High-quality grand unified theories with three generations*

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Abstract: We extend the unitary groups beyond the $SU(5)$ and $SU(6)$ to determine possible grand unified theories that give rise to three-generational Standard Model fermions without simple repetitions. By demanding asymptotic free theories at short distances, we find gauge groups of $SU(7)$, $SU(8)$, and $SU(9)$, together with their anomaly-free irreducible representations, are such candidates. Two additional gauge groups of $SU(10)$ and $SU(11)$ can also achieve the generational structure without asymptotic freedom. We also deduce that these models can solve the Peccei-Quinn (PQ) quality problem, which is intrinsic in the axion models, with the leading PQ-breaking operators determined from the symmetry requirement.

Keywords: GUT, flavor, Peccei-Quinn quality

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I. INTRODUCTION

Grand unified theories (GUTs) [1, 2] were proposed to unify all fundamental interactions described by the standard model (SM). In addition to the aesthetic aspect of achieving the gauge coupling unification in its supersymmetric (SUSY) extension [3], it is pragmatic to conjecture the zeroth law of GUT, namely, *a successful GUT could address all intrinsic SM puzzles and as many physical issues beyond the SM as possible, with all necessary but minimal set of fields determined by symmetry.* One of such longstanding puzzles that has not been well answered is the existence of three generational SM fermions, as well as their mass hierarchies in the framework of GUTs. Some of the previous efforts in addressing the SM fermion masses in GUTs include Refs. [4-14]. In a seminal paper [15], Georgi suggested to extend the minimal $SU(5)$ into larger simple Lie groups $SU(N)$ (with $N \geq 7$), and developed his three laws of GUTs. Instead of the simple repetition of a set of anomaly-free irreducible representations (irreps) three times, it is argued that the three-generational structure arises from different anti-symmetric irreps of the $SU(N)$.

In this paper, we investigate the possible non-minimal GUTs beyond the $SU(5)$ and $SU(6)$ that can give rise to three-generational SM fermions. The number of generations can be easily obtained according to the counting method in terms of the $SU(5)$ irreps as given in Ref. [15]. It turns out that the SM fermion generations n_g can already become three or beyond for the $SU(7)$ group [16]. Historically, the number of the SM fermion generations n_g was also considered to be beyond three [17]. Meanwhile, the direct searches for the fourth-generational quarks at the Large Hadron Collider (LHC) have already excluded this possibility [18-20]. Therefore, only the non-minimal GUTs with their anomaly-free irreps that lead to $n_g = 3$ cases will be considered in our study. In Georgi's third law, he decided that no individual irrep of the GUT group should appear more than once. Accordingly, he found that the minimal GUT group that give rise to $n_g = 3$ is $SU(11)$, with a total number of 1023 left-handed fermions [15]. Obviously, the third law prevent the three-generational structure through the simple repetition of the anomaly-free irreps. However, this may be a too strong constraint and was not usually adopted in the later studies. In our discussions, we modify Georgi's third law in a different version proposed by Christensen and Shrock [21] in the study on the dynamical origin of the SM fermion masses. A different perspective can be formed, such that the global symmetries can usually emerge once the original third law was abandoned, such as in the $SU(9)$ GUT [22]. This can be advantageous at least in two aspects. In the first advantage, the emergent global symmetry, with its breaking, can be a mechanism that elucidated the lightness of the Higgs boson, as discussed by Dvali [23] in the context of the SUSY $SU(6)$. The other advantage is

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that the global $U(1)$ symmetry can be identified as the Peccei-Quinn (PQ) symmetry [24] for the strong CP problem. The emergent PQ symmetry, together with both the gauge and the global symmetries, can usually constrain the mass dimensions of the PQ-breaking operators and lead to a high-quality axion [22, 25-29]. Two recent examples include the axion from the $SO(10)$ [30] and the SUSY $SU(6)$ GUT [31].

The remainder of this paper is organized as follows. In Sec. II, we review Georgi's guidelines for building the non-minimal GUTs that can lead to three generations of SM fermions without simple repetition. Some other relevant results of the gauge anomaly cancellation, Higgs representations, and PQ quality are also setup in this section. Sec. III presents the core of this work. We analyze all possible $SU(N)$ GUTs (up to $SU(11)$) and their anomaly-free fermion contents that can lead to three generational SM fermions according to Georgi's counting. The PQ charge assignments to Higgs fields and the corresponding PQ-breaking operators will be presented. We summarize our results and make discussions in Sec. IV.

II. SOME GENERAL REMARKS

A. Lie group representations and Georgi's guidelines

To facilitate the discussion, we express the fermion representations under the $SU(N)$ GUT group in terms of the set of rank-k anti-symmetric irrep of $[N,k]$ as follows

$$\{f_{\rm L}\}_{SU(N)} = \sum_{k=0}^{N-1} n_k\,[N,k]\,, \tag{1}$$

with n_k being the multiplicity. Obviously, $k=0$ corresponds to the singlet representation, and $k=1$ corresponds to the fundamental representation, etc. The singlet representations contribute neither to the gauge anomaly, nor to the renormalization group equations (RGEs). Throughout the discussion, we always denote the conjugate representation, such that $\overline{[N,k]} = [N,N-k]$. It will be also useful to use a compact vector notation of

$$\vec{n} \equiv (n_0\,,\ldots,n_{N-1})\,. \tag{2}$$

For a given rank-k anti-symmetric irrep of $[N,k]$, its dimension and trace invariants are

$$\dim([N,k]) = \frac{N!}{k!(N-k)!}\,, \tag{3}$$

$$T([N,k]) = \frac{(N-2)!}{2(k-1)!(N-k-1)!}\,. \tag{4}$$

From Cartan's classification, it is well-known that the only possible Lie groups for non-minimal GUTs beyond the $SU(5)$ or $SO(10)$ are

$$SU(N)\ (N\geq 6)\,,\quad SO(4k+2)\ (k\geq 3)\,,\quad E_6\,. \tag{5}$$

Because the exceptional E_6 group has a fixed rank, it is impossible to consider further extensions. For any irrep under these Lie groups, one can always decompose it under the subgroup of the $SU(5)$. For example, the fundamental representation of the $SU(N)$ can be decomposed as

$$[N,1] = (N-5)\times[5,0]\oplus[5,1]\,. \tag{6}$$

The decompositions of the higher irreps can be obtained by `LieART` [32]. For an $SU(N)$ GUT, its fermion contents can be generally decomposed in terms of the $SU(5)$ irreps as follows

$$\{f_{\rm L}\}_{SU(N)} = n_0\,[5,0]+n_1\,[5,1]+n_2\,[5,2]+n_3\,[5,3]+n_4\,[5,4]\,. \tag{7}$$

The anomaly cancellation condition leads to the following relation to the multiplicities

$$n_1+n_2 = n_3+n_4\,. \tag{8}$$

In Ref. [15], Georgi argued that the counting of the SM fermion generations is equivalent to the counting of the multiplicity of the residual $SU(5)$ irreps of $[5,2]\oplus[5,4]$, which is

$$n_g = n_2-n_3 = n_4-n_1\,. \tag{9}$$

Note that the counting of the SM fermion generations in Eq. (9) does not rely on the realistic gauge symmetry breaking patterns. Based on Georgi's counting, it turned out that any GUT with orthogonal groups larger than the $SO(10)$ essentially leads to $n_g=0$. This can be understood by decomposing the 16-dimensional $SO(10)$ Weyl fermions under the $SU(5)$ as

$$\mathbf{16_F} = \mathbf{1_F}\oplus\overline{\mathbf{5_F}}\oplus\mathbf{10_F}\,. \tag{10}$$

The Weyl fermions from larger orthogonal groups are always decomposed under the $SO(10)$ in pairs of $\mathbf{16_F}\oplus\overline{\mathbf{16_F}}$, and this can only lead to $n_g=0$.

Georgi's third law requires that not any representation of $[N,k]$ should appear more than once, which means $n_k=0$ or $n_k=1$ in Eq. (1). This leads to a consequence that no global symmetry can emerge from the corresponding fermion setup. Instead, we adopt an alternative criterion by Christensen and Shrock [21], namely, the greatest common divisor of $\{n_k\}$ is not greater than unity. Therefore, one can expect the global symmetry of

$$\mathcal{G}_{\rm global}=\prod_{\{k'\}}[SU(n_{k'})\otimes U(1)_{k'}],\tag{11}$$

for all irreps of $[N,k']$ that appear more than once. This can be viewed as a generalization of the global symmetry in the rank-2 anti-symmetric theory of $SU(N+4)$ by Dimopoulos, Raby, and Susskind (DRS) [33]. The $U(1)$ components of the global symmetry (11) can be identified as the global PQ symmetry, which are likely to lead to high-quality axion [31]. In this regard, the modified criterion of the fermion assignments is likely to solve the long-standing PQ quality problem [22, 25-29] in the framework of GUT.

B. Gauge anomaly cancellation

To have an anomaly-free non-minimal GUT, we have to solve the following Diophantine equation

$$\vec{n}\cdot\vec{A}=0,\tag{12}$$

with the N-dimensional anomaly vector [34, 35] being

$$\vec{A}=(A([N,0]),A([N,1]),\ldots,A([N,N-1])),\tag{13}$$

$$A([N,0])=0,\ \ A([N,k])=\frac{(N-2k)(N-3)!}{(N-k-1)!(k-1)!}\ \ (k>0).\tag{14}$$

The property that the anomaly of a given irrep and its conjugate cancel each other is apparent in Eq. (14). In addition, the self-conjugate representations must be anomaly-free, such that $A\left(\left[N,\frac{N}{2}\right]\right)=0$ for N being even. Thus, the anomaly vector can be expressed as

$$\vec{A}=\left(0,A([N,1]),\ldots,A\left(\left[N,\left[\frac{N}{2}\right]\right]\right),\right.$$
$$\left.-A\left(\left[N,\left[\frac{N}{2}\right]\right]\right),\ldots,-A([N,1])\right).\tag{15}$$

In practice, one has to decompose the $SU(N)$ fermion representations from $[N,1]$ to $\left[N,\left[\frac{N}{2}\right]\right]$ under the $SU(5)$, to count the generations.

C. The Higgs representations

Once the fermion contents are determined for a particular non-minimal GUT, the Higgs fields can be determined by the following criteria

1. The GUT symmetry breaking of $SU(N)\to SU(m)\otimes SU(N-m)\otimes U(1)$ with $m=\left[\frac{N}{2}\right]$ is always assumed at its first stage, which requires an adjoint Higgs field [36]. The other possible symmetry breaking of $SU(N)\to SU(N-1)$ (with $N\geq 6$) at the first stage is very likely to lower the proton lifetime predictions, and thus bring tension with the current experimental constraint to the proton lifetime from the Super-Kamionkande [37].

2. All possible gauge-invariant Yukawa couplings, which also respect the global symmetry in Eq. (11), can be formed.

3. Higgs fields for achieving any intermediate symmetry breaking stages are necessary, where their proper irreps contain the SM-singlet directions.

4. Only the Higgs fields with the minimal dimensions are taken into account.

Before proceeding to the more realistic models, we display the Higgs fields in the $SU(6)$ GUT as an example. Its minimal anomaly-free fermion contents and decomposition under the $SU(5)$ are

$$\{f_L\}_{SU(6)}=[6,2]\oplus 2\times[6,5]=\mathbf{15_F}\oplus 2\times\overline{\mathbf{6_F}}$$
$$=2\times[5,0]\oplus[5,1]\oplus[5,2]\oplus 2\times[5,4].\tag{16}$$

The $[5,1]$, one of the $[5,4]$, as well as two singlets of $[5,0]$, obtain their masses at an intermediate symmetry-breaking scale. The remaining fermions contain precisely one generational SM fermions of $[5,2]\oplus[5,4]\approx[5,2]\oplus\overline{[5,1]}$. Apparently, the minimal $SU(6)$ GUT has a global symmetry of $SU(2)_{\overline{\mathbf{6_F}}}\otimes U(1)_{\rm PQ}$ and is a one-generational model according to Georgi's counting. The gauge-invariant and $SU(2)_{\overline{\mathbf{6_F}}}$-invariant Yukawa coupling can be expressed as follows:

$$-\mathcal{L}_Y=\overline{\mathbf{6_F}}^{\rho}\mathbf{15_F}\overline{\mathbf{6_{H}}}_{\rho}+\mathbf{15_F}\mathbf{15_F}\mathbf{15_H}$$
$$+\epsilon_{\rho\sigma}\overline{\mathbf{6_F}}^{\rho}\overline{\mathbf{6_F}}^{\sigma}(\mathbf{15_H}+\mathbf{21_H})+{\rm H.c.},\tag{17}$$

with the minimal set of Higgs fields. Of course, a 35-dimensional adjoint Higgs field is necessary to achieve the first-stage GUT symmetry breaking of $SU(6)\to SU(3)_c\otimes SU(3)_W\otimes U(1)_X$ [31, 38, 39]. It turns out that the VEVs from Higgs fields of $\left(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\rho}\subset\overline{\mathbf{6_H}}_{\rho}$, $\left(\mathbf{1},\mathbf{6},+\frac{2}{3}\right)_{\mathbf{H}}\subset\mathbf{21_H}$ can be responsible for the intermediate symmetry breaking of $SU(3)_c\otimes SU(3)_W\otimes U(1)_X\to SU(3)_c\otimes SU(2)_W\otimes U(1)_Y$ [31].

D. The asymptotic freedom (AF)

The GUTs with their earliest versions are usually asymptotically free above the unification scale. However, there was no definite answer to whether the AF should be retained. An alternative criterion is to have an asymptotic safe theory, which reaches a fixed point at the short distance [40, 41]. In general, the analysis of the asymptotic safe theories involves the RGEs of gauge couplings, as well as Yukawa and Higgs self couplings. This can only be performed for the individual theory by specifying the symmetry breaking patterns. In the non-minimal GUTs, the AF is likely to be violated because the trace invariants of the rank-2 and rank-3 anti-symmetric representations scale as $T([N,2])\sim N$ and $T([N,3])\sim N^2$. Previously, this was also considered in the $SU(11)$ model [42], but with only fermions taken into account. In our discussions below, we study the short-distance behavior for non-minimal GUTs up to $SU(11)$, with their minimal fermion setup. It turns out that the minimal models in $SU(10)$ and $SU(11)$ violate the AF, and thus careful analysis of their unification couplings and scales are required for these two cases. The one-loop β coefficients are obtained by including both fermions and Higgs fields as follows

$$b_1 = -\frac{11}{3}C_2(\mathcal{G}) + \frac{4}{3}\kappa\sum_f T(\mathcal{R}_f) + \frac{1}{3}\eta\sum_s T(\mathcal{R}_s), \qquad (18)$$

with $\kappa = 1\,(1/2)$ for Dirac (Weyl) fermions, and $\eta = 1\,(1/2)$ for complex (real) scalars. For the adjoint Higgs fields, we always consider them to be real for the non-SUSY case. The AF can be determined by whether $b_1 < 0$ or not.

E. The PQ quality and axion

The global PQ symmetry has an intrinsic problem known as the PQ quality [22, 25-29]. In general, global symmetries are not fundamental, but emerge with the underlying gauge theories. They are believed to be broken by quantum gravity effects in the form of the following dimension-$2m+n$ operator

$$\mathcal{O}_{\not{\rm PQ}}^{d=2m+n} = k\frac{|\Phi|^{2m}\Phi^n}{M_{\rm pl}^{2m+n-4}}. \qquad (19)$$

The size of the PQ-breaking is constrained such that the minima of the QCD effective potential induced by axion should satisfy $|\langle a/f_a\rangle| \lesssim 10^{-10}$, which generates a PQ quality constraint of

$$\frac{f_a^d}{M_{\rm pl}^{d-4}} \lesssim 10^{-10}\,\Lambda_{\rm QCD}^4. \qquad (20)$$

Consequently, the mass dimension in Eq. (19) should be $d \gtrsim 9$, to obtain a reasonable axion decay constant $f_a \sim \mathcal{O}(10^{12})$ GeV without excessively fine tuning the coefficient k [26-28] in Eq. (19). Without knowing the underlying symmetry origin of the Φ field, there is generally no reason to forbid any PQ-breaking operators with $d \lesssim 9$.

Previous studies of the axion in the GUT [43-46] were made in both $SU(5)$ and $SO(10)$, where the global PQ symmetry was introduced manually. Therefore, the issue of PQ quality remains present. Recent discussions on the PQ quality problem in the frame of GUT include the $SO(10)$ [30] and $SU(6)$ [31] cases. In the $SO(10)$ GUT, the author adopted the generational symmetry in the limit of vanishing Yukawa couplings. A dimension-9 gauge-invariant operator for generating a high-quality axion was found, which is made up of Higgs fields for the intermediate symmetry breaking. In the minimal $SU(6)$ GUT, it already possesses a global DRS symmetry, as expressed in Eq. (11). With the SUSY extensions, the authors [31] found a dimension-6 operator that led to a high-quality axion. Therefore, it becomes suggestive that the GUTs beyond the minimal versions are likely to solve the PQ-quality problem, with their local and emergent global symmetries. In the context of GUTs, the PQ-breaking operators can be formed by Higgs fields that develop vacuum expectation values (VEVs) at both the electroweak (EW) scale of $v_{\rm EW}$ and the PQ symmetry-breaking scale of f_a. This further alleviates the PQ quality constraint in Eq. (20), even when $d < 9$. For example, in the minimal SUSY $SU(6)$ GUT [31], such a PQ-breaking operator turns out to be $\mathcal{O}_{\rm cancel\not{PQ}}^{d=6} = (\epsilon_{\alpha\beta}\overline{\mathbf{6_H}}^{\alpha}\overline{\mathbf{6_H}}^{\beta}\mathbf{15_H})^2$, with $\alpha = 1,2$. The PQ quality constraint can be fulfilled when setting $\langle\overline{\mathbf{6_H}}^{1}\rangle \sim \langle\mathbf{15_H}\rangle \simeq v_{\rm EW}$ and $\langle\overline{\mathbf{6_H}}^{2}\rangle \simeq f_a$. A natural question can be raised on whether the PQ quality constraint can be generally satisfied in the non-minimal GUTs with $n_g = 3$. Remarkably, we find this generally holds with proper assignment of the PQ charges to the Higgs fields.

The probes of the axion rely on the axion-photon effective coupling of

$$C_{a\gamma\gamma} = \frac{E}{N_{SU(3)_c}} - 1.92. \qquad (21)$$

For the GUTs, there is a universal prediction to the factor $E/N_{SU(3)_c} = 8/3$. The color anomaly factor of $N_{SU(3)_c}$ relates the axion decay constant with the associate symmetry-breaking scale as $v_{\rm SB} = |2N_{SU(3)_c}|f_a$, and also determines the domain wall number as $N_{\rm DW} = 2N_{SU(3)_c}$. In practice, it is unnecessary to derive the factor by analyzing the symmetry breaking patterns. Instead, this can be obtained by using the 't Hooft anomaly matching condition [47] of

$$N_{SU(3)_c} = N_{SU(N)} = \sum_{\mathbf{F}} T(\mathcal{R}_{\mathbf{F}})\mathrm{PQ}(\mathcal{R}_{\mathbf{F}}). \qquad (22)$$

Notice that in our current study, the physical axion does not emerge at the GUT scale of $\sim 10^{16}$ GeV. Instead, it originates from the phases of Higgs fields that are responsible for the intermediate symmetry breaking scale, with necessary orthogonality conditions imposed. One of such examples can be found in the minimal SUSY $SU(6)$ GUT [31]. We focus on the PQ-breaking operators in the non-minimal GUTs, while the constructions of the physical axion in the specific GUT model will be left for future work.

III. THE RESULTS

In this section, we obtain our results for the $SU(N)$ GUTs that lead to $n_g = 3$. Examples include $SU(7)$, $SU(8)$, and $SU(9)$ groups, where the AF can be achieved. We also deduce that the higher groups of $SU(10)$ and $SU(11)$ with their minimal irreps cannot achieve the AF condition. For each case, we also look for the possible gauge-invariant and PQ-breaking operators. With proper PQ charge assignment at the GUT scale, we demonstrate that the PQ quality problem can be generally avoided in each model.

A. The $SU(7)$

For the $SU(7)$ group, the anomaly vector in Eq. (13) is reads:

$$\vec{A} = (0,1,3,2,-2,-3,-1). \tag{23}$$

The decompositions of the $SU(7)$ irreps under the $SU(5)$ are expressed as:

$$\mathbf{7} = [7,1] = 2\times[5,0]\oplus[5,1], \tag{24a}$$

$$\mathbf{21} = [7,2] = [5,0]\oplus 2\times[5,1]\oplus[5,2], \tag{24b}$$

$$\mathbf{35} = [7,3] = [5,1]\oplus 2\times[5,2]\oplus[5,3.]. \tag{24c}$$

There are two possibilities for $n_g = 3$, namely,

$$\begin{aligned}\{f_L\}^{\rm A}_{SU(7)} =& 2\times[7,2]\oplus[7,3]\oplus 8\times[7,6]\\ =& 2\times\mathbf{21_F}\oplus\mathbf{35_F}\oplus 8\times\overline{\mathbf{7}}_{\mathbf{F}}, \quad \dim_{\mathbf{F}} = 133,\\ \mathcal{G}^{\rm A}_{\rm global} =& [SU(8)_{\overline{7}_F}\otimes U(1)_{\rm PQ}]\otimes[SU(2)\otimes U(1)'],\end{aligned} \tag{25a}$$

$$\begin{aligned}\{f_L\}^{\rm B}_{SU(7)} =& [7,2]\oplus 2\times[7,3]\oplus 7\times[7,6]\\ =& \mathbf{21_F}\oplus 2\times\mathbf{35_F}\oplus 7\times\overline{\mathbf{7}}_{\mathbf{F}}, \quad \dim_{\mathbf{F}} = 140,\\ \mathcal{G}^{\rm B}_{\rm global} =& [SU(7)_{\overline{7}_F}\otimes U(1)_{\rm PQ}]\otimes[SU(2)\otimes U(1)'].\end{aligned} \tag{25b}$$

Because the number of fermions in two cases only differ by less than 10, we determine to consider both possibilities. Note that a recent study [48] suggested that the $SU(7)$ model can suppress the proton decay with the proper embedding of the SM fermions.

The Higgs sector of two $SU(7)$ models is determined by the fermions and global symmetries in Eq. (25) as follows

$$\{H\}^{\rm A}_{SU(7)} = 8\times\overline{\mathbf{21_H}}\oplus\mathbf{7_H}\oplus 2\times\mathbf{21_H}\oplus\mathbf{35_H}[\oplus\mathbf{48_H}], \tag{26a}$$

$$\{H\}^{\rm B}_{SU(7)} = 7\times\overline{\mathbf{7}}_{\mathbf{F}}\oplus\mathbf{7_H}\oplus 2\times\mathbf{21_H}\oplus\mathbf{35_H}[\oplus\mathbf{48_H}]. \tag{26b}$$

Here and below, we adopt the square brackets to denote the real adjoint Higgs fields for the GUT scale symmetry breaking. By using the fermions and Higgs fields in Eqs. (25) and (26), we deduce that $b_1^{\rm A} = -5$ and $b_1^{\rm B} = -\frac{55}{6}$. Hence, both the $SU(7)$-A and the $SU(7)$-B model are asymptotically free. The gauge-invariant Yukawa couplings are

$$\begin{aligned}-\mathcal{L}_{\rm Y}^{\rm A} =& \sum_{\rho=1}^{8}\overline{\mathbf{7}}_{\mathbf{F}}^{\rho}\mathbf{35_F}\overline{\mathbf{21}_{\mathbf{H}\rho}} + \sum_{\dot\rho=1,2}\mathbf{21}_{\mathbf{F}}^{\dot\rho}\mathbf{35_F}\mathbf{21}_{\mathbf{H}\dot\rho}\\ &+\epsilon_{\dot\rho\dot\sigma}\mathbf{21_F}^{\dot\rho}\mathbf{21_F}^{\dot\sigma}\mathbf{35_H} + \mathbf{35_F}\mathbf{35_F}\mathbf{7_H} + {\rm H.c.},\end{aligned} \tag{27a}$$

$$\begin{aligned}-\mathcal{L}_{\rm Y}^{\rm B} =& \sum_{\rho=1}^{7}\overline{\mathbf{7}}_{\mathbf{F}}^{\rho}\mathbf{21_F}\overline{\mathbf{7}_{\mathbf{H}\rho}} + \sum_{\dot\rho=1,2}\mathbf{35}_{\mathbf{F}}^{\dot\rho}\mathbf{21_F}\mathbf{21}_{\mathbf{H}\dot\rho}\\ &+\epsilon_{\dot\rho\dot\sigma}\mathbf{35_F}^{\dot\rho}\mathbf{35_F}^{\dot\sigma}\mathbf{7_H} + \mathbf{21_F}\mathbf{21_F}\mathbf{35_H} + {\rm H.c.}.\end{aligned} \tag{27b}$$

We assign the PQ charges for all $SU(7)$ fermions and Higgs fields in Table 1. The PQ charges cannot be uniquely determined from the PQ neutrality of the Yukawa couplings (27). Therefore, we assign the PQ charges by removing the possible dangerous PQ-breaking operators with low mass dimensions. In $SU(7)$-A, one may assign ${\rm PQ}(\mathbf{21_F}) = q_1$ and ${\rm PQ}(\mathbf{35_F}) = q_2$. Accordingly, it is easy to find two following PQ-breaking operators

$$\begin{aligned}\mathcal{O}_{\rm PQ}^{d=5} =& \epsilon^{\dot\rho\dot\sigma}\mathbf{21}_{\mathbf{H}\dot\rho}\mathbf{21}_{\mathbf{H}\dot\sigma}\mathbf{7_H}^3, \quad \Delta{\rm PQ} = -2q_1 - 8q_2\\ \mathcal{O}_{\rm PQ}^{d=3} =& \epsilon^{\dot\rho\dot\sigma}\mathbf{21}_{\mathbf{H}\dot\rho}\overline{\mathbf{35}_{\mathbf{H}\dot\sigma}}\mathbf{7_H}, \quad \Delta{\rm PQ} = -2q_1 - 4q_2.\end{aligned} \tag{28}$$

These two operators would better be PQ-neutral owing to their mass dimensions, and this leads to $q_1 = q_2 = 0$. Similarly, in the $SU(7)$-B and larger groups below, we find the corresponding PQ charge assignments.

The color and electromagnetic anomaly factors and domain wall numbers according to Eq. (22) are given by

Table 1. The PQ charge assignments and the representations of the fermions and Higgs fields under their global symmetries for two $SU(7)$ unification models.

$SU(7)$-A	$\overline{\mathbf{7_F}}$	$\mathbf{21_F}$	$\mathbf{35_F}$	$\overline{\mathbf{21_H}}$	$\mathbf{7_H}$	$\mathbf{21_H}$	$\overline{\mathbf{35_H}}$	$\mathbf{35_H}$
PQ charges	1	0	0	−1	0	0	0	0
$SU(8)_{\overline{\mathbf{7_F}}}$	$\square$	1	1	$\bar{\square}$	1	1	1	1
$SU(2)$	1	$\square$	1	1	1	$\bar{\square}$	$\bar{\square}$	1
$SU(7)-\mathrm{B}$	$\overline{\mathbf{7_F}}$	$\mathbf{35_F}$	$\mathbf{21_F}$	$\overline{\mathbf{7_H}}$	$\mathbf{35_H}$	$\overline{\mathbf{21_H}}$	$\mathbf{21_H}$	$\mathbf{7_H}$
PQ charges	1	0	0	−1	0	0	0	0
$SU(7)_{\overline{\mathbf{7_F}}}$	$\square$	1	1	$\bar{\square}$	1	1	1	1
$SU(2)$	1	$\square$	1	1	1	$\bar{\square}$	$\bar{\square}$	1

$$SU(7)-\mathrm{A}\ :\ N_{SU(3)_c}=4\,,\quad E=\frac{32}{3}\,,\quad N_{\rm DW}=8\,,\tag{29a}$$

$$SU(7)-\mathrm{B}\ :\ N_{SU(3)_c}=\frac{7}{2}\,,\quad E=\frac{28}{3}\,,\quad N_{\rm DW}=7\,,\tag{29b}$$

for two models. The leading gauge-invariant PQ-breaking operators become[1)]

$$SU(7)-\mathrm{A}\ :\mathcal{O}^{d=10}_{\Delta\mathrm{PQ}=-8}=(\overline{\mathbf{21_H}}_\rho)^8\cdot\mathbf{7_H}^2=\epsilon^{A_1\ldots A_7}\epsilon^{B_1\ldots B_7}\epsilon^{\rho_1\ldots\rho_8}\times(\overline{\mathbf{21_H}}_{\rho_1})_{[A_1B_1]}\ldots(\overline{\mathbf{21_H}}_{\rho_8})_{[A_8B_8]}\mathbf{7_H}^{[A_8}\mathbf{7_H}^{B_8]}\,,\tag{30a}$$

$$SU(7)-\mathrm{B}\ :\mathcal{O}^{d=7}_{\Delta\mathrm{PQ}=-7}=(\overline{\mathbf{7_H}}_\rho)^7=\epsilon^{A_1\ldots A_7}\epsilon^{\rho_1\ldots\rho_7}(\overline{\mathbf{7_H}}_{\rho_1})_{A_1}\ldots(\overline{\mathbf{7_H}}_{\rho_7})_{A_7}\,.\tag{30b}$$

For the leading PQ-breaking operator in the $SU(7)$-A model, its mass dimension of 10 can guarantee the PQ-quality constraint even if all Higgs fields of $\overline{\mathbf{21_H}}_\rho$ and $\mathbf{7_H}$ develop their VEVs at $\sim f_a$. For the $SU(7)$-B model, the leading PQ-breaking operator has a mass dimension 7. To obtain a consistent axion decay constant of $f_a\gtrsim 10^8\,\mathrm{GeV}$, it is necessary that one of the $\overline{\mathbf{7_H}}_\rho$ develops its VEV for the electroweak symmetry breaking (EWSB).

B. The $SU(8)$

For the $SU(8)$ group, the anomaly vector in Eq. (13) is reads:

$$\vec{A}=(0,1,4,5,0,-5,-4,-1)\,.\tag{31}$$

The decompositions of the $SU(8)$ irreps under the $SU(5)$ are expressed as:

$$\mathbf{8}=[8,1]=3\times[5,0]\oplus[5,1],\tag{32a}$$

$$\mathbf{28}=[8,2]=3\times[5,0]\oplus3\times[5,1]\oplus[5,2],\tag{32b}$$

$$\mathbf{56}=[8,3]=[5,0]\oplus3\times[5,1]\oplus3\times[5,2]\oplus[5,3].\tag{32c}$$

The possibility for $n_g=3$ with the minimal anomaly-free fermion content is given by

$$\begin{aligned}\{f_L\}_{SU(8)}&=[8,2]\oplus[8,3]\oplus9\times[8,7]\\&=\mathbf{28_F}\oplus\mathbf{56_F}\oplus9\times\overline{\mathbf{8_F}}\,,\mathrm{dim}_{\mathbf{F}}=156\,,\\\mathcal{G}_{\rm global}&=SU(9)_{\overline{\mathbf{8_F}}}\otimes U(1)_{\rm PQ}.\end{aligned}\tag{33}$$

The Higgs sector of the $SU(8)$ is determined by the fermions and the global symmetries in Eq. (33) as follows[2)]

$$\{H\}_{SU(8)}=9\times\overline{\mathbf{8_H}}\oplus9\times\overline{\mathbf{28_H}}\oplus\mathbf{28_H}\oplus\mathbf{56_H}\oplus\mathbf{70_H}[\oplus\mathbf{63_H}]\,.\tag{34}$$

By adopting the fermions and Higgs fields in Eqs. (33) and (34), we find that $b_1=-\frac{2}{3}<0$. Thus, the minimal $SU(8)$ model is asymptotically free. The gauge-invariant Yukawa couplings are

$$\begin{aligned}-\mathcal{L}_{\rm Y}=&\sum_{\rho=1}^{9}(\overline{\mathbf{8_F}}^\rho\mathbf{28_F}\overline{\mathbf{8_H}}_\rho+\overline{\mathbf{8_F}}^\rho\mathbf{56_F}\overline{\mathbf{28_H}}_\rho)\\&+\mathbf{56_F}\mathbf{56_F}\mathbf{28_H}+\mathbf{28_F}\mathbf{56_F}\mathbf{56_H}+\mathbf{28_F}\mathbf{28_F}\mathbf{70_H}+\mathrm{H.c.}\,,\end{aligned}\tag{35}$$

We assign PQ charges for all $SU(8)$ fields in Table 2, with the same argument in the $SU(7)$ models. The color and electromagnetic anomaly factors and the domain wall numbers according to Eq. (22) are given by

$$SU(8)\ :\ N_{SU(3)_c}=\frac{9}{2}\,,\quad E=12\,,\quad N_{\rm DW}=9\,.\tag{36}$$

Table 2. The PQ charge assignments and $SU(9)_{\rm F}$ representations of the fermions and Higgs fields in the minimal $SU(8)$ unification model.

	$9\times\overline{\mathbf{8_F}}$	$\mathbf{28_F}$	$\mathbf{56_F}$	$9\times\overline{\mathbf{8_H}}$	$9\times\overline{\mathbf{28_H}}$	$\mathbf{28_H}$	$\mathbf{56_H}$	$\mathbf{70_H}$
PQ charges	1	0	0	−1	0	0	0	0
$SU(9)_{\overline{\mathbf{8_F}}}$	$\square$	1	1	$\bar{\square}$	1	1	1	1

1) Throughout the context, we use the square brackets to anti-symmetrize the indices.
2) In the $SU(8)$, the 70-dimensional irrep is self-conjugate.

The leading PQ-breaking operators in the $SU(8)$ model is

$$SU(8)\ : \mathcal{O}^{d=12}_{\Delta PQ=-9} = (\overline{\mathbf{8}_{\mathbf{H}\rho}})^9 \cdot \mathbf{28_H} \cdot \mathbf{56_H} \cdot \mathbf{70_H} = \epsilon^{A_1\ldots A_8}\epsilon_{B_1\ldots B_8}\epsilon^{\rho_1\ldots\rho_8\delta}(\overline{\mathbf{8}_{\mathbf{H}\rho_1}})_{A_1}\ldots(\overline{\mathbf{8}_{\mathbf{H}\rho_8}})_{A_8}(\overline{\mathbf{8}_{\mathbf{H}\delta}})_{B_0} \times (\mathbf{28_H})^{[B_0B_1]}(\mathbf{56_H})^{[B_2B_3B_4]}(\mathbf{70_H})^{[B_5B_6B_7B_8]}. \tag{37}$$

C. The $SU(9)$

For the $SU(9)$ group, the anomaly vector in Eq. (13) is expressed as:

$$\vec{A} = (0,1,5,9,5,-5,-9,-5,-1). \tag{38}$$

The decompositions of the $SU(9)$ irreps under the $SU(5)$ are the following

$$\mathbf{9} = [9,1] = 4\times[5,0]\oplus[5,1], \tag{39a}$$

$$\mathbf{36} = [9,2] = 6\times[5,0]\oplus4\times[5,1]\oplus[5,2], \tag{39b}$$

$$\mathbf{84} = [9,3] = 4\times[5,0]\oplus6\times[5,1]\oplus4\times[5,2]\oplus[5,3], \tag{39c}$$

$$\mathbf{126} = [9,4] = [5,0]\oplus4\times[5,1]\oplus6\times[5,2]\oplus4\times[5,3]\oplus[5,4]. \tag{39d}$$

The possibility for $n_g=3$ with the minimal anomaly-free fermion content is given by

$$\{f_L\}_{SU(9)} = [9,3]\oplus9\times[9,8] = \mathbf{84_F}\oplus9\times\overline{\mathbf{9_F}},\quad \dim_{\mathbf{F}} = 165,\quad \mathcal{G}_{\rm global} = SU(9)_{\overline{\mathbf{9}_\mathbf{F}}}\otimes U(1)_{\rm PQ}. \tag{40}$$

Another possibility with more fermions of $[9,2]\oplus[9,4]\oplus10\times[9,8]$ will not be considered here. Larger fermion contents of $2\times[9,2]\oplus2\times[9,4]\oplus[9,6]\oplus11\times[9,8]$ were previously mentioned in Ref. [5].

The Higgs sector of the $SU(9)$ is determined by the fermions and the global symmetries in Eq. (40) as follows

$$\{H\}_{SU(9)} = 9\times\overline{\mathbf{36}_\mathbf{H}}\oplus\mathbf{84_H}[\oplus\mathbf{80_H}]. \tag{41}$$

By using the fermions and Higgs fields in Eqs. (40) and (41), we deduce that $b_1 = -\frac{15}{2} < 0$. Hence, the minimal $SU(9)$ model is asymptotically free. The gauge-invariant Yukawa couplings are

$$-\mathcal{L}_Y = \overline{\mathbf{9}_\mathbf{F}}^{\rho}\mathbf{84_F}\overline{\mathbf{36}_{\mathbf{H}\rho}} + \mathbf{84_F}\mathbf{84_F}\mathbf{84_H} + \text{H.c.}, \tag{42}$$

and we assign the PQ charges in Table 3. Naively, the PQ charge of the $\mathbf{84_H}$ can be arbitrary according to the Yukawa couplings (42). Consequently, a dimension-3 PQ-breaking operator of $\mathcal{O}^{d=3}_{\rm PQ} = (\mathbf{84_H})^3 = \epsilon_{A_1B_1C_1A_2B_2C_2A_3B_3C_3} \times(\mathbf{84_H})^{[A_1B_1C_1]}(\mathbf{84_H})^{[A_2B_2C_2]}(\mathbf{84_H})^{[A_3B_3C_3]}$ can emerge, which is dangerous from the dimensional counting. Therefore, we determine that $\mathrm{PQ}(\mathbf{84_H})=0$. The color and electromagnetic anomaly factors and the domain wall numbers according to Eq. (22) are given by

$$SU(9)\ :\ N_{SU(3)_c} = \frac{9}{2},\quad E=12,\quad N_{\rm DW}=9. \tag{43}$$

The leading dimension-9 PQ-breaking operator in the $SU(9)$ model is

$$\mathcal{O}^{d=9}_{\Delta PQ=-9} = (\overline{\mathbf{36}_\mathbf{H}})^9 = \epsilon^{A_1\ldots A_9}\epsilon^{B_1\ldots B_9}\epsilon^{\rho_1\ldots\rho_9}(\overline{\mathbf{36}_{\mathbf{H}\rho_1}})_{[A_1B_1]}\ldots(\overline{\mathbf{36}_{\mathbf{H}\rho_9}})_{[A_9B_9]}. \tag{44}$$

According to the dimension counting in Ref. [27], this is likely to produce a high-quality axion.

D. The $SU(10)$

For the $SU(10)$ group, the anomaly vector in Eq. (13) is expressed as:

Table 3. The PQ charge assignments and $SU(9)_{\rm F}$ representations of the fermions and Higgs fields in the minimal $SU(9)$ unification model.

	$9\times\overline{\mathbf{9}_\mathbf{F}}$	$\mathbf{84_F}$	$9\times\overline{\mathbf{36}_\mathbf{H}}$	$\mathbf{84_H}$
PQ charges	1	0	−1	0
$SU(9)_{\overline{\mathbf{9}_\mathbf{F}}}$	$\square$	1	$\overline{\square}$	1

$$\vec{A} = (0,1,6,14,14,0,-14,-14,-6,-1). \tag{45}$$

The decompositions of the $SU(10)$ irreps under the $SU(5)$ are the following

$$\mathbf{10} = [10,1] = 5\times[5,0]\oplus[5,1], \tag{46a}$$

$$\mathbf{45} = [10,2] = 10\times[5,0]\oplus5\times[5,1]\oplus[5,2], \tag{46b}$$

$$\mathbf{120} = [10,3] = 10\times[5,0]\oplus10\times[5,1]\oplus5\times[5,2]\oplus[5,3], \tag{46c}$$

$$\mathbf{210} = [10,4] = 5\times[5,0]\oplus10\times[5,1]\oplus10\times[5,2]\oplus5\times[5,3]\oplus[5,4]. \tag{46d}$$

The possibility for $n_g=3$ with the minimal anomaly-free fermion content is given by

$$\begin{aligned}\{f_L\}_{SU(10)} =& [10,3]\oplus[10,8]\oplus 8\times[10,9]\\ =& \mathbf{120_F}\oplus\overline{\mathbf{45_F}}\oplus 8\times\overline{\mathbf{10_F}}\,,\quad \dim_{\mathbf{F}}=245\,,\\ \mathcal{G}_{\rm global} =& SU(8)_{\overline{\mathbf{10_F}}}\otimes U(1)_{\rm PQ}\,.\end{aligned}\tag{47}$$

The Higgs sector of the $SU(10)$ is determined by the fermions and the global symmetries in Eq. (47) as follows

$$\{H\}_{SU(10)}=8\times\overline{\mathbf{45_H}}\oplus 8\times\mathbf{120_H}\oplus\overline{\mathbf{10_H}}\oplus\mathbf{210_H}[\oplus\mathbf{99_H}]\,.\tag{48}$$

By using the fermions and Higgs fields in Eqs. (47) and (48), we deduce that $b_1=223/6$. Hence, the minimal $SU(10)$ model is not asymptotically free.

The gauge-invariant Yukawa couplings are

$$\begin{aligned}-\mathcal{L}_Y =& \overline{\mathbf{10_F}}^{\rho}\mathbf{120_F}\overline{\mathbf{45_H}}_{\rho}+\overline{\mathbf{10_F}}^{\rho}\overline{\mathbf{45_F}}\mathbf{120_H}_{\rho}\\ &+(\overline{\mathbf{45_F}}\overline{\mathbf{45_F}}+\mathbf{120_F}\mathbf{120_F})\mathbf{210_H}\\ &+\overline{\mathbf{45_F}}\mathbf{120_F}\overline{\mathbf{10_H}}+\text{H.c.}\,,\end{aligned}\tag{49}$$

and we assign the PQ charges in Table 4. Consequently, three dimension-5 PQ-breaking operators of $\mathcal{O}^{d=5}_{\rm PQ}=(\overline{\mathbf{10_H}})^4\mathbf{210_H}$, $\mathcal{O}^{d=5}_{\rm PQ}=(\overline{\mathbf{10_H}})^2(\mathbf{210_H})^3$, and $\mathcal{O}^{d=5}_{\rm PQ}=(\mathbf{210_H})^5$ can emerge if $\text{PQ}(\overline{\mathbf{10_H}})\neq 0$ or $\text{PQ}(\mathbf{210_H})\neq 0$. Therefore, we determine that $\text{PQ}(\overline{\mathbf{10_H}})=0$ and $\text{PQ}(\mathbf{210_H})=0$. The color and electromagnetic anomaly factor and the domain wall numbers according to Eq. (22) are given by

$$\begin{aligned}SU(10)\ :\ N_{SU(3)_c} =& 4\,,\\ E =& \frac{32}{3}\,,\\ N_{\rm DW} =& 8\,.\end{aligned}\tag{50}$$

Two dimension-12 PQ-breaking operators in the $SU(10)$ model are expressed as:

$$\begin{aligned}\mathcal{O}^{d=12}_{\Delta\rm PQ=-8} =& (\overline{\mathbf{45_H}})^8(\overline{\mathbf{10_H}})^2(\mathbf{210_H})^2\\ =& \epsilon^{A_1B_1\ldots A_5B_5}\epsilon^{\rho_1\ldots\rho_8}(\overline{\mathbf{45_H}}_{\rho_1})_{[A_1B_1]}\ldots(\overline{\mathbf{45_H}}_{\rho_5})_{[A_5B_5]}\\ &\times(\overline{\mathbf{45_H}}_{\rho_6})_{[A_6B_6]}(\overline{\mathbf{45_H}}_{\rho_7})_{[A_7B_7]}(\overline{\mathbf{45_H}}_{\rho_8})_{[A_8B_8]}\\ &\times(\overline{\mathbf{10_H}})_{C_1}(\overline{\mathbf{10_H}})_{C_2}(\mathbf{210_H})^{[A_6A_7A_8C_1]}(\mathbf{210_H})^{[B_6B_7B_8C_2]}\,,\end{aligned}\tag{51a}$$

$$\begin{aligned}\mathcal{O}^{d=12}_{\Delta\rm PQ=-8} =& (\mathbf{120_H})^8(\overline{\mathbf{10_H}})^2(\mathbf{210_H})^2\\ =& \epsilon^{A_1\ldots A_8A_9A_{10}}\epsilon^{B_1\ldots B_8B_9B_{10}}\epsilon^{C_1\ldots C_8C_9C_{10}}\epsilon^{\rho_1\ldots\rho_8}\\ &\times(\mathbf{120_H}_{\rho_1})^{[A_1B_1C_1]}\ldots(\mathbf{120_H}_{\rho_8})^{[A_8B_8C_8]}(\overline{\mathbf{10_H}})_{D_1}\\ &\times(\overline{\mathbf{10_H}})_{D_2}(\mathbf{210_H})^{[D_1A_9B_9C_9]}(\mathbf{210_H})^{[D_2A_{10}B_{10}C_{10}]}\,.\end{aligned}\tag{51b}$$

Table 4. The PQ charge assignments and the $SU(8)_{\overline{\mathbf{10_F}}}$ representations of the fermions and Higgs fields in the minimal $SU(10)$ unification model.

	$8\times\overline{\mathbf{10_F}}$	$\overline{\mathbf{45_F}}$	$\mathbf{120_F}$	$8\times\overline{\mathbf{45_H}}$	$8\times\mathbf{120_H}$	$\overline{\mathbf{10_H}}$	$\mathbf{210_H}$
PQ charges	1	0	0	−1	−1	0	0
$SU(8)_{\overline{\mathbf{10_F}}}$	$\square$	1	1	$\overline{\square}$	$\overline{\square}$	1	1

E. The $SU(11)$

For the $SU(11)$ group, the anomaly vector in Eq. (13) is reads:

$$\vec{\mathcal{A}}=(0,1,7,20,28,14,-14,-28,-20,-7,-1)\,.\tag{52}$$

The decompositions of the $SU(11)$ irreps under the $SU(5)$ are expressed as:

$$\mathbf{11}=[11,1]=6\times[5,0]\oplus[5,1]\,,\tag{53a}$$

$$\mathbf{55}=[11,2]=15\times[5,0]\oplus 6\times[5,1]\oplus[5,2]\,,\tag{53b}$$

$$\begin{aligned}\mathbf{165} =& [11,3]=20\times[5,0]\oplus 15\times[5,1]\oplus 6\\ &\times[5,2]\oplus[5,3]\,,\end{aligned}\tag{53c}$$

$$\begin{aligned}\mathbf{330} =& [11,4]=15\times[5,0]\oplus 20\times[5,1]\oplus 15\\ &\times[5,2]\oplus 6\times[5,3]\oplus[5,4]\,,\end{aligned}\tag{53d}$$

$$\begin{aligned}\mathbf{462} =& [11,5]=7\times[5,0]\oplus 15\times[5,1]\oplus 20\\ &\times[5,2]\oplus 15\times[5,3]\oplus 6\times[5,4]\,.\end{aligned}\tag{53e}$$

The possibility for $n_g=3$ with the minimal anomaly-free fermion content is given by

$$\begin{aligned}\{f_L\}_{SU(11)} =& [11,3]\oplus 2\times[11,9]\oplus 6\times[11,10]\\ =& \mathbf{165_F}\oplus 2\times\overline{\mathbf{55_F}}\oplus 6\times\overline{\mathbf{11_F}}\,,\quad \dim_{\mathbf{F}}=341\,,\\ \mathcal{G}_{\rm global} =& [SU(6)_{\overline{\mathbf{11_F}}}\otimes U(1)_{\rm PQ}]\otimes[SU(2)\otimes U(1)']\,.\end{aligned}\tag{54}$$

The Higgs sector of the $SU(11)$ is determined by the fermions and the global symmetries in Eq. (54) as follows

$$\{H\}_{SU(11)}=6\times\overline{\mathbf{55_H}}\oplus 2\times\overline{\mathbf{11_H}}\oplus\mathbf{330_H}\oplus\mathbf{462_H}[\oplus\mathbf{120_H}]\,.\tag{55}$$

By using the fermions and Higgs fields in Eqs. (54) and (55), we deduce that $b_1=83/3$. The gauge-invariant Yukawa couplings are

$$-\mathcal{L}_Y = \sum_{\rho=1}^{6} \overline{\mathbf{11_F}}^{\rho} \mathbf{165_F} \overline{\mathbf{55_{H}}}_{\rho} + \sum_{\dot\rho=1,2} \overline{\mathbf{55_F}}^{\dot\rho} \mathbf{165_F} \overline{\mathbf{11_{H}}}_{\dot\rho} + \epsilon_{\dot\rho\dot\sigma} \overline{\mathbf{55_F}}^{\dot\rho} \overline{\mathbf{55_F}}^{\dot\sigma} \mathbf{330_H} + \mathbf{165_F}\mathbf{165_F}\mathbf{462_H} + \text{H.c.}. \quad (56)$$

We assign the PQ charges in Table 5. Consequently, a possible dimension-5 PQ-breaking operator of $\mathcal{O}_{\rm PQ}^{d=5} = (\mathbf{330_H})^3(\mathbf{462_H})^2$ may emerge. Accordingly, we consider $\mathrm{PQ}(\mathbf{330_H}) = 0$ and $\mathrm{PQ}(\mathbf{462_H}) = 0$. The color and electromagnetic anomaly factors and the domain wall numbers according to Eq. (22) are given by

$$SU(11)\ :\ N_{SU(3)_c} = 3\,,\quad E = 8\,,\quad N_{\rm DW} = 6\,. \quad (57)$$

The leading PQ-breaking operators in the $SU(11)$ model is deduced to be

$$\begin{aligned}\mathcal{O}_{\rm PQ}^{d=9} =& (\overline{\mathbf{55_H}})^6(\mathbf{330_H})^3\\ =& \epsilon_{A_1B_1\ldots A_6}\epsilon^{\rho_1\cdots\rho_6}(\overline{\mathbf{55_{H}}}_{\rho_1})_{[A_1B_1]}\ldots(\overline{\mathbf{55_{H}}}_{\rho_6})_{[A_6C]}\\ &\times \epsilon_{D_1E_1\ldots E_3F_3}(\mathbf{330_H})^{[D_1E_1F_1G_1]}(\mathbf{330_H})^{[D_2E_2F_2G_2]}\\ &\times (\mathbf{330_H})^{[D_3E_3F_3C]}\,. \end{aligned} \quad (58)$$

IV. CONCLUSIONS

We have studied a set of non-minimal GUTs that can lead to the observed three generational SM fermions according to Georgi's counting. With the origin of the generational structure, these models themselves are promising in answering the most puzzling question of the SM fermion mass hierarchies. Our results suggest four such models that achieve the AF property at short distances, and two more that may be considered with further studies. The obtained results are summarized in Table 6. The other important feature of the non-minimal GUTs in our study originates from their global symmetries, which was also previously noted in the $SU(6)$ model [23, 31]. Although the $SU(6)$ model benefits from a global DRS symmetry, the SUSY extension was inevitable, to produce a high-quality PQ symmetry. In six non-minimal GUTs of the current study, the sizes of the PQ-breaking effects due to the quantum gravity are generally under better control due to the gauge symmetries and the associated global DRS symmetries. It is thus reasonable to expect the long-standing PQ quality problem can be avoided in non-minimal GUTs with $n_g = 3$, where the emergent global DRS symmetries are general.

Table 5. The PQ charge assignments and $SU(6)_{\overline{\mathbf{11_F}}}$ representations of the fermions and Higgs fields in the minimal $SU(11)$ unification model.

	$6\times\overline{\mathbf{11_F}}$	$2\times\overline{\mathbf{55_F}}$	$\mathbf{165_F}$	$6\times\overline{\mathbf{55_H}}$	$2\times\overline{\mathbf{11_H}}$	$\mathbf{330_H}$	$\mathbf{462_H}$
PQ charges	1	0	0	−1	−1	0	0
$SU(6)_{\overline{\mathbf{11_F}}}$	□	1	1	$\bar{□}$	$\bar{□}$	1	1

Table 6. The non-minimal GUTs with $n_g = 3$, their one-loop β coefficients, the color anomaly factors, and the PQ-breaking operators.

Models	$\dim_{\mathbf{F}}$	b_1	$N_{SU(3)_c}$	PQ-breaking
$SU(7)$-A	133	−5	4	$(\overline{\mathbf{21_{H}}}_{\rho})^8\cdot(\mathbf{7_H})^2$
$SU(7)$-B	140	$-\frac{55}{6}$	7/2	$(\overline{\mathbf{7_{H}}}_{\rho})^7$
$SU(8)$	156	$-\frac{2}{3}$	9/2	$(\overline{\mathbf{8_{H}}}_{\rho})^9\cdot\mathbf{28_H}\cdot\mathbf{56_H}\cdot\mathbf{70_H}$
$SU(9)$	165	$-\frac{15}{2}$	9/2	$(\overline{\mathbf{36_H}})^9$
$SU(10)$	245	$\frac{223}{6}$	4	$(\overline{\mathbf{45_H}})^8(\overline{\mathbf{10_H}})^2(\mathbf{210_H})^2$
				$(\mathbf{120_H})^8(\overline{\mathbf{10_H}})^2(\mathbf{210_H})^2$
$SU(11)$	341	$\frac{83}{3}$	3	$(\overline{\mathbf{55_H}})^6(\mathbf{330_H})^3$

Evidently, we expect the following studies to be performed for specific models: (i) viable symmetry breaking patterns, (ii) SM fermion mass hierarchies and their mixings, and (iii) physical axion mass predictions and their related experimental searches. A recent study on the $SU(6)$ toy model [39] suggested that the bottom quark and tau lepton masses can be naturally suppressed to the top quark mass through the seesaw-like mass matrices with their heavy fermion partners. Such heavy fermion partners for both the down-type quarks and charged leptons are general in the non-minimal GUTs with $n_g = 3$. Furthermore, the non-minimal GUTs with $n_g = 3$ can naturally lead to multiple symmetry breaking scales between the $\Lambda_{\rm GUT}$ and the EW scale. In general, we expect that the observed fermion mass hierarchies among these three generations can be realized with the appropriate symmetry breaking pattern in the non-minimal GUT with $n_g = 3$. Ultimately, one has to analyze the gauge coupling unifications for the viable symmetry breaking patterns and predict the proton lifetime. Because our models possess several intermediate scales, this usually requires the two-loop RGEs, together with the matching conditions and mass threshold effects [49, 50].

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