sum_s p_{is}(\alpha)\}$ of the PNSM.

B. The $SO(6)$ -reduced matrix elements

The $SO(6)$ -reduced matrix elements of an arbitrary tensor $T_{v_2 q_2 L_2 M_2}^{v_2}$ are defined by the generalized Wigner-Eckart theorem

$$\begin{aligned} & \langle v_3 v_3 q_3 L_3 \| T_{v_2 q_2 L_2 M_2}^{v_2} \| v_1 v_1 q_1 L_1 \rangle \\ &= \left\langle \begin{matrix} v_1 & v_2 \\ v_1 q_1 L_1 & v_2 q_2 L_2 \end{matrix} \middle| \begin{matrix} v_3 \\ v_3 q_3 L_3 \end{matrix} \right\rangle \langle v_3 \| T_{v_2 q_2 L_2 M_2}^{v_2} \| v_1 \rangle, \end{aligned} \quad (28)$$

where $\left\langle \begin{matrix} v_1 & v_2 \\ v_1 q_1 L_1 & v_2 q_2 L_2 \end{matrix} \middle| \begin{matrix} v_3 \\ v_3 q_3 L_3 \end{matrix} \right\rangle$ are the $SO(6)$ Clebsch-Gordan coefficients. The latter, according to the Racah's lemma [34], can be factorized in terms of the isoscalar factors that appear in each step of the reduction chain (6). Taking the correspondence $(\lambda, \mu) \leftrightarrow (v, \nu)$ given by Eq. (7), the $SO(6)$ Clebsch-Gordan coefficients can be written in a more convenient way as

$$\begin{aligned} & \left\langle \begin{matrix} v_1 & v_2 \\ v_1 q_1 L_1 & v_2 q_2 L_2 \end{matrix} \middle| \begin{matrix} v_3 \\ v_3 q_3 L_3 \end{matrix} \right\rangle = \left\langle \begin{matrix} v_1 & v_2 \\ (\lambda_1, \mu_1) & (\lambda_2, \mu_2) \end{matrix} \middle| \begin{matrix} v_3 \\ (\lambda_3, \mu_3) \end{matrix} \right\rangle \\ & \times \langle (\lambda_1, \mu_1) q_1 L_1; (\lambda_2, \mu_2) q_2 L_2 \| (\lambda_3, \mu_3) q_3 L_3 \rangle, \end{aligned} \quad (29)$$

where $\left\langle \begin{matrix} v_1 & v_2 \\ (\lambda_1, \mu_1) & (\lambda_2, \mu_2) \end{matrix} \middle| \begin{matrix} v_3 \\ (\lambda_3, \mu_3) \end{matrix} \right\rangle$ and $\langle (\lambda_1, \mu_1) q_1 L_1; (\lambda_2, \mu_2) q_2 L_2 \| (\lambda_3, \mu_3) q_3 L_3 \rangle$ are the $SO(6) \supset SU(3)$ and $SU(3) \supset SO(3)$ isoscalar factors, respectively. Computer codes [35, 36] exist for the calculation of the latter. From another side, by using the isomorphism $SO(6) \approx SU(4)$ and the fact that the $SO(6)$ tensors $T^{(v)}$ correspond to the $SU(4)$ tensors of the type $T^{[v, v, 0, 0]}$, the calculation of the $SO(6) \supset SU(3)$ isoscalar factors is equivalent to the calculation of the $SU(4) \supset SU(3)$ isoscalar factors, for which a computer code is also available [37] (Actually, the code given in [37] calculates the full $SU(4) \supset SU(3) \otimes U(1) \supset SU(2) \otimes U(1) \supset U(1)$ Clebsch-Gordan coefficients. The structure of the code, however, allows easily to adapt the latter for the calculation of the $SU(4) \supset SU(3)$ isoscalar factors by using the factorization property of the $SU(4)$ Clebsch-Gordan coefficients and modifying the line 143). Therefore, matrix elements of any operators of interest can be calculated from their $SO(6)$ -reduced matrix elements.

The $SO(6)$ -reduced matrix elements of the basic $SO(6)$ tensor Q are given by [12]

$$\langle v' | Q | v \rangle = \sqrt{\frac{v+1}{2v+6}} \delta_{v', v+1} + \sqrt{\frac{v+3}{2v+2}} \delta_{v', v-1}. \quad (30)$$

The $SO(6)$ -reduced matrix elements of other, more complicated tensors can be calculated utilizing these elementary matrix elements. For example, the $SO(6)$ -re-

duced matrix elements $[Q \times Q]^{v=2}$, which are relevant to the mass quadrupole tensor operators $Q_{ij}(\alpha, \beta) \rightarrow T_{vq2m}^{v=2} = r^2 Y_{vq2m}^{v=2} \equiv r^2 [Q \times Q]_{vq2m}^{v=2}$, are given by the standard recoupling technique:

$$\langle v' | [Q \times Q]_{vq2m}^{v=2} | v \rangle = \sum_c U(v; 1; v'; 1 \| v_c; 2) \langle v' | Q | v_c \rangle \langle v_c | Q | v \rangle. \quad (31)$$

The $SO(6)$ recoupling coefficients $U(v; 1; v'; 1 \| v_c; 2)$ are equivalent to the two-rowed $SU(4)$ Racah coefficients $U([v, v]_4; [1, 1]_4; [v', v']_4; [1, 1]_4 \| [v_c, v_c]_4; [2, 2]_4)$. The two-rowed or two-column $U(d)$ Racah recoupling coefficients, as well as other $U(d)$ invariant quantities, are equivalent to those of the $U(2)$ group, as shown, for example, by Li and Paldus in Ref. [38]. For convenience, we present some of the relevant PNSM $SO(6) \approx SU(4)$ recoupling coefficients in Table 1.

Table 1. Some $SO(6)$ Racah recoupling coefficients.

$U(v; 1; v+2; 1 \ v+1; 2) = 1$
$U(v; 1; v; 1 \ v+1; 2) = -(-1)^v \left[\frac{v(v+2)}{2(2+v(v+3))} \right]^{1/2}$
$U(v; 1; v; 1 \ v-1; 2) = -(-1)^v \left[\frac{v(v+2)}{2v(v+1)} \right]^{1/2}$
$U(v; 2; v+4; 2 \ v+2; 4) = 1$
$U(v; 2; v; 2 \ v; 4) = \left[\frac{2(v-1)(v+3)}{3v(v+2)} \right]^{1/2}$
$U(v; 2; v+2; 2 \ v; 2) = \left[\frac{v(v+1)}{2(2+3v(v+3))} \right]^{1/2}$
$U(v; 2; v+2; 2 \ v+2; 2) = - \left[\frac{(v+3)(v+4)}{2(v^2+5v+6)} \right]^{1/2}$
$U(v; 2; v; 2 \ v+2; 4) = \left[\frac{(v+3)v(v-1)}{6(v^3+6v^2+11v+6)} \right]^{1/2}$
$U(v; 2; v; 2 \ v+2; 2) = \left[\frac{(v+3)v}{2(v^2+3v+2)} \right]^{1/2}$
$U(v; 2; v; 2 \ v+2; 0) = (-1)^v \left[\frac{v+3}{3(v+1)} \right]^{1/2}$
$U(v; 2; v; 2 \ v-2; 4) = \left[\frac{(v-1)(v^2+5v+6)}{6v(v^2-1)} \right]^{1/2}$
$U(v; 2; v; 2 \ v-2; 2) = - \left[\frac{(v-1)(v+2)}{2v(v+1)} \right]^{1/2}$
$U(v; 2; v; 2 \ v-2; 0) = \left[\frac{v-1}{3(v+1)} \right]^{1/2}$
$U(v; 2; v; 2 \ v; 4) = \left[\frac{2(v-1)(v+3)}{3v(v+2)} \right]^{1/2}$
$U(v; 2; v-2; 2 \ v-2; 2) = (-1)^{-v} \left[\frac{v-2}{2v} \right]^{1/2}$

In practice, for the calculation of the matrix elements of various algebraic interactions, it turns out to be more convenient to use the $SO(6)$ -reduced matrix elements of the symplectic generators, which in turn are obtained

from the basic creation $B_i^\dagger(\alpha)$ and annihilation $B_i(\alpha)$ operators, respectively. The $SO(6)$ -reduced matrix elements of these operators can be obtained by considering the equivalent chain

$$U(6) \supset SO(6), \tag{32}$$

$$E \quad \nu$$

whose branching rules for the case of fully symmetric irreps are given by [39]

$$[E]_6 = \bigoplus_{\nu=E, E-2, \dots, 0(1)} (\nu, 0, 0)_6 = \bigoplus_{i=0}^{\lfloor \frac{E}{2} \rfloor} (E-2i)_6. \tag{33}$$

The $SO(6)$ -reduced matrix elements are therefore obtained by applying the Wigner-Eckart theorem with respect to this chain

$$\langle E' \nu' ||| B_i^\dagger(\alpha) ||| E \nu \rangle = \langle E' ||| B_i^\dagger(\alpha) ||| E \rangle \begin{matrix} [E'] & [1] \\ \nu' & 1 \end{matrix} \begin{matrix} [E] \\ \nu \end{matrix} \begin{matrix} [1] \\ 1 \end{matrix} \begin{matrix} [E'] \\ \nu' \end{matrix}. \tag{34}$$

Using the elementary (one-particle) $U(d) \supset SO(d)$ isoscalar factors, given in Ref. [40], one readily obtains for $d = 6$:

$$\langle E+1, \nu+1 ||| B_i^\dagger(\alpha) ||| E \nu \rangle = \sqrt{\frac{(\nu+1)(E+\nu+6)}{(2\nu+6)}}, \tag{35}$$

$$\langle E+1, \nu-1 ||| B_i^\dagger(\alpha) ||| E \nu \rangle = \sqrt{\frac{(E-\nu+2)(\nu+3)}{(2\nu+2)}}. \tag{36}$$

However, this result is valid only for the scalar $\langle \sigma \rangle = 0$ irreducible representation of $Sp(12, R)$, which corresponds to the doubly closed-shell nuclei; hence, it is not physically interesting. For non-scalar irreps $\langle \sigma \rangle \neq 0$, the ket vectors $|E \nu \bar{\eta}\rangle$ need to be replaced by the $U(6)$ -coupled basis states

$$|\sigma n \rho E \nu \bar{\eta}\rangle = [P^{(n)}(F) \times |\sigma\rangle]_{\bar{\eta}}^{\rho E \nu} \tag{37}$$

along the chain

$$Sp(12, R) \supset U(6) \supset SO(6), \tag{38}$$

$$\sigma \quad n \rho \quad E \quad \nu$$

where $\bar{\eta}$ labels the basis of an $SO(6)$ irrep ν , classified further by the other subgroups along the chain (6). The basic creation $B_i^\dagger(\alpha)$ and annihilation $B_i(\alpha)$ operators couple the even and odd irreps of $Sp(12, R)$ and belong to the larger collective algebra $wsp(12, R) = [hw(6)]sp(12, R)$

[9], which is the semidirect sum of Heisenberg-Weyl algebra $hw(6) = \{B_i^\dagger(\alpha), B_i(\alpha), I\}$ and symplectic algebra $sp(12, R)$. Then, in order to calculate the matrix elements of $B_i^\dagger(\alpha)$ and $B_i(\alpha)$, one must consider the representation theory of the $wsp(12, R)$ which basis states can be represented as

$$|\langle \langle \sigma \rangle \rangle l \langle \omega \rangle n \rho E \nu \bar{\eta}\rangle = [P^{(n)}(F) \times [Q^{(l)}(B_i^\dagger(\alpha)) \times |\sigma\rangle]_{\bar{\eta}}^{(\omega) \rho E \nu}] \tag{39}$$

that are classified according to the reduction chain

$$wsp(12, R) \supset sp(12, R) \supset u(6) \supset so(6). \tag{40}$$

$$\langle \langle \sigma \rangle \rangle \quad l \quad \langle \omega \rangle \quad n \rho \quad E \quad \nu$$

For more details, we refer the reader to Ref. [9]. For the present purposes, it is sufficient to restrict the construction of the basis states to the form

$$|\sigma n \rho E \nu \bar{\eta}\rangle = [Q^{(n)}(B_i^\dagger(\alpha)) \times |\sigma\rangle]_{\bar{\eta}}^{\rho E \nu}. \tag{41}$$

Then, using a standard recoupling technique, the $SO(6)$ -reduced matrix elements of the raising generators $B_i^\dagger(\alpha)$ of $wsp(12, R)$ along the basis (41) can be written as

$$\begin{aligned} & \langle \sigma' n' \rho' E' \nu' ||| B_i^\dagger(\alpha) ||| \sigma n \rho E \nu \rangle \\ &= \langle \sigma' n' \rho' E' \nu' ||| B_i^\dagger(\alpha) ||| \sigma n \rho E \nu \rangle \begin{matrix} [E'] & [1] \\ \nu' & 1 \end{matrix} \begin{matrix} [E] \\ \nu \end{matrix} \begin{matrix} [1] \\ 1 \end{matrix} \begin{matrix} [E'] \\ \nu' \end{matrix} \\ &= U([\sigma]_6; [n]_6; [E']_6; [1]_6 \parallel [E]_6; [n']_6) \\ & \times \langle \sigma' ||| B_i^\dagger(\alpha) ||| \sigma \rangle \begin{matrix} [E'] & [1] \\ \nu' & 1 \end{matrix} \begin{matrix} [E] \\ \nu \end{matrix} \begin{matrix} [1] \\ 1 \end{matrix} \begin{matrix} [E'] \\ \nu' \end{matrix}, \end{aligned} \tag{42}$$

where the corresponding $U(6)$ Racah coefficient is equal to 1. Using the latter, we obtain the following expressions for the $SO(6)$ -reduced matrix elements of the creation operators $B_i^\dagger(\alpha)$:

$$\begin{aligned} & \langle \sigma' n+1, \rho', E+1, \nu+1 ||| B_i^\dagger(\alpha) ||| \sigma n \rho E \nu \rangle \\ &= \sqrt{\frac{(n+1)(\sigma+n+\nu+6)(\nu+1)}{(\sigma+n+1)(2\nu+6)}}, \end{aligned} \tag{43}$$

$$\begin{aligned} & \langle \sigma' n+1, \rho', E+1, \nu-1 ||| B_i^\dagger(\alpha) ||| \sigma n \rho E \nu \rangle \\ &= \sqrt{\frac{(n+1)(\sigma+n-\nu+2)(\nu+3)}{(\sigma+n+1)(2\nu+2)}}. \end{aligned} \tag{44}$$

where we have also used the relation $E = \sigma + n$. We also observe that the quantum number n is related to the p quantum number via $n = 2p$; there exists a correspond-

ence between the harmonic oscillator basis states $|n\rangle$ and the $SU(1,1)$ basis $|\lambda_\nu, p\rangle$. To observe this, one can relabel the basis states $|n\rangle$ of the harmonic oscillator by the $SU(1,1)$ quantum numbers $\{|\lambda_\nu, p\rangle; p = 0, 1, 2, \dots\}$. Then, one readily obtains the following correspondence

$$|n\rangle = |\lambda_\nu = \frac{6}{2}, p = \frac{n}{2}\rangle, \quad (45)$$

for even n , and

$$|n\rangle = |\lambda_\nu = \frac{8}{2}, p = \frac{n-1}{2}\rangle, \quad (46)$$

for odd n values, respectively. Conversely, taking $|\lambda_\nu, p\rangle = |n\rangle$ with $n = \lambda_\nu + 2p - 6/2$ and using Eqs. (2)–(4) and (8)–(10), one obtains for $n = 0, 1, 2, \dots$ the standard oscillator expressions

$$S_+^{(\lambda_\nu)}|n\rangle = \frac{1}{2}\sqrt{(n+1)(n+2)}|n+2\rangle, \quad (47)$$

$$S_-^{(\lambda_\nu)}|n\rangle = \frac{1}{2}\sqrt{n(n-1)}|n-2\rangle, \quad (48)$$

$$S_0^{(\lambda_\nu)}|n\rangle = \frac{1}{2}\left(n + \frac{6}{2}\right)|n\rangle, \quad (49)$$

which are special cases ($\lambda_\nu = 6/2$ and $\lambda_\nu = 8/2$) of the general expressions (12)–(14).

The corresponding $SO(6)$ -reduced matrix elements of the annihilation operators are obtained by the relation

$$\begin{aligned} &\langle \sigma' n' \rho' E' \nu' ||| B_i(\alpha) ||| \sigma n \rho E \nu \rangle \\ &= \sqrt{\frac{\dim(\nu)}{\dim(\nu')}} \langle \sigma n \rho E \nu ||| B_i^\dagger(\alpha) ||| \sigma' n' \rho' E' \nu' \rangle, \end{aligned} \quad (50)$$

where $\dim(\nu) = (\nu+3)(\nu+2)^2(\nu+1)/12$ is the dimension of the $SO(6)$ irrep ν . The result is

$$\begin{aligned} &\langle \sigma' n-1, \rho', E-1, \nu-1 ||| B_i(\alpha) ||| \sigma n \rho E \nu \rangle \\ &= \sqrt{\frac{n(\sigma+n+\nu+4)(\nu+2)(\nu+3)}{(\sigma+n)(\nu+1)(2\nu+4)}}, \end{aligned} \quad (51)$$

$$\begin{aligned} &\langle \sigma' n-1, \rho', E-1, \nu+1 ||| B_i(\alpha) ||| \sigma n \rho E \nu \rangle \\ &= \sqrt{\frac{n(\sigma+n-\nu)(\nu+1)(\nu+2)}{(\sigma+n)(\nu+3)(2\nu+4)}}. \end{aligned} \quad (52)$$

Using a standard recoupling technique, with the help of Eqs. (43)–(52) and the values of the $SO(6)$ Racah coefficients given in Table 1, one readily obtains the $SO(6)$ -reduced matrix elements of the symplectic generators (2), (3), and (4):

$$\langle \sigma n+2, \rho', E+2, \nu+2 ||| F_{ij}(\alpha, \beta) ||| \sigma n \rho E \nu \rangle = \left[\frac{(n+1)(n+2)(\sigma+n+\nu+6)(\sigma+n+\nu+8)}{(\sigma+n+1)(\sigma+n+2)} \frac{(\nu+1)(\nu+2)}{(2\nu+6)(2\nu+8)} \right]^{1/2}, \quad (53)$$

$$\langle \sigma n-2, \rho', E-2, \nu-2 ||| G_{ij}(\alpha, \beta) ||| \sigma n \rho E \nu \rangle = \left[\frac{n(n-1)(\sigma+n+\nu+2)(\sigma+n+\nu+4)}{(\sigma+n)(\sigma+n-1)} \frac{(\nu+2)^2(\nu+3)}{\nu(2\nu+2)(2\nu+4)} \right]^{1/2}, \quad (54)$$

$$\begin{aligned} \langle \sigma n \rho E \nu + 2 ||| A_{ij}(\alpha, \beta) ||| \sigma n \rho E \nu \rangle &= \frac{1}{2} \left[\frac{n(\nu+2)}{(\sigma+n)} \sqrt{\frac{(\sigma+n+\nu+6)(\sigma+n-\nu)(\nu+1)}{(\nu+3)(2\nu+4)(2\nu+8)}} \right. \\ &\quad \left. + \frac{(n+1)}{(\sigma+n+1)(2\nu+6)} \sqrt{\frac{(\sigma+n+\nu+6)(\sigma+n-\nu)(\nu+1)(\nu+2)(\nu+3)}{(\nu+4)}} \right], \end{aligned} \quad (55)$$

$$\begin{aligned} \langle \sigma n \rho E \nu - 2 ||| A_{ij}(\alpha, \beta) ||| \sigma n \rho E \nu \rangle &= \frac{1}{2} \left[\frac{n(\nu+2)}{(\sigma+n)} \sqrt{\frac{(\sigma+n+\nu+4)(\sigma+n-\nu+2)(\nu+3)}{(\nu+1)2\nu(2\nu+4)}} \right. \\ &\quad \left. + \frac{(n+1)}{(\sigma+n+1)(2\nu+2)} \sqrt{\frac{(\sigma+n-\nu+2)(\sigma+n+\nu+4)(\nu+1)(\nu+2)(\nu+3)}{\nu}} \right], \end{aligned} \quad (56)$$

$$\langle \sigma n \rho E v \| \| A_{ij}(\alpha, \beta) \| \| \sigma n \rho E v \rangle = -\frac{1}{2} \left[\frac{n(\sigma+n+v+4)(v+2)}{(\sigma+n)(2v+4)(v+1)} \sqrt{\frac{v(v+3)}{2}} + \frac{n(\sigma+n-v)(v+2)}{(\sigma+n)(2v+4)} \sqrt{\frac{v(v+1)(v+4)}{2(2+v(v+3))(v+3)}} \right. \\ \left. + \frac{(n+1)(\sigma+n-v+2)}{(\sigma+n+1)(2v+2)} \sqrt{\frac{v(v+3)}{2}} + \frac{(n+1)(\sigma+n+v+6)}{(\sigma+n+1)(2v+6)} \sqrt{\frac{v(v+1)(v+3)(v+4)}{2(2+v(v+3))}} \right]. \tag{57}$$

The coherent state theory of symplectic algebras [41-46] implies that the matrix elements of the raising and lowering symplectic operators (i.e., Eqs. (53) and (54)) are required to be multiplied by the *K*-matrix. However, in the limit of large representations $\langle \sigma \rangle$ of $Sp(12, R)$ or in the multiplicity free case ($\rho = \rho' = 1$), which are the cases of practical importance, the *K*-matrix reduces to simple normalization coefficients (a diagonal *K*-matrix):

$$\sqrt{\Delta \Omega(\sigma n' E'; n E)} = \sqrt{\Omega(\sigma n' E') - \Omega(\sigma n E)},$$

where

$$\Omega(\sigma n E) = \frac{1}{4} \sum_{a=1}^6 [2E_a^2 - n_a^2 + 14(E_a - n_a) - 2a(2E_a - n_a)] \tag{43}$$

Therefore, the final expressions for all matrix elements in which the raising and lowering symplectic generators *F*'s and *G*'s enter must be multiplied by the proper normalization factors.

From the matrix elements of the symplectic generators, one can obtain the *SO*(6)-reduced matrix elements of different algebraic interactions. For example, we present the results for the *SO*(6)-reduced matrix elements of the

$$A^2(\alpha, \beta) \cdot F^2(\alpha, \beta) \simeq [A^2(\alpha, \beta) \times F^2(\alpha, \beta)]_{v=\pm 2q=1l=0m=0}^{v=2} \\ F^2(\alpha, \beta) \cdot F^2(\alpha, \beta) \simeq [F^2(\alpha, \beta) \times F^2(\alpha, \beta)]_{\pm 4100}^4 \text{ and} \\ G^2(\alpha, \beta) \cdot F^2(\alpha, \beta) \simeq [G^2(\alpha, \beta) \times F^2(\alpha, \beta)]_{\pm 4100}^4$$

tensor operators:

$$\langle \sigma n + 2, \rho', E + 2, v + 2 \| \| A^2(\alpha, \beta) \cdot F^2(\alpha, \beta) \| \| \sigma n \rho E v \rangle \\ = \sqrt{\frac{5(n+1)(n+2)(\sigma+n+v+6)(\sigma+n+v+8)(v+1)(v+2)(v+3)(v+4)}{(\sigma+n+1)(\sigma+n+2)2(v^2+5v+6)(2v+6)(2v+8)}} \\ \times \left(\frac{(n+2)(\sigma+n+v+8)(v+4)}{(\sigma+n+2)(2v+8)(v+3)} \sqrt{\frac{(v+2)(v+5)}{2}} + \frac{(n+2)(\sigma+n-v)(v+4)}{(\sigma+n+2)(2v+8)} \sqrt{\frac{(v+2)(v+3)(v+6)}{2(2+(v+2)(v+5))(v+5)}} \right. \\ \left. + \frac{(n+3)(\sigma+n-v+2)}{(\sigma+n+3)(2v+6)} \sqrt{\frac{(v+2)(v+5)}{2}} + \frac{(n+3)(\sigma+n+v+10)}{(\sigma+n+3)(2v+10)} \sqrt{\frac{(v+2)(v+3)(v+5)(v+6)}{2(2+(v+2)(v+5))}} \right), \tag{58}$$

$$\langle \sigma n + 4, \rho', E + 4, v + 4 \| \| F^2(\alpha, \beta) \cdot F^2(\alpha, \beta) \| \| \sigma n \rho E v \rangle \\ = \sqrt{\frac{5(n+1)(n+2)(n+3)(n+4)(\sigma+n+v+6)(\sigma+n+v+8)(\sigma+n+v+10)(\sigma+n+v+12)}{(\sigma+n+1)(\sigma+n+2)(\sigma+n+3)(\sigma+n+4)(2v+6)(2v+8)(2v+10)(2v+12)}} \\ \times \sqrt{(v+1)(v+2)(v+3)(v+4)}, \tag{59}$$

$$\langle \sigma n \rho E v \| \| G^2(\alpha, \beta) \cdot F^2(\alpha, \beta) \| \| \sigma n \rho E v \rangle \\ = \frac{(n+1)(n+2)(\sigma+n+v+6)(\sigma+n+v+8)(v+4)}{(\sigma+n+1)(\sigma+n+2)(2v+6)(2v+8)} \sqrt{\frac{5(v-1)v(v+1)(v+3)(v+5)}{6(v^3+6v^2+11v+6)}} \\ + \frac{(n+1)(n+2)(\sigma+n-v+2)(\sigma+n-v+4)}{(\sigma+n+1)(\sigma+n+2)(2v)(2v+2)} \sqrt{\frac{5(v-1)v(v+2)(v+3)(v^2+5v+6)}{6(v+1)}}. \tag{60}$$

C. The full matrix elements

The full $SO(3)$ -reduced matrix elements of physical observables of interest are obtained by multiplying the $SO(6)$ -reduced matrix elements with the proper $SO(6) \supset SU(3)$ and $SU(3) \supset SO(3)$ isoscalar factors. For example, for the $SO(3)$ -reduced matrix elements of the raising symplectic generators we have

$$\begin{aligned} & \langle \sigma n + 2, \rho', E + 2, \nu + 2, \nu + 2, q' L' \| F^{2m}(\alpha, \beta) \| \sigma n \rho E \nu \nu q L \rangle \\ &= \langle \sigma n + 2, \rho', E + 2, \nu + 2 \| F^{2m}(\alpha, \beta) \| \sigma n \rho E \nu \rangle \\ & \times \left\langle \begin{matrix} \nu & 2 \\ (\lambda, \mu) & (2, 0) \end{matrix} \middle| \begin{matrix} \nu + 2 \\ (\lambda + 2, \mu) \end{matrix} \right\rangle \\ & \times \langle (\lambda, \mu) q L; (2, 0) 2 \| (\lambda + 2, \mu) q' L' \rangle, \end{aligned} \quad (61)$$

where the $SO(6)$ -reduced matrix elements of $F^{2m}(\alpha, \beta)$ are given by Eq. (53). Similarly, one can obtain the matrix elements of other physical operators of interest.

IV. APPLICATION

A. Energy spectrum

In practical calculation, another realization of the $SO(6)$ algebra becomes more convenient. It is obtained using the transformation

$$\begin{aligned} a_j^\dagger &= \frac{1}{\sqrt{2}} \left(-i B_j^\dagger(p) + B_j^\dagger(n) \right), \\ b_j^\dagger &= \frac{1}{\sqrt{2}} \left(i B_j^\dagger(p) + B_j^\dagger(n) \right) \end{aligned} \quad (62)$$

and their conjugate counterparts

$$\begin{aligned} a_j &= \frac{1}{\sqrt{2}} \left(i B_j(p) + B_j(n) \right), \\ b_j &= \frac{1}{\sqrt{2}} \left(-i B_j(p) + B_j(n) \right). \end{aligned} \quad (63)$$

Then, one obtains an alternative realization of $SO(6)$ [32, 33, 47, 48]:

$$\tilde{q}^{2M} = \sqrt{3} [A^{2M}(a, a) - A^{2M}(b, b)], \quad (64)$$

$$Y^{1M} = \sqrt{2} [A^{1M}(a, a) + A^{1M}(b, b)], \quad (65)$$

$$M = a^\dagger \cdot a - b^\dagger \cdot b = N_a - N_b, \quad (66)$$

In this realization, the creation operators a_j^\dagger and b_j^\dagger carry

the $SU(3)$ irreps $(1, 0)$ and $(0, 1)$, respectively. We recall that both $B_j^\dagger(p)$ and $B_j^\dagger(n)$ were transformed as two independent $(1, 0)$ $SU(3)$ tensor operators. Correspondingly, the symplectic operators $F^{2m}(a, a)$, $F^{2m}(b, b)$, and $F^{2m}(a, b)$ transform as $(2, 0)$, $(0, 2)$, and $(1, 1)$ $SU(3)$ tensors, whereas $A^{1m}(a, b)$ and $A^{1m}(b, a)$ – as $(2, 0)$ and $(0, 2)$ $SU(3)$ tensors, respectively. Then the construction of the basis (shown in Table 1 of Ref. [10]) along the chain (6) becomes straightforward. For instance: $|E = 2, \nu = 2, \nu = 2, qLM\rangle \simeq F^{LM}(a, a)|0\rangle$, $|E = 2, \nu = 2, \nu = 0, qLM\rangle \simeq F^{LM}(a, b)|0\rangle$, $|E = 2, \nu = 2, \nu = -2, qLM\rangle \simeq F^{LM}(b, b)|0\rangle$, where $|0\rangle = |E = 0, \nu = 0, \nu = 0, q = 1, L = 0, M = 0\rangle$.

The starting point of the present application is the following dynamical symmetry Hamiltonian [10]:

$$\begin{aligned} H &= 2S_0^{(\lambda)} + \Lambda \Lambda^2 + BC_2[S U_{pn}(3)] \\ &+ aC_2[SO(3)] + b(C_2[SO(3)])^2, \end{aligned} \quad (67)$$

in which the first term $2S_0^{(\lambda)}$, as mentioned before, represents the harmonic oscillator mean field that defines the shell structure. The eigenvalues of this Hamiltonian are the energies

$$\begin{aligned} E(n, \nu, \lambda, \mu, L) &= n\hbar\omega + A\nu(\nu + 4) \\ &+ \frac{2}{3}B(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) \\ &+ aL(L + 1) + b[L(L + 1)]^2. \end{aligned} \quad (68)$$

To illustrate the present theory, we apply it to two nuclei with various collective properties, namely ^{106}Ru and ^{158}Gd with the characteristic ratios $E_{4_1^+}/E_{2_1^+} \simeq 2.65$ and 3.26 [49], respectively. These two isotopes were chosen as examples of nuclei with different collective structures. The purpose of the present work is not to provide a detailed and accurate description of their low-lying collective spectra that are observed experimentally; instead, we aim to illustrate the application of the developed computational technique. They are chosen as nuclei having different collective properties and not specifically as possible candidates for $SO(6)$ symmetry (the case of ^{106}Ru , whose energy ratio is close to the γ -unstable Wilets-Jean [22] model value 2.50). As demonstrated in Ref. [10], the present version of the PNSM represents a microscopic shell-model counterpart of the Bohr-Mottelson [11] collective model, having three classical submodel limits – the γ -unstable Wilets-Jean model [22], the rigid-rotor model of Ui [50], and the harmonic vibrator model. Therefore, it was suggested that these three limiting types of nuclear collective behavior can be described simultaneously within the framework of the present shell-model coupling scheme of the PNSM. However, this is beyond the scope of the present paper.

Detailed application of the present approach to each type of nuclear structure behavior, corresponding to a certain submodel of the BM [11] model, will be separately provided in future publications.

Because in heavy mass regions, the spin-orbit interaction is strong, we use the pseudo- $SU(3)$ scheme [51-53] to determine the relevant irreducible representations of $Sp(12,R)$. To obtain the leading $SU(3)$ irrep for these nuclei, i.e. the one which is maximally deformed and has maximal value of the $SU(3)$ second-order Casimir operator, at the observed quadrupole deformation, one first fills the pseudo-oscillator Nilsson levels of the valence shells with protons and neutrons pairwise, from bottom up in energy. For instance, in the case of ^{106}Ru , we have 8 valence normal-parity neutrons in the pseudo- fp shell, which produce the $(10,4)$ irreducible representation for the neutron subsystem. Using the Nilsson model ideas [54, 55], however, the appropriate prolate-deformed $SU(3)$ irrep would be $(18,0)$ that is obtained from the $(10,4)$ one. For the proton subsystem, the valence protons completely fill the pseudo- ds shell; consequently, they produce the scalar irrep $(0,0)$ of $SU(3)$. Computer codes are available [56, 57] for determining the corresponding Pauli allowed $SU(3)$ irreducible representations in a given major shell for a given number of valence nucleons. For the highest-weight $SU(3)$ states considered here, one can also manually obtain the relevant leading irreps. In the second step, one needs to couple the proton and neutron $SU(3)$ subirreps to obtain the relevant $SU_{pn}(3)$ irreducible representation of the combined proton-neutron nuclear system. For ^{106}Ru , this is trivially done because $(18,0) \otimes (0,0) \rightarrow (18,0)$. The relevant $Sp(12,R)$ irreducible representation is fixed by the requirement that the lowest-weight $U(6)$ irrep $\sigma = [\sigma_1, \dots, \sigma_6]_6$ of the symplectic bandhead should contain the $SU(3)$ leading irrep $(18,0)$. Thus, the Nilsson model ideas [54, 55] and shell-model considerations based on the pseudo- $SU(3)$ scheme for ^{106}Ru provide the irreducible representation $\langle \sigma \rangle = \langle 33 + \frac{105}{2}, 15 + \frac{105}{2}, 15 + \frac{105}{2}, 15 + \frac{105}{2}, 15 + \frac{105}{2}, 15 + \frac{105}{2} \rangle$ of $Sp(12,R)$, which corresponds to the lowest-weight $U(6)$ irreducible representation $\sigma = [33, 15, 15, 15, 15, 15]_6 \equiv [18]_6$. The latter, according to Eqs. (7) and (33), contains the lowest-energy $SU(3)$ irrep $(18,0)$ of the ground state (when there is no $SU(3)$ mixing) that is embedded in the $SO(6)$ irreducible representation $\nu_0 = 18$. The set of $SU_{pn}(3)$ irreps within the $SO(6)$ irrep $\nu_0 = 18$ is given by $(18,0), (17,1), (16,2), \dots, (10,8), (9,9), (8,10), \dots, (2,16), (1,17), (0,18)$. Similarly, for ^{158}Gd , we obtain [10] the irreducible representation $\langle \sigma \rangle = \langle 63 + \frac{157}{2}, 27 + \frac{157}{2}, 27 + \frac{157}{2}, 27 + \frac{157}{2}, 27 + \frac{157}{2}, 27 + \frac{157}{2} \rangle$ of $Sp(12,R)$, which corresponds to the lowest-weight $U(6)$ irreducible representation $\sigma = [63, 27, 27,$

$27, 27, 27]_6 \equiv [36]_6$. Likewise, the shell model considerations for ^{158}Gd give the lowest-energy $SU(3)$ irrep $(36,0)$, which according to Eqs. (7) and (33), is contained in the $SO(6)$ irreducible representation $\nu_0 = 36$. The full set of $SU_{pn}(3)$ multiplets that is contained in the $SO(6)$ irrep $\nu_0 = 36$, according to Eq. (7), is as follows: $(36,0), (35,1), (34,2), \dots, (19,17), (18,18), (17,19), \dots, (2,34), (1,35), (0,36)$. Thus, in the pure $SU(1,1) \otimes SO(6)$ dynamical symmetry limit of the PNSM, the first excited β and γ bands in ^{158}Gd belong to the $SU_{pn}(3)$ irrep $(34,2)$. Similarly, the γ band in ^{106}Ru belongs to the $SU_{pn}(3)$ irrep $(16,2)$.

Using Eq. (68), we compare the excitation energies of the ground and γ bands in ^{106}Ru and those of the ground, β , and γ bands in ^{158}Gd , with the experiment shown in Fig. 1. The adopted values of the model parameters in MeV are $A = 0, B = -0.0324, a = 0.032,$ and $b = -0.00005$ for ^{106}Ru and $A = 0, B = -0.026, a = 0.0134,$ and $b = -0.0000093$ for ^{158}Gd . For comparison, in the case of ^{106}Ru , the results of the general collective model (GCM) with a collective potential including terms up to sixth order (with 8 free parameters) are given as well [58]. This figure shows a good agreement with the experimental data for the excitation energies of the bands under consid-

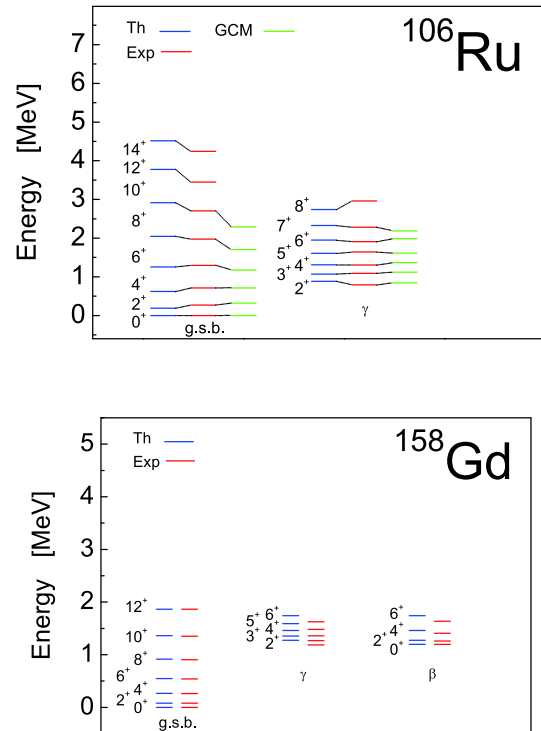


Fig. 1. (color online) Comparison of the excitation energies of the ground and γ bands in ^{106}Ru and ground, β , and γ bands in ^{158}Gd via experiments according to Eq. (68). For comparison, in the case of ^{106}Ru , the results of the general collective model (GCM) with a collective potential including terms up to the sixth order are given as well [58].

eration for the two nuclei.

B. Transition probabilities

The transition probabilities are given, by definition, by the $SO(3)$ -reduced matrix elements of the corresponding transition operator. Here we restrict ourselves to the

$$B(E2; L_i \rightarrow L_f) = \frac{2L_f + 1}{2L_i + 1} |\langle f || T^{E2} || i \rangle|^2 = \frac{2L_f + 1}{2L_i + 1} \left(\frac{eZ}{A-1} \right)^2 \left(k \sqrt{3} \langle (\lambda, \mu) q L_i; (1, 1) 2 || (\lambda, \mu) q' L_f \rangle \right)^2 \times \sqrt{2 \langle C_2[S U_{pn}(3)] \rangle}, \quad (69)$$

where $\langle C_2[S U_{pn}(3)] \rangle = \frac{2}{3}(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu)$ is the eigenvalue of the $SU_{pn}(3)$ second-order Casimir operator. Additionally, because the relevant irreps in the PNSM for heavy nuclei are determined by using the pseudo- $SU(3)$ scheme [51–53], a constant factor k enters the expression, which relates the real quadrupole operator with its pseudo- $SU(3)$ counterpart, i.e., $\tilde{q}^{2m} = k\tilde{q}_{\text{pseudo}}^{2m}$. For rare-earth and Ru regions, $k \approx 1.20$ and $k \approx 1.25$, respectively. In present calculations, no effective charge is used (i.e., $e = 1$).

Using Eq. (69), in Fig. 2, we compare the calculated intraband $B(E2)$ transition probabilities between the states of the ground band in ^{158}Gd with experiment [49] (dashed green line). We observe that theoretical values overestimate the experimental data. For ^{106}Ru , the experimental value is known only for the transition $B(E2; 2_1^+ \rightarrow 0_1^+) = 66$ W.u. [49]. The theoretical value we obtained in the pure $SU_{pn}(3)$ dynamical symmetry limit is 95.76 W.u. The results obtained for the $B(E2)$ transition probabilities in the two nuclei suggest that in order to reduce the theoretical values, a horizontal mixing of differ-

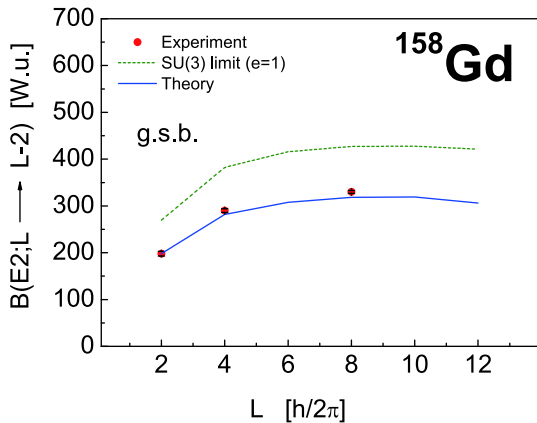


Fig. 2. (color online) Calculated intraband $B(E2)$ values in Weisskopf units between the states of the ground band in ^{158}Gd obtained in the pure $SU(3)$ limit (dashed green line) and mixed representation calculations (continuous blue curve). No effective charge is used.

$B(E2)$ transition strengths between the states of the ground band only. We use $T^{E2} = (eZ/(A-1))\tilde{q}^{2m}$ as the $E2$ transition operator, where \tilde{q}^{2m} is given by Eq. (64). Because \tilde{q}^{2m} is a generator of the $SU_{pn}(3)$ group, we obtain the well-known result for the $B(E2)$ transition probabilities

ent $SU(3)$ multiplets within the corresponding $SO(6)$ irreducible representation ν_0 is required.

C. Mixed $SU(3)$ calculations

To obtain better agreement with the experiment on $B(E2)$ transition probabilities, we use the following simple Hamiltonian

$$H_{\text{mix}} = h(G^2(a, a) \cdot F^2(b, b) + G^2(b, b) \cdot F^2(a, a)) \quad (70)$$

that mixes different $SU(3)$ multiplets within the maximal seniority $SO(6)$ representation ν_0 contained in the corresponding symplectic bandhead. Then, for example, the $SO(3)$ -reduced matrix elements of the tensor $G^2(a, a) \cdot F^2(b, b) = \frac{2\sqrt{5}}{3} [G^2(a, a) \times F^2(b, b)]_{-4100}^4$ are given by

$$\begin{aligned} & \langle \sigma \rho E \nu \nu' q' L || G^2(a, a) \cdot F^2(b, b) || \sigma \eta \rho E \nu \nu q L \rangle \\ &= \langle \sigma \eta \rho E \nu || [G^2(a, a) \times F^2(b, b)]_{-4100}^4 || \sigma \eta \rho E \nu \rangle \\ & \times \frac{2\sqrt{5}}{3} \left\langle \begin{matrix} \nu \\ (\lambda, \mu) \end{matrix} \begin{matrix} 4 \\ (0, 4) \end{matrix} \middle| \begin{matrix} \nu \\ (\lambda-2, \mu+2) \end{matrix} \right\rangle \\ & \times \langle (\lambda, \mu) q L; (0, 4) 0 || (\lambda-2, \mu+2) q' L \rangle, \end{aligned} \quad (71)$$

where the entering $SO(6)$ -reduced matrix elements are given by Eq. (60). The $SO(3)$ -reduced matrix elements of the tensor operator $G^2(b, b) \cdot F^2(a, a) = \frac{2\sqrt{5}}{3} [G^2(b, b) \times F^2(a, a)]_{4100}^4$ are similarly expressed. We recall that the corresponding $SO(6) \approx SU(4) \supset SU(3)$ and $SU(3) \supset SO(3)$ isoscalar factors are computed numerically by using the existing codes [37] and [35, 36], respectively.

We diagonalize the full Hamiltonian consisting of Eqs. (67) and (70) in the collective subspace spanned by the $SO(6)$ irreducible representation ν_0 . Additionally, because there is a prolate-oblate degeneracy related with the conjugate $SU_{pn}(3)$ multiplets (λ, μ) and (μ, λ) contained within the corresponding $SO(6)$ irreducible representations (cf. Table 1 of Ref. [10]), we consider only the $SU(3)$ multiplets (λ, μ) with $\lambda \geq \mu$ in the diagonalization.

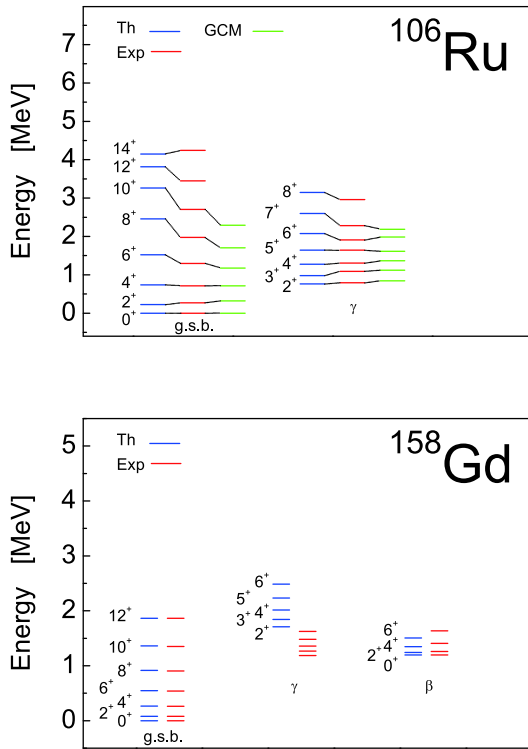


Fig. 3. (color online) Comparison of the excitation energies of the ground and γ bands in ^{106}Ru and those of the ground, β , and γ bands in ^{158}Gd via experiments according to Eqs. (67) and (70). The values of the model parameters in MeV are as follows: $A=0$, $B=-0.0924$, $a=0.004$, $b=0$, and $h=-0.23$ for ^{106}Ru and $A=0$, $B=-0.0127$, $a=0.0134$, $b=0.0000093$, and $h=-0.1384$ for ^{158}Gd , respectively. For ^{106}Ru , the results of the general collective model (GCM) with a collective potential including terms up to the sixth order are given as well [58].

The diagonalization results for the low-lying excitation spectra in both nuclei are shown in Fig. 3; the intraband $B(E2)$ transition strengths between the states of the ground band in ^{158}Gd are shown in Fig. 2 with the continuous blue curve. We observe that the agreement on the transition probabilities, compared to the pure $SU(3)$ limit, is improved. For ^{106}Ru , the experimental value $B(E2; 2_1^+ \rightarrow 0_1^+) = 66$ W.u. [49] is also reproduced in the diagonalization without using an effective charge.

In Figs. 4 and 5 we present the $SU(3)$ decomposition of the wave functions of the ground and γ bands in ^{106}Ru , and the ground, β , and γ bands in ^{158}Gd for different angular momentum values. These figures show that the $SU(3)$ symmetry is severely broken; however, the mixing of different $SU(3)$ multiplets is performed in an approximately coherent way in which the squared amplitudes are almost L -independent. This is accurate for the low angular momenta for which the Coriolis and centrifugal forces are not significantly strong, especially for the case of a well deformed nucleus ^{158}Gd . An interesting picture

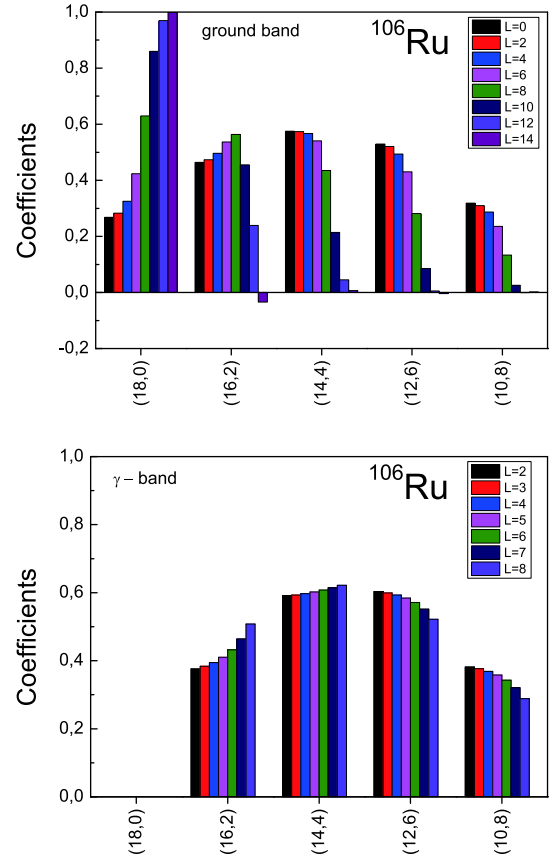


Fig. 4. (color online) The $SU(3)$ decomposition of the wave functions of the ground and γ bands in ^{106}Ru for different angular momentum values.

for the $SU(3)$ decomposition coefficients is observed for ^{158}Gd , which resembles the eigenfunctions of the simple one-dimensional harmonic oscillator given by the Hermite polynomials. Figures 4 and 5 thus indicate a new kind of symmetry, called quasi-dynamical symmetry [59]. This symmetry is associated with the mathematical concept of embedded representations [60]. We also note that all the $SU(3)$ multiplets that contribute to the structure of collective states, for both nuclei, belong to a single $SO(6)$ irreducible representation – namely that of maximum seniority ν_0 of the corresponding bandhead structure. Thus, the results for the microscopic structure of the rotational states of the lowest collective bands in the two nuclei under consideration reveal, in addition to the good $SO(6)$ symmetry, the presence also of an approximate $SU(3)$ quasi-dynamical symmetry for the low angular momenta, in the sense given in Refs. [27, 59].

V. CONCLUSIONS

The purpose of this work is to develop a computational technique for the shell-model reduction chain $Sp(12, R) \supset SU(1, 1) \otimes SO(6) \supset U(1) \otimes SU_{pn}(3) \otimes SO(2) \supset SO(3)$ of the PNSM. This dynamical symmetry limit was re-

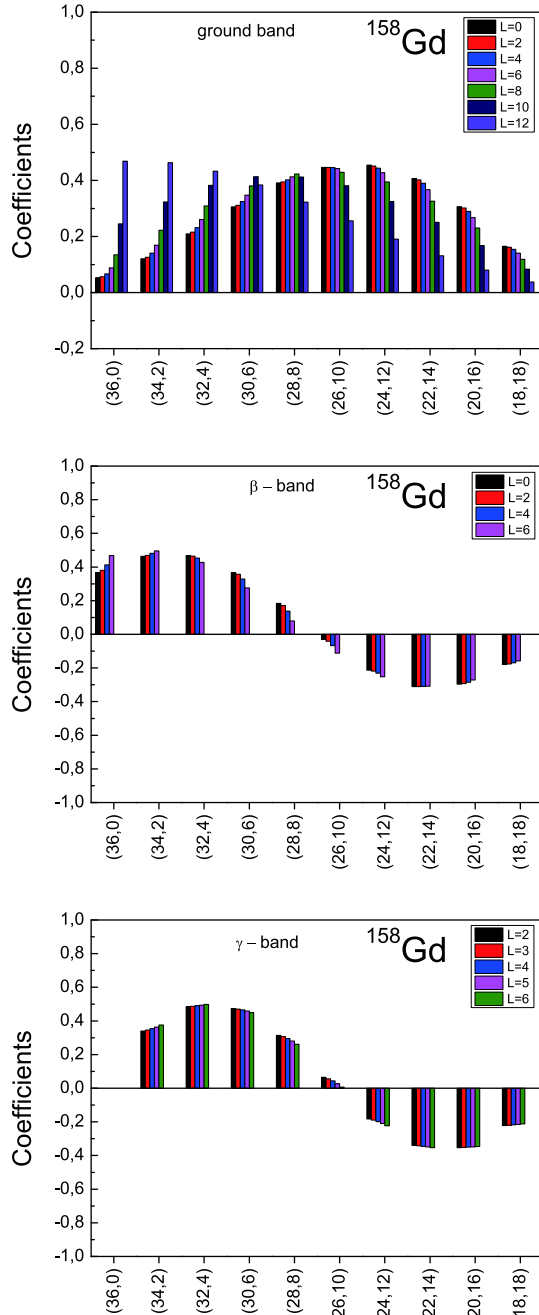


Fig. 5. (color online) The $SU(3)$ decomposition of the wave functions of the ground, β , and γ bands in ^{158}Gd for different angular momentum values.

cently shown to correspond to a microscopic shell-model version of the generalized quadrupole-monopole Bohr-Mottelson collective model by embedding the latter in the microscopic shell-model theory [10].

The shell-model coupling scheme, defined by the considered reduction chain, provides an alternative basis for the diagonalization of different Hamiltonians of the algebraic and polynomial type in the many-particle position and momentum coordinates of the nuclear system. In the present paper, however, we consider simple algebraic interactions, which are expressed as simple functions of the symplectic generators of $Sp(12, R)$ dynamical group. Closed analytical expressions along the $Sp(12, R) \supset SU(1, 1) \otimes SO(6) \supset U(1) \otimes SU_{pn}(3) \otimes SO(2) \supset SO(3)$ dynamical symmetry chain are obtained for the matrix elements of the basic building blocks of the PNSM and the $Sp(12, R)$ symplectic generators, which in turn allows computing the matrix elements of other physical operators of interest. In particular, the reduced matrix elements of the tensor operators $A^2(\alpha, \beta) \cdot F^2(\alpha, \beta) \simeq [A^2(\alpha, \beta) \times F^2(\alpha, \beta)]_{\gamma=\pm 2100}^{\nu=2}$ and $F^2(\alpha, \beta) \cdot F^2(\alpha, \beta) \simeq [F^2(\alpha, \beta) \times F^2(\alpha, \beta)]_{\pm 4100}^4$, which couple the microscopic shell-model states from different oscillator shells, are also explicitly provided. The matrix elements of other, more complicated interactions can be obtained utilizing the analytical expressions presented here. In this way, the computational technique developed in the present paper generally provides us with the required algebraic tool for performing realistic symplectic-based shell-model calculations of nuclear collective excitations.

Two simple examples are given which illustrate the application of the theory to the determination of the microscopic shell-model structure of the observed low-energy rotational states in ^{106}Ru and ^{158}Gd , exhibiting different collectivity in their spectra. The $SU(1, 1) \otimes SO(6)$ shell-model coupling scheme of the PNSM will extend its applicability in describing the collective properties of different heavy mass nuclei. This variant of the PNSM provides an interesting and relevant framework for exploring the nuclear collective dynamics. More detailed application of the proton-neutron symplectic-based shell-model theory to the description of low-lying excited states in different nuclei with varying collective properties will be presented elsewhere.

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